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A GENERAL FORMULA ON THE CONJUGATE OF THE DIFFERENCE OF FUNCTIONS

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ABSTRACT. Given an arbitrary function $g:X \to (-\infty, +\infty]$ and a lower-semicontinuous convex function $h:X \to (-\infty, +\infty]$, we give the general expression of the conjugate $(g - h)^*$ of $g - h$ in terms of g^* and h^*. As a consequence, we get Toland's duality theorem:

$$\inf_{x \in X} \{g(x) - h(x)\} = \inf_{x^* \in X^*} \{h^*(x^*) - g^*(x^*)\}.$$

1. **Introduction**. The conjugacy operation which associates a function f^* defined on X^* with every function f defined on X is a key-tool in formulating variational principles in nonconvex optimization and calculus of variations as well as in deriving duality schemes for such problems. It has also been proved to be a useful device in other areas like statistics, through the Cramer transformation for example. More precisely, let μ be a probability measure on X and $L_\mu:X^* \to]0, +\infty]$ its Laplace transform, that is:

$$\forall x^* \in X^* \qquad L_\mu(x^*) = \int_X e^{\langle x^*, x\rangle} \mathrm{d}\mu(x). \tag{1.1}$$

The Cramer transform C_μ of μ is defined on X as:

$$\forall x \in X \qquad C_\mu(x) = \sup_{x^* \in X^*} \{\langle x^*, x\rangle - \operatorname{Log} L_\mu(x^*)\}. \tag{1.2}$$

So, C_μ is nothing else than the conjugate function of Log L_μ. See [2] for a detailed account of the use of Cramer transforms in studying large deviations in statistics.

Calculating f^* in terms of f_i^* when f has been constructed from other functions f_i constitute the body of calculus rules on conjugate functions. As for example, the conjugate of the infimal convolution of f_1 and f_2 is the sum of the conjugate functions of f_1 and f_2. Our aim here is to give the *exact* expression of the conjugate of $f = g - h$, h convex, in its most general setting. We thereby shed a new light on Toland's duality results associating

$$\inf_{x \in X} \{g(x) - h(x)\} \quad \text{and} \quad \inf_{x^* \in X^*} \{h^*(x^*) - g^*(x^*)\}.$$

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2. **The conjugate of the difference of two functions**. As is usual in the context of convex analysis, we work in the setting of two locally convex (real) topological vector spaces X and X^* *paired in separating duality* by a bilinear form we denote by $\langle\, ,\rangle$ (cf., e.g. [6, §6], [5, §6.3], [8, §3]). Given $f: X \to \overline{\mathbb{R}}$ ($\overline{\mathbb{R}}$ denotes the extended set of real numbers), the *conjugate function* $f^*: X^* \to \overline{\mathbb{R}}$ is defined as:

$$(2.1) \qquad \forall x^* \in X^* \qquad f^*(x^*) = \sup_{x \in X} \{\langle x, x^* \rangle - f(x)\}.$$

In particular, the infimum of f over X is nothing more than $-f^*(0)$. We recall that f is equal to its biconjugate f^{**} $(= (f^*)^*)$ whenever f is convex, lower-semicontinuous and proper (f proper means that f does not take the value $-\infty$ and is not constantly equal to $+\infty$). The class of such functions sometimes is denoted by $\Gamma_0(X)$ in the literature ([6, 3]). When dealing with difference of functions which possibly take the value $+\infty$, we are confronted with ambiguities like $(+\infty) + (-\infty)$. We adopt, as does Moreau ([6]), the following rules on $\overline{\mathbb{R}}$:

$$(2.2) \qquad \begin{array}{l} (+\infty) \dot{+} (-\infty) = (-\infty) \dot{+} (-\infty) = (+\infty). \\ (+\infty) \dot{-} (+\infty) = (+\infty) \dot{+} (-\infty). \end{array}$$

We assume throughout that both g and h are *proper* functions of X into $(-\infty, +\infty]$ and we set:

$$(2.3) \qquad \forall x \in X \;\; f(x) = g(x) \dot{-} h(x).$$

There is actually no loss of generality in supposing that g and h are proper. For the purposes of minimizing f, we note that:

$$(2.4) \qquad \inf_{x \in X} f(x) = \inf_{x \in \operatorname{dom} g} \{g(x) - h(x)\},$$

where dom g denotes the set where g is finite.

PROPOSITION 2.1.

$$(2.5) \qquad \forall x^* \in X^* \quad f^*(x^*) \geq \sup_{h^*(y^*) \in \mathbb{R}} \{g^*(x^* + y^*) - h^*(y^*)\},$$

with $\sup\limits_{\phi} = -\infty$ *by convention.*

PROOF.

$$f^*(x^*) = \sup_{x \in \operatorname{dom} g} \{\langle x, x^* \rangle - [g(x) - h(x)]\} \geq \langle x, x^* + y^* \rangle - g(x)$$
$$+ h(x) - \langle x, y^* \rangle \quad \text{for all } x \in \operatorname{dom} g \text{ and all } y^* \in \operatorname{dom} h^*.$$

Thus

$$f^*(x^*) \dot{+} \{\langle x, y^* \rangle - h(x)\} \geq \langle x, x^* + y^* \rangle - g(x)$$
$$\text{for all } x \in \operatorname{dom} g \text{ and all } y^* \in Y^*.$$

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Whence

$$f^*(x^*) \dot{+} h^*(y^*) \geq g^*(x^* + y^*) \quad \text{whenever } h^*(y^*) \in \mathbb{R},$$

and the announced inequality is proved. □

To get equality in (2.5), we assume more on h, more precisely that h is *convex, lower-semicontinuous and proper* ($h \in \Gamma_0(X)$).

THEOREM 2.2. *If $h \in \Gamma_0(X)$ then*

$$(2.6) \qquad \forall x^* \in X^* \quad f^*(x^*) = \sup_{y^* \in \operatorname{dom} h^*} \{g^*(x^* + y^*) - h^*(y^*)\}.$$

PROOF. dom h^* is nonempty since $h \in \Gamma_0(X)$. Suppose now there is $r \in \mathbb{R}$ such that

$$(2.7) \qquad g^*(x^* + y^*) - h^*(y^*) \leq r \quad \text{for all } y^* \in \operatorname{dom} h^*.$$

If not there is nothing to prove to get (2.6).

We infer from (2.7):

$$g^*(x^* + y^*) \leq h^*(y^*) + r \quad \text{for all } y^* \in Y^*.$$

Taking the conjugate of both sides of the above inequality yields:

$$g^{**}(y) - \langle y, x^* \rangle \geq h^{**}(y) - r = h(y) - r \quad \text{for all } y \in X.$$

Now, since $g^{**} \leq g$, we derive:

$$\langle y, x^* \rangle - [g(y) - h(y)] \leq r \quad \text{for all } y \in \operatorname{dom} g.$$

Whence $(g - h)^*(x^*) \leq r$. Q.E.D. □

By just making $x^* = 0$ in Proposition 2.1 and Theorem 2.2 we get Toland's results ([9, Sections 2.1 and 3.1], [10, Section 2.1]).

COROLLARY 2.3.

$$\operatorname{Inf}_{x \in X} \{g(x) \dot{-} h(x)\} \leq \inf_{x^* \in X^*} \{h^*(x^*) \dot{-} g^*(x^*)\}.$$

Equality holds whenever $h \in \Gamma_0(X)$.

We have shown that:

$$(2.8) \quad \inf_{x \in \operatorname{dom} g} \{g(x) - h(x)\} = \inf_{x^* \in \operatorname{dom} h^*} \{h^*(x^*) - g^*(x^*)\} \quad \text{whenever } h \in \Gamma_0(X).$$

By just using that sup $(\cdot) = -$ inf $(-\cdot)$, we also derive:

$$(2.9) \quad \sup_{x \in \operatorname{dom} h} \{g(x) - h(x)\} = \sup_{x^* \in \operatorname{dom} g^*} \{h^*(x^*) - g^*(x^*)\} \quad \text{whenever } g \in \Gamma_0(X).$$

REMARK 1. Toland's result (2.8) has been obtained here as a consequence of the formula (2.6) giving $(g - h)^*$. Conversely, one may wonder whether one can get (2.6) having the relation (2.8) or (2.9) at our disposal. The answer is yes and the proof comes as follows:

$$(g - h)^*(x^*) = \sup_{x \in \operatorname{dom} g} \{\langle x, x^* \rangle - [g(x) - h(x)]\}$$
$$= \sup_{x \in \operatorname{dom} g} \{h(x) - [g(x) - \langle x, x^* \rangle]\}.$$

It then remains to apply (2.9) with the functions $x \to h(x)$ and $x \to g(x) - \langle x, x^* \rangle$.

REMARK 2. An alternate proof of (2.6) can be offered assuming that h is subdifferentiable on X ([4]). Both proofs are actually simple and concise.

COMMENTS. Pshenichnyi ([7]) should be credited with having considered firstly the conjugate of the difference of convex functions. He proved the formula of Theorem 2.2 assuming that both g and h are finite-valued convex functions. The way he proved that was first deriving the relation (2.9) and then deducing the general formula as it is sketched in Remark 1.

In two successive papers ([9], [10]), Toland proved and exploited thoroughly the equality (2.8). The mapping which assigns $h^* - g^*$ to $g - h$, g and $h \in \Gamma_0(X)$, is sometimes referred to as Toland's involution. The duality schemes Toland proposed in the context of variational problems have been further developed by Auchmuty ([1]).

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NOTE ADDED ON PROOFS: The equalities (2.8) and (2.9) also appear in the following papers:

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