II

The Baouendi–Treves approximation formula

In this chapter we prove what is probably the most important single result in the theory of locally integrable structures. It states that in a small neighborhood of a given point of the domain of a locally integrable structure $\mathcal{L}$, any solution of the equation $\mathcal{L}u = 0$ may be approximated by polynomials in a set of a finite number of homogeneous solutions as soon as the solutions in that set are chosen with linearly independent differentials and the number of them is equal to the corank of $\mathcal{L}$. Such a set is called a complete set of first integrals of the locally integrable structure.

The proof is relatively simple for classical solutions and depends on the construction of a suitable approximation of the identity modeled on the kernel of the heat equation as shown in Section II.1. The extension to distribution solutions is carried out in Section II.2. Section II.3 studies the convergence of the formula in some of the standard spaces used in analysis: Lebesgue spaces L^p, $1 \leq p < \infty$; Sobolev spaces; Hölder spaces; and (localizable) Hardy spaces h^p, $0 < p < \infty$. The last section is devoted to applications.

II.1 The approximation theorem

Since the approximation formula is of a local nature it will be enough to restrict our attention to a locally integrable structure $\mathcal{L}$ defined in an open subset Ω of $\mathbb{R}^N$ over which $\mathcal{L}^\perp$ is spanned by the differentials $\mathrm{d}Z_1, \dots, \mathrm{d}Z_m$ of m smooth functions $Z_j \in C^\infty(\Omega)$, $j = 1, \dots, m$, at every point of Ω. Thus, if n is the rank of $\mathcal{L}$, we recall that $N = n + m$.

Given a distribution $u \in \mathcal{D}'(\Omega)$ we say that u is a homogeneous solution of $\mathcal{L}$ and write $\mathcal{L}u = 0$ if

$$Lu = 0 \quad \text{on } U$$

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for every local section L of $\mathcal{L}$ defined on an open subset $U \subset \Omega$. Simple examples of homogeneous solutions of $\mathcal{L}$ are the constant functions and also the functions $Z_1, \ldots, Z_m$, since $LZ_j = \langle \mathrm{d}Z_j, L\rangle = 0$ because $\mathrm{d}Z_j \in \mathcal{L}^\perp$, $j = 1, \ldots, m$. By the Leibniz rule, any product of smooth homogeneous solutions is again a homogeneous solution, so a polynomial with constant coefficients in the m functions Z_j, i.e., a function of the form

$$P(Z) = \sum_{|\alpha|\leq d} c_\alpha Z^\alpha, \quad \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m, \quad c_\alpha \in \mathbb{C}, \tag{II.1}$$

is also a homogeneous solution. The approximation theorem states that any distribution solution u of $\mathcal{L}u = 0$ is the weak limit of polynomial solutions such as (II.1).

Theorem II.1.1. *Let $\mathcal{L}$ be a locally integrable structure on Ω and assume that $\mathrm{d}Z_1, \ldots, \mathrm{d}Z_m$ span $\mathcal{L}^\perp$ at every point of Ω. Then, for any $p \in \Omega$, there exist two open sets U and W, with $p \in U \subset \overline{U} \subset W \subset \Omega$, such that*

(i) every $u \in \mathcal{D}'(W)$ that satisfies $\mathcal{L}u = 0$ on W is the limit in $\mathcal{D}'(U)$ of a sequence of polynomial solutions $P_j(Z_1, \ldots, Z_m)$:

$$u = \lim_{j\to\infty} P_j \circ Z \quad \textit{in } \mathcal{D}'(U);$$

(ii) if $u \in C^k(W)$ the convergence holds in the topology of $C^k(U)$, $k = 0, 1, 2, \ldots, \infty$.

Some well-known approximation results in analysis are particular cases of Theorem II.1.1.

Example II.1.2. Let $\mathcal{L}$ be the locally integrable structure generated over an open set $\Omega \subset \mathbb{C}$ by the Cauchy–Riemann vector field

$$\overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad z = x + iy.$$

Then a distribution solution of $\overline{\partial}u = 0$ is just a holomorphic function and the theorem simply states that any holomorphic function can be locally approximated by polynomials in the complex variable z.

Later we will give several applications of the approximation theorem but we wish to point out already one interesting consequence. Assume that two points $p, q \in U$ are such that $Z(p) = Z(q)$ and let $u \in C^0(\Omega)$ satisfy $\mathcal{L}u = 0$. Then $P \circ Z(p) = P \circ Z(q)$ for any polynomial P in m variables and, by the uniform approximation of u on U by polynomials in Z, it follows that $u(p) = u(q)$. The *fibers of Z in U* are, by definition, the equivalence classes of the equivalence relation defined by ‘$p \sim q$ if and only if $Z(p) = Z(q)$’. Thus, every solution $u \in C^0(\Omega)$ of $\mathcal{L}u = 0$ is constant on the fibers of Z. In particular, if the differentials of $Z_1^\#, \dots, Z_m^\#$ span $\mathcal{L}^\perp$ over Ω it follows that $Z^\# = (Z_1^\#, \dots, Z_m^\#)$ is constant on the fibers of Z in U. Applying the theorem with $Z^\#$ in the place of Z we may as well find a neighborhood $U^\# \subset U$ of p such that Z is constant on the fibers of $Z^\#$ in $U^\#$, which shows that the fibers of Z and the fibers of $Z^\#$ on $U^\#$ are identical. Thus, in the sense of germs of sets at p, the equivalence classes defined by Z and those defined by any other $Z^\# = (Z_1^\#, \dots, Z_m^\#)$ such that $\mathrm{d}Z_1^\#, \dots, \mathrm{d}Z_m^\#$ generates $\mathcal{L}^\perp$ coincide. This independence of the particular choice of Z allows us to talk about the *germs at p of the fibers of $\mathcal{L}$* which are invariants of the structure.

The fact that u is constant on the fibers of Z in U when $\mathcal{L}u = 0$, $u \in C^0(\Omega)$, may be expressed by saying that there exists a function $\widehat{u} \in C^0(Z(U))$ such that $u = \widehat{u} \circ Z$. Thus, any continuous solution of $\mathcal{L}u = 0$ can be factored as the composition with Z of a continuous function defined on a subset of $\mathbb{C}^m$. In general, the set $Z(U)$ may be irregular but if it happens to be a submanifold of $\mathbb{C}^m$, then $\widehat{u}$ will satisfy in the weak sense the induced Cauchy–Riemann equations on $Z(U)$. Hence, at a conceptual level, the theorem links the study of solutions of $\mathcal{L}u = 0$ to solutions of the induced Cauchy–Riemann equations on certain sets of $\mathbb{C}^m$.

We will prove Theorem II.1.1 in several steps. The first step consists of taking convenient local coordinates in a neighborhood of p. Applying Corollary I.10.2, there exists a local coordinate system vanishing at p,

$$\{x_1, \dots, x_m, t_1, \dots, t_n\}$$

and smooth, real-valued functions $\phi_1, \dots, \phi_m$ defined in a neighborhood of the origin and satisfying

$$\phi_k(0,0) = 0, \quad \mathrm{d}_x\phi_k(0,0) = 0, \quad k = 1, \dots, m,$$

such that the functions Z_k, $k = 1, \dots, m$, may be written as

$$Z_k(x,t) = x_k + i\phi_k(x,t), \quad k = 1, \dots, m, \tag{II.2}$$

on a neighborhood of the origin. To do so we need to assume that the real parts of $\mathrm{d}Z_1, \dots, \mathrm{d}Z_m$ are linearly independent, for which we might have to replace Z_j by iZ_j for some of the indexes $j \in \{1, \dots, m\}$. Notice that this will not change the conclusion of the theorem. Thus, we may choose a number R such that if

$$V = \{q: \quad |x(q)| < R, \ |t(q)| < R\}$$

then (II.2) holds in a neighborhood of $\overline{V}$ and we may assume that

$$\left\| \left(\frac{\partial \phi_j(x,t)}{\partial x_k} \right) \right\| < \frac{1}{2}, \quad (x,t) \in \overline{V}, \tag{II.3}$$

where the double bar indicates the norm of the matrix $\phi_x(x,t) = (\partial \phi_j(x,t)/\partial x_k)$ as a linear operator in $\mathbb{R}^m$. Modifying the functions ϕ_k's off a neighborhood of $\overline{V}$ may assume without loss of generality that the functions $\phi_k(x,t)$, $k = 1, \dots, m$, are defined throughout $\mathbb{R}^N$, have compact support and satisfy (II.3) everywhere, that is

$$\left\| \left(\frac{\partial \phi_j(x,t)}{\partial x_k} \right) \right\| < \frac{1}{2}, \quad (x,t) \in \mathbb{R}^N. \tag{II.3$'$}$$

Modifying also $\mathcal{L}$ off a neighborhood of $\overline{V}$ we may assume as well that the differentials $\mathrm{d}Z_j$, $j = 1, \dots, m$, given by (II.2), span $\mathcal{L}^\perp$ over $\mathbb{R}^N$. Of course, the new structure $\mathcal{L}$ and the old one coincide on V so any conclusion we draw about the new $\mathcal{L}$ on V will hold as well for the original $\mathcal{L}$. We will make use of the vector fields L_j, $j = 1, \dots, n$ and M_k, $k = 1, \dots, m$ entirely analogous to those introduced in Chapter I after Corollary I.10.2, with the only difference that here they are defined throughout $\mathbb{R}^N$. We recall from Chapter I that the vector fields

$$M_k = \sum_{\ell=1}^{m} \mu_{k\ell}(x,t) \frac{\partial}{\partial x_\ell}, \quad k = 1, \dots, m$$

are characterized by the relations

$$M_k Z_\ell = \delta_{k\ell}, \quad k, \ell = 1, \dots, m$$

and that the vector fields

$$L_j = \frac{\partial}{\partial t_j} - i \sum_{k=1}^{m} \frac{\partial \phi_k}{\partial t_j}(x,t) M_k, \quad j = 1, \dots, n$$

are linearly independent and satisfy $L_j Z_k = 0$, for $j = 1, \dots, n$, $k = 1, \dots, m$. Hence, $L_1, \dots, L_n$ span $\mathcal{L}$ at every point while the $N = n + m$ vector fields

$$L_1, \dots, L_n, M_1 \dots, M_m$$

are pairwise commuting and span $\mathbb{C}T_p(\mathbb{R}^N)$, $p \in \mathbb{R}^N$. Since

$$\mathrm{d}Z_1, \dots, \mathrm{d}Z_m, \mathrm{d}t_1, \dots, \mathrm{d}t_n \quad \text{span } \mathbb{C}T^*\mathbb{R}^N$$

the differential $\mathrm{d}w$ of a C^1 function $w(x,t)$ may be expressed in this basis. In fact, we have

$$\mathrm{d}w = \sum_{j=1}^{n} L_j w \,\mathrm{d}t_j + \sum_{k=1}^{m} M_k w \,\mathrm{d}Z_k \tag{II.4}$$

which may be checked by observing that $L_j Z_k = 0$ and $M_k t_j = 0$ for $1 \leq j \leq n$ and $1 \leq k \leq m$, while $L_j t_k = \delta_{jk}$ for $1 \leq j, k \leq n$ and $M_k Z_j = \delta_{jk}$ for $1 \leq j, k \leq m$ ($\delta_{jk} =$ Kronecker delta).

We now choose the open set W as any fixed neighborhood of $\overline{V}$ in Ω. In proving the theorem we will assume initially that u is a smooth homogeneous solution of $\mathcal{L}u = 0$ defined in W with continuous derivatives of all orders, i.e., $u \in C^\infty(W)$ satisfies on W the overdetermined system of equations

$$\begin{cases} L_1 u = 0, \\ L_2 u = 0, \\ \cdots\cdots\cdots \\ L_n u = 0. \end{cases} \tag{II.5}$$

Given such u we define a family of functions $\{E_\tau u\}$ that depend on a real parameter τ, $0 < \tau < \infty$, by means of the formula

$$E_\tau u(x, t) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} \mathrm{e}^{-\tau[Z(x,t)-Z(x',0)]^2} u(x', 0) h(x') \det Z_x(x', 0)\, \mathrm{d}x'$$

which we now discuss. For $\zeta = (\zeta_1, \ldots, \zeta_m) \in \mathbb{C}^m$ we will use the notation $[\zeta]^2 = \zeta_1^2 + \cdots + \zeta_m^2$, which explains the meaning of $[Z(x, t) - Z(x', 0)]^2$ in the formula. The function $h(x) \in C_c^\infty(\mathbb{R}^m)$ satisfies $h(x) = 0$ for $|x| \geq R$ and $h(x) = 1$ in a neighborhood of $|x| \leq R/2$ (recall that R was introduced right before (II.3) in the definition of the set V). Note that since u is assumed to be defined in a neighborhood of $\overline{V}$, the product $u(x', 0)h(x')$ is well-defined on $\mathbb{R}^m$, compactly supported, and of class C^∞. Since Z has m components we may regard Z_x as the $m \times m$ matrix $(\partial Z_j/\partial x_k)$ and denote by $\det Z_x$ its determinant. Furthermore, since the exponential in the integrand is an entire function of $(Z_1, \ldots, Z_m)$, the chain rule shows that it satisfies the homogenous system of equations (II.5) and the same holds for $E_\tau u(x, t)$ by differentiation under the integral sign. The second step of the proof will be to show that $E_\tau u(x, t) \to u(x, t)$ as $\tau \to \infty$ uniformly for $|x| < R/4$ and $|t| < T < R$ if T is conveniently small. Once this is proved we may approximate in the C^∞ topology the exponential $\mathrm{e}^{-\tau[\zeta]^2}$ (for fixed large τ) by the partial sum of degree k, $P_k(\zeta)$, of its Taylor series on a fixed polydisk that contains the set $\{\sqrt{\tau}(Z(x, t) - Z(x', 0)) : |x|, |x'| < R, |t| < R\}$, so replacing the exponential in the definition of E_τ by $P_k(Z(x, t) - Z(x', 0))$ we will find polynomials in $Z(x, t)$ that approximate $E_\tau u(x, t)$ in the C^∞ topology for $|x| < R/4$ and $|t| < T$ when k is large. Hence, from now on we fix our attention on the convergence of $E_\tau u \to u$. We consider the following modification of the operator E_τ:

$$G_\tau u(x,t) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} e^{-\tau[Z(x,t)-Z(x',t)]^2} u(x',t) h(x') \det Z_x(x',t)\, dx'.$$

Notice that in the trivial case in which the functions ϕ_k, $k = 1, \ldots, m$, vanish identically so $Z(x,t) = x$ and $\det Z_x = 1$, G_τ is just the convolution of $u(x,0)h(x)$ with a Gaussian in $\mathbb{R}^m$, which is a well-known approximation of the identity as $\tau \to \infty$. In general, the functions ϕ_k do not vanish but they are relatively small because they vanish at the origin and (II.3′) holds, so G_τ is still an approximation of the identity. The idea is then to prove that $G_\tau u \to u$ and then estimate the difference $R_\tau u = G_\tau u - E_\tau u$ using the fact that $\mathcal{L}u = 0$.

LEMMA II.1.3. *Let B be an $m \times m$ matrix with real coefficients and norm $\|B\| < 1$ and set $A = I + iB$ where I is the identity matrix. Then*

$$\det A \int_{\mathbb{R}^m} e^{-[Ax]^2}\, dx = \pi^{m/2}.$$

PROOF. We may write $[Ax]^2 = {}^tAAx \cdot x$ (the dot indicates the standard inner product in $\mathbb{R}^m$ and also its extension as a $\mathbb{C}$-bilinear form to $\mathbb{C}^m$) so $e^{-[Ax]^2} = e^{-Cx\cdot x}$ where the matrix $C = {}^tAA$ has positive definite real part $\Re C = I - {}^tBB$ because $\|B\| < 1$. It is then known that (see, e.g., **[H2,** page 85**]**)

$$\int_{\mathbb{R}^m} e^{-Cx\cdot x}\, dx = \pi^{m/2}(\det C)^{-1/2}$$

where the branch of the square root is chosen so $(\det C)^{1/2} > 0$ when C is real. Since $\det C = (\det A)^2$ the proof is complete. □

Set $h(x)u(x,t)\det Z_x(x,t) = v(x,t)$. For (x,t) fixed, the matrix $Z_x(x,t) = I + i\phi_x(x,t)$ satisfies the hypotheses of the lemma in view of (II.3′). Thus, we may write

$$h(x)u(x,t) = \pi^{-m/2} \int_{\mathbb{R}^m} e^{-[Z_x(x,t)x']^2} v(x,t)\, dx'.$$

Introducing the change of variables $x' \mapsto x + \tau^{-1/2}x'$ in the integral that defines $G_\tau u$ we get

$$G_\tau u(x,t) = \pi^{-m/2} \int_{\mathbb{R}^m} e^{-\tau[Z(x,t)-Z(x+\tau^{-1/2}x',t)]^2} v(x+\tau^{-1/2}x',t)\, dx'.$$

Then

$$G_\tau u(x,t) - h(x)u(x,t) = I_\tau + J_\tau,$$

where

$$I_\tau(x,t) = \pi^{-m/2} \int_{\mathbb{R}^m} \mathrm{e}^{-[Z_x(x,t)x']^2} (v(x+\tau^{-1/2}x',t) - v(x,t))\,\mathrm{d}x'$$

and

$$J_\tau(x,t) = \pi^{-m/2}$$
$$\int_{\mathbb{R}^m} \left(\mathrm{e}^{-\tau[Z(x,t)-Z(x+\tau^{-1/2}x',t)]^2} - \mathrm{e}^{-[Z_x(x,t)x']^2}\right) v(x+\tau^{-1/2}x',t)\,\mathrm{d}x'.$$

To estimate I_τ we observe that $|\mathrm{e}^{-[Z_x(x,t)x']^2}| = \mathrm{e}^{-|x'|^2 + |\phi_x(x,t)x'|^2} \le \mathrm{e}^{-3|x'|^2/4}$ in view of (II.3′). We also observe that $|\nabla_x v(x,t)|$ is bounded in $\mathbb{R}^m \times \{|t| \le R\}$ because v vanishes for large x, so the mean value theorem gives

$$|I_\tau(x,t)| \le C\tau^{-1/2} \int_{\mathbb{R}^m} \mathrm{e}^{-3|x'|^2/4} |x'|\,\mathrm{d}x' \le C'\tau^{-1/2},$$

showing that $|I_\tau(x,t)| \to 0$ as $\tau \to \infty$ uniformly on $\mathbb{R}^m \times \{|t| \le R\}$. To estimate J_τ we first observe that $|\mathrm{e}^{-\tau[Z(x,t)-Z(x+\tau^{-1/2}x',t)]^2} - \mathrm{e}^{-[Z_x(x,t)x']^2}| \le 2\mathrm{e}^{-3|x'|^2/4}$, so

$$|J_\tau(x,t)| \le C \int_{|x'|<K} |\mathrm{e}^{-\tau[Z(x,t)-Z(x+\tau^{-1/2}x',t)]^2} - \mathrm{e}^{-[Z_x(x,t)x']^2}|\,\mathrm{d}x'$$
$$+ C\exp(-K^2/2).$$

Thus, to show that $|J_\tau(x,t)| \to 0$ uniformly we need only estimate the integral on $|x'| < K$ for any large K. When $|x'| \le K$ and $|t| \le R$, the Leibniz quotient $\zeta_1 = (Z(x,t) - Z(x+\tau^{-1/2}x',t))/\tau^{-1/2}$ converges to $\zeta_2 = -Z_x(x,t)x'$ uniformly in x as $\tau \to \infty$ in view of (II.3′), which also implies that $\Re[\zeta_1]^2 \ge 0$ and $\Re[\zeta_2]^2 \ge 0$. Since $\mathrm{e}^{-\zeta}$ is a Lipschitz function on $\Re\zeta \ge 0$ and $|[\zeta_1]^2 - [\zeta_2]^2| \le C\tau^{-1/2}$ (note that ζ_2 remains bounded as $(x,t) \in \mathbb{R}^N$ and $|x'| \le K$), we have

$$|J_\tau(x,t)| \le CK^m\tau^{-1/2} + C\exp(-K^2/2),$$

which shows that $J_\tau(x,t) \to 0$ uniformly for $x \in \mathbb{R}^m$ and $|t| \le R$ as $\tau \to \infty$. Thus, $G_\tau u(x,t) \to h(x)u(x,t)$ uniformly and the limit $h(x)u(x,t) = u(x,t)$ for $|x| < R/2$.

We will now estimate the remainder $R_\tau = G_\tau - E_\tau$ by means of Stokes' theorem. The fact that u satisfies the system (II.5)—which was not used to prove that $G_\tau u \to hu$—is essential at this point. For $(x,t) \in \mathbb{R}^N$ fixed consider the m-form on $\mathbb{R}^N$ given by

$$\begin{aligned}\omega(x',t') &= (\tau/\pi)^{m/2} \mathrm{e}^{-\tau[Z(x,t)-Z(x',t')]^2} u(x',t')h(x')\,\mathrm{d}Z(x',t') \\ &= v(x',t')\,\mathrm{d}Z(x',t'),\end{aligned}$$

where $\mathrm{d}Z = \mathrm{d}Z_1 \wedge \cdots \wedge \mathrm{d}Z_m$. Hence, we may write

$$G_\tau u(x,t) = \int_{\mathbb{R}^m \times \{t\}} \omega \quad \text{and} \quad E_\tau u(x,t) = \int_{\mathbb{R}^m \times \{0\}} \omega,$$

observing that the pullback of $\mathrm{d}Z(x', t')$ to a slice $\{t = c = \text{const.}\}$ is given by $\det Z_x(x', c)\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m$. Keeping in mind that ω vanishes identically for $|x'| > R$ and invoking Stokes' theorem, we have

$$G_\tau u(x,t) - E_\tau u(x,t) = \int_{\mathbb{R}^m \times [0,t]} \mathrm{d}\omega$$

where $[0, t]$ denotes the segment joining the origin of $\mathbb{R}^n$ to the point $t \in \mathbb{R}^n$. To compute $\mathrm{d}\omega$ we will take advantage of expression (II.4). We have $\mathrm{d}\omega = \mathrm{d}v \wedge \mathrm{d}Z$ so the only terms in (II.4) that matter here are those that do not contain $\mathrm{d}Z_j$, $j = 1, \ldots, m$, i.e., $\mathrm{d}\omega = \sum_{j=1}^n L_j v\, \mathrm{d}t_j \wedge \mathrm{d}Z$. Since the exponential factor in v is an entire function of $Z_1, \ldots, Z_n$, and thus satisfies (II.5) as well as u, we obtain

$$R_\tau u(x,t) = (\tau/\pi)^{m/2} \sum_{j=1}^n \int_{\mathbb{R}^m \times [0,t]} \mathrm{e}^{-\tau[Z(x,t)-Z(x',t')]^2} u(x', t') L_j h(x')\, \mathrm{d}t_j \wedge \mathrm{d}Z(x', t').$$

Assume now that $|x| \le R/4$ and $|t| \le T$, where T will be chosen momentarily. We wish to estimate the exponential factor

$$|\mathrm{e}^{-\tau[Z(x,t)-Z(x',t')]^2}| = \mathrm{e}^{\tau(|\phi(x,t)-\phi(x',t')|^2 - |x-x'|^2)}.$$

We have

$$\begin{aligned} |\phi(x,t) - \phi(x',t')| &\le |\phi(x,t) - \phi(x',t)| + |\phi(x',t) - \phi(x',t')| \\ &\le \frac{1}{2}|x - x'| + C|t - t'| \\ &\le \frac{1}{2}|x - x'| + CT \end{aligned}$$

because $t' \in [0, t]$ and $|t| \le T$. Hence,

$$|\phi(x,t) - \phi(x',t')|^2 \le \frac{1}{2}|x - x'|^2 + 2\lambda T^2$$

and

$$\left|\mathrm{e}^{-\tau[Z(x,t)-Z(x',t')]^2}\right| = \mathrm{e}^{\tau(2\lambda T^2 - |x-x'|^2/2)},$$

where λ is a bound that depends only on ϕ and does not depend on u. Since $L_j h$ vanishes for $|x'| \le R/2$ we have that $|x'| \ge R/2$ in all integrands in the expression of R_τ, so $|x - x'| \ge R/4$ and

$$|R_\tau u(x,t)| \le C\mathrm{e}^{\tau(2\lambda T^2 - R^2/32)}.$$

We may now choose T small enough so as to achieve $|R_\tau u(x,t)| \le C\mathrm{e}^{-\tau R^2/33}$. This proves that $|R_\tau u(x,t)| \to 0$ uniformly on $U = \{|x| \le R/4\} \times \{|t| \le T\}$. Summing up, we have found a neighborhood of the origin U such that for any C^∞-solution u of (II.5) defined in W, $E_\tau u \to u$ uniformly on U, which partially proves part (i) of the theorem for very regular distributions.

The third step is to prove part (ii) of the theorem for $k = \infty$ (the cases $1 \le k < \infty$ will be proved later). The main tool is the use of commutation formulas for the vector fields M_k with G_τ.

Lemma II.1.4. *For $u \in C^1(W)$ and $k = 1, \dots, m$, the following identity holds:*

$$\begin{aligned} M_k G_\tau u(x,t) - G_\tau M_k u(x,t) &= [M_k, G_\tau]u(x,t) \\ = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} &\mathrm{e}^{-\tau[Z(x,t)-Z(x',t)]^2} u(x',t) M_k h(x') \det Z_x(x',t)\,\mathrm{d}x'. \end{aligned} \tag{II.6}$$

Proof. By the symmetry in the variables x and x' of the expression

$$Z_j(x,t) - Z_j(x',t), \quad j = 1, \dots, m$$

we have

$$\begin{aligned} \delta_{jk} &= M_k(x,t,D_x)(Z_j(x,t) - Z_j(x',t))\big) \\ &= -M_k(x',t,D_{x'})(Z_j(x,t) - Z_j(x',t))\big). \end{aligned}$$

Thus, if $F(\zeta)$ is an entire holomorphic function and we set

$$f(x,x',t) = F(Z(x,t) - Z(x',t))$$

we also have, by the chain rule,

$$M_k(x,t,D_x) f(x,t,t') = -M_k(x',t,D_{x'}) f(x,t,t').$$

Applying this to $F(\zeta) = \mathrm{e}^{-\tau[\zeta]^2}$ we get, after differentiation under the integral sign that

$$\begin{aligned} M_k G_\tau u(x,t) = &-(\tau/\pi)^{m/2} \\ &\int_{\mathbb{R}^m} M_k(x',t,D_{x'})(\mathrm{e}^{-\tau[Z(x,t)-Z(x',t)]^2}) u(x',t) h(x')\,\mathrm{d}Z(x',t) \end{aligned}$$

where we have used the fact that the pullback to any slice $t' =$ const. of the m-form $\mathrm{d}Z_1 \wedge \cdots \wedge \mathrm{d}Z_n$ is given by $\det Z_x(x',t)\,\mathrm{d}x'$. Next, using the 'integration by parts' formula

$$\int_{\mathbb{R}^m} M_k v\, w\,\mathrm{d}Z = -\int_{\mathbb{R}^m} v\, M_k w\,\mathrm{d}Z \tag{II.7}$$

which is valid if v and w are of class C^1 and one of them has compact support, we get

$$M_k G_\tau u(x,t) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} \mathrm{e}^{-\tau[Z(x,t)-Z(x',t)]^2} \big(M_k u(x',t)h(x') + u(x',t)M_k h(x')\big)\,\mathrm{d}Z(x',t)$$

which proves (II.6). To complete the proof we show that (II.7) holds. Consider the exact m-form defined by

$$\begin{aligned}\omega_k &= \mathrm{d}(uv\,\mathrm{d}Z_1 \wedge \cdots \wedge \widehat{\mathrm{d}Z_k} \wedge \cdots \wedge \mathrm{d}Z_m)\\ &= \mathrm{d}(uv) \wedge \mathrm{d}Z_1 \wedge \cdots \wedge \widehat{\mathrm{d}Z_k} \wedge \cdots \wedge \mathrm{d}Z_m\end{aligned}$$

where the hat indicates that the factor $\mathrm{d}Z_k$ has been omitted. The pullback of ω_k to the slice $\{t\} \times \mathbb{R}^m$ is exact, so

$$\int_{\{t\}\times\mathbb{R}^m} \omega_k = 0. \tag{II.8}$$

Using (II.4) to compute $\mathrm{d}(uv)$ and observing that the pullback to the slice of terms that contain a factor $\mathrm{d}t_j$ vanish, we get

$$\omega_k|_{\{t\}\times\mathbb{R}^m} = (-1)^{k+1}(vM_k u + uM_k v)\,\mathrm{d}Z|_{\{t\}\times\mathbb{R}^m},$$

so (II.8) implies (II.7). □

Next we prove for the L_j commutation formulas analogous to (II.6). We write

$$\begin{aligned}L_j &= \frac{\partial}{\partial t_j} - i\sum_{k=1}^m \frac{\partial \phi_k}{\partial t_j}(x,t)M_k,\\ &= \frac{\partial}{\partial t_j} + \sum_{k=1}^m \lambda_{jk}\frac{\partial}{\partial x_k}, \qquad j = 1,\dots,n.\end{aligned}$$

We start with a technical lemma.

Lemma II.1.5.

$$\frac{\partial \det Z_x}{\partial t_j} + \sum_{k=1}^m \frac{\partial(\lambda_{jk}\det Z_x)}{\partial x_k} \equiv 0, \quad j = 1,\dots,n. \tag{II.9}$$

Proof. Note that (II.9) says that the vector field $(\det Z_x)L_j$ is divergence free, i.e., $\operatorname{div}((\det Z_x)L_j) = 0$, or that ${}^tL_j(\det Z_x) = 0$ where ${}^tL_j$ is the transpose of L_j. Take a test function $v(x,t)$ and consider the compactly supported exact form

$$\begin{aligned}\omega_j &= \mathrm{d}\big(v\,\mathrm{d}Z\wedge \mathrm{d}t_1\wedge\cdots\wedge\widehat{\mathrm{d}t_j}\wedge\cdots\wedge\mathrm{d}t_n\big)\\ &= \mathrm{d}v\wedge\mathrm{d}Z\wedge\mathrm{d}t_1\wedge\cdots\wedge\widehat{\mathrm{d}t_j}\wedge\cdots\wedge\mathrm{d}t_n\\ &= (-1)^{m+j-1}L_j v\,\mathrm{d}Z\wedge\mathrm{d}t\\ &= (-1)^{m+j-1}L_j v\,(\det Z_x)\,\mathrm{d}x\wedge\mathrm{d}t\end{aligned}$$

whose integral over $\mathbb{R}^N$ vanishes, that is,

$$\int_{\mathbb{R}^N} L_j v(\det Z_x)\,\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^N} v\,{}^tL_j(\det Z_x)\,\mathrm{d}x\mathrm{d}t = 0.$$

Since v is arbitrary, ${}^tL_j(\det Z_x)\equiv 0$ and (II.9) is proved. □

If $\tilde g(\zeta,t)$ is a smooth function on $\mathbb{C}^m\times\mathbb{R}^n$ that is holomorphic with respect to ζ and we set $g(x,t)=\tilde g(Z(x,t),t)$ we have, by the chain rule, that

$$L_j g(x,t) = \frac{\partial \tilde g}{\partial t_j}(Z(x,t),t)$$

because $L_jZ_k=0$, $k=1,\dots,m$. To take advantage of this fact we may write $G_\tau u(x,t) = (\tau/\pi)^{m/2}(\tilde G_\tau u)(Z(x,t),t)$, where

$$\tilde G_\tau u(\zeta,t) = \int_{\mathbb{R}^m} \mathrm{e}^{-\tau[\zeta-Z(x',t)]^2}u(x',t)h(x')\det Z_x(x',t)\,\mathrm{d}x',$$

so

$$L_jG_\tau u(x,t) = (\tau/\pi)^{m/2}\frac{\partial\tilde G_\tau u}{\partial t_j}(Z(x,t),t).$$

To compute the right-hand side of the last identity we write $e_\tau(\zeta,x',t) = \mathrm{e}^{-\tau[\zeta-Z(x',t)]^2}$, differentiate with respect to t_j under the integral sign, and observe that

$$\begin{aligned}\frac{\partial(e_\tau uh\det Z_x)}{\partial t_j} &= \frac{\partial(e_\tau uh)}{\partial t_j}\det Z_x + e_\tau uh\frac{\partial(\det Z_x)}{\partial t_j}\\ &= \det Z_xL_j(e_\tau uh) - \det Z_x\sum_{k=1}^m\lambda_{jk}\frac{\partial}{\partial x_k}(e_\tau uh)\\ &\quad + e_\tau uh\frac{\partial(\det Z_x)}{\partial t_j}.\end{aligned}$$

Note that the integral over $\mathbb{R}^m$ of the second term of the right-hand side may be written, after integration by parts, as

$$\int e_\tau uh\sum_{k=1}^m\frac{\partial}{\partial x_k}(\lambda_{jk}\det Z_x)\,\mathrm{d}x$$

so the integral of the second and third terms together yields

$$\int e_\tau uh\left(\frac{\partial(\det Z_x)}{\partial t_j}+\sum_{k=1}^{m}\frac{\partial}{\partial x_k}(\lambda_{jk}\det Z_x)\right)\mathrm{d}x=0,$$

in view of (II.9). Since $L_j(e_\tau)=0$, we also have that $\det Z_x L_j(e_\tau uh)=\det Z_x e_\tau(L_j u)h+\det Z_x e_\tau u L_j h$. This shows that

$$\frac{\partial}{\partial t_j}\tilde{G}_\tau u(\zeta,t)=\tilde{G}_\tau L_j u(\zeta,t)+\int e_\tau(\zeta,x',t)(u(L_j h)\det Z_x)(x',t)\,\mathrm{d}x'.$$

When $\zeta=Z(x,t)$ we obtain

LEMMA II.1.6. *For $u\in C^1(W)$ and $j=1,\ldots,m$ the following identity holds:*

$$\begin{aligned}L_j G_\tau u(x,t)-G_\tau L_j u(x,t)&=[L_j,G_\tau]u(x,t)\\&=(\tau/\pi)^{m/2}\int_{\mathbb{R}^m}\mathrm{e}^{-\tau[Z(x,t)-Z(x',t)]^2}u(x',t)L_j h(x')\det Z_x(x',t)\,\mathrm{d}x'.\end{aligned}\qquad\text{(II.10)}$$

□

Let us assume now that $u\in C^\infty(W)$ satisfies $\mathcal{L}u=0$ and we wish to prove that $E_\tau u(x,t)\to u(x,t)$ in $C^\infty(U)$. We have already proved that $G_\tau u\to hu$ uniformly in $\{|t|\le T\}\times\mathbb{R}^m$. Since $L_j M_k u=M_k L_j u=0$, $1\le j\le n$, $1\le k\le m$, $M_k u$ is a smooth solution of the system, so we also have that $G_\tau M_k u\to hM_k u$ uniformly on $\{|t|\le T\}\times\mathbb{R}^m$. Now, the expression (II.6) of $[M_k,G_\tau]u$ is almost identical to that of G_τ, the only difference being that h has been replaced by $M_k h$, so $[M_k,G_\tau]u\to(M_k h)u$. Restricting our attention to U where $h=1$ and $M_k h=0$, we conclude that $M_k G_\tau u=G_\tau M_k u+[M_k,G_\tau]u\to M_k u$ uniformly on U as $\tau\to\infty$. A similar conclusion can be obtained for $L_j G_\tau u$ using (II.10) instead of (II.6), that is, $L_j G_\tau u\to L_j u$ uniformly on U. Since any first-order derivative D may expressed as a linear combination with smooth coefficients of the M_k's and the L_j's, we see that $DG_\tau u\to Du$ uniformly on U. This shows that $G_\tau u\to u$ in $C^1(U)$. Of course, the argument can be iterated for higher-order derivatives to conclude that $G_\tau u\to u$ in $C^\infty(U)$.

II.2 Distribution solutions

We continue the proof of Theorem II.1.1, keeping the notations of Section II.1. In order to extend the arguments of the previous section to a distribution $u\in\mathcal{D}'(W)$ such that $\mathcal{L}u=0$—which is the fourth step of the proof of Theorem II.1.1—it is enough to check the following facts:

(a) $E_\tau u$ is well-defined for $u\in\mathcal{D}'(W)$;
(b) $G_\tau u$ is well-defined for $u\in\mathcal{D}'(W)$;

(c) $G_\tau u \to u$ in $\mathcal{D}'(U)$ as $\tau \to \infty$ for $u \in \mathcal{D}'(W)$;
(d) $R_\tau u = G_\tau u - E_\tau u \to 0$ in $\mathcal{D}'(U)$ as $\tau \to \infty$ for $u \in \mathcal{D}'(W)$.

We start by observing that since u satisfies the system of equations (II.5) on a neighborhood of $\overline{V}$, the wave front set $WF(u)$ of u is contained in the characteristic set of $\mathcal{L}$ and therefore does not intersect the set

$$\{(x, t, 0, \tau) \in \mathbb{R}^N \times \mathbb{R}^N, \quad |x|, |t| < R', \tau \neq 0\},$$

for some $R' > R$. Thus, $WF(hu)$ is contained in the same set and, in particular, the restriction of u to W belongs to

$$C^\infty(\{|t| \leq R\}; \mathcal{D}'(\{|x| < R\})).$$

On the connection between wave front sets and restrictions of distributions, we refer to [**H2**, chapter VIII]. Moreover, since $V = \{|x| < R\} \times \{|t| < R\}$ is relatively compact in W we may assume that $t \mapsto u(\cdot, t)$ is a continuous function with values in the L^2 based local Sobolev space $L^{2,s}_{\text{loc}}(B_R)$ of order s, for all $|t| \leq R$ and some real s, where B_R denotes the ball of radius R centered at the origin of $\mathbb{R}^m$ (for the definition of local Sobolev spaces see Section II.3.2 below). Thus, for any $|t| \leq R$, the trace $u(\cdot, t)$ is well-defined and belongs to $L^{2,s}_{\text{loc}}(B_R)$. Then, $E_\tau u(x, t)$ (resp. $G_\tau u(x, t)$) is well-defined if we interpret the integral as duality between the distribution $u(\cdot, 0)$ and the test function $(\tau/\pi)^{m/2} e^{-\tau[Z(x,t)-Z(x',0)]^2} h(x') \det Z_x(x', 0)$ (resp. $u(\cdot, t)$ and the test function $(\tau/\pi)^{m/2} e^{-\tau[Z(x,t)-Z(x',t)]^2} h(x') \det Z_x(x', t)$). This takes care of (a) and (b). To prove (d), it is convenient to express $R_\tau u$ by a reinterpretation of the formula obtained for smooth u using Stokes' theorem. We point out that the formula could also have been written as

$$R_\tau u(x, t) = \int_{[0,t]} \sum_{j=1}^n r_j(x, t, t', \tau)\, dt'_j, \tag{II.11}$$

where

$$r_j(x, t, t', \tau) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} e^{-\tau[Z(x,t)-Z(x',t')]^2} u(x', t') L_j h(x') \det Z_x(x', t')\, dx' \tag{II.12}$$

and $[0, t]$ denotes the straight segment joining 0 to t. In other words, by integrating first in x we may express the integral of an $m+1$-form over the cell $\mathbb{R}^m \times [0, t]$ as the integral of a 1-form over the segment $[0, t]$. In this form, Stokes' theorem is just a restatement of the fundamental theorem of calculus for a 1-form. To prove this claim, write for fixed ζ and τ

$$g(t') = \tilde{G}_\tau u(\zeta, t') = \int_{\mathbb{R}^m} e^{-\tau[\zeta - Z(x',t')]^2} u(x', t') h(x') \det Z_x(x', t')\, dx'.$$

Then,

$$g(t) - g(0) = \int_{[0,t]} \sum_{j=1}^{n} \frac{\partial g}{\partial t'_j}(t')\,\mathrm{d}t'_j. \tag{$*$}$$

To compute the derivatives of g we write $e_\tau(\zeta, x', t) = \mathrm{e}^{-\tau[\zeta - Z(x',t)]^2}$, differentiate with respect to t'_j under the integral sign, and recall that

$$\begin{aligned}\frac{\partial(e_\tau uh \det Z_x)}{\partial t'_j} &= \frac{\partial(e_\tau uh)}{\partial t'_j}\det Z_x + e_\tau uh \frac{\partial(\det Z_x)}{\partial t'_j} \\ &= \det Z_x L_j(e_\tau uh) - \det Z_x \sum_{k=1}^{m} \lambda_{jk} \frac{\partial}{\partial x_k}(e_\tau uh) \\ &\quad + e_\tau uh \frac{\partial(\det Z_x)}{\partial t'_j},\end{aligned}$$

a fact we already used in the proof of (II.10). Once again, the integral over $\mathbb{R}^m$ of the second term of the right-hand side may be written, after integrating by parts, as

$$\int e_\tau uh \sum_{k=1}^{m} \frac{\partial}{\partial x_k}(\lambda_{jk}\det Z_x)\,\mathrm{d}x$$

so the integral of the second and third terms together yields

$$\int e_\tau uh \left(\frac{\partial(\det Z_x)}{\partial t_j} + \sum_{k=1}^{m} \frac{\partial}{\partial x_k}(\lambda_{jk}\det Z_x) \right) \mathrm{d}x = 0,$$

in view of (II.9). Since $L_j(e_\tau u) = 0$, we also have that $\det Z_x L_j(e_\tau uh) = \det Z_x e_\tau u L_j h$. This shows that

$$\frac{\partial g}{\partial t'_j}(t') = \tilde{r}_j(\zeta, t', \tau) \tag{$**$}$$

where

$$\tilde{r}_j(\zeta, t', \tau) = \int_{\mathbb{R}^m} \mathrm{e}^{-\tau[\zeta - Z(x',t')]^2} u(x', t') L_j h(x') \det Z_x(x', t')\,\mathrm{d}x'.$$

Hence, $(*)$ for $\zeta = Z(x,t)$ gives an alternative proof of the fact that $R_\tau u = G_\tau u - E_\tau u$ as given by (II.11) and (II.12). Notice that (II.12) makes sense if $u \in C^\infty(\{|t| \le R\}; \mathcal{D}'(\{|x| < R\}))$ as soon as we change the integral symbol by the duality pairing between the distribution $u(\cdot, t')$ and the appropriate test function; furthermore, $R_\tau u = G_\tau u - E_\tau u$ is still given by (II.11) and (II.12) in the case of distribution solutions since $(**)$ is easily seen to remain valid in this case. Note also that $R_\tau u(x,t)$ is a smooth function of (x,t). We will prove a stronger form of (d).

PROPOSITION II.2.1. *Let $u \in \mathcal{D}'(W)$ satisfy the system* (II.5). *Then,*

$$R_\tau u(x,t) \to 0 \quad in\ C^\infty(U). \tag{II.13}$$

PROOF. We already saw that the exponential in (II.12) may be majorized by $\mathrm{e}^{-c\tau}$ for some positive constant $c > 0$ when $|x| < R/4$, $|x'| \geq R/2$, $|t| < T$ and $t' \in [0, T]$. Let Δ_x denote the Laplacian in $\mathbb{R}^m$. For $k \in \mathbb{Z}_+$ we may write

$$\begin{aligned} L_j h(x')\, u(x',t') \det Z_x(x',t') =& \chi(x')(1-\Delta_{x'})^k (1-\Delta_{x'})^{-k}\big(L_j h(x') \\ & u(x',t') \det Z_x(x',t')\big), \end{aligned}$$

where $\chi(x')$ is a cut-off function that vanishes for $|x'| \leq R/4$ such that $\chi(x')L_j h(x') = L_j h(x')$. Let us write

$$v_j(x',t') = (1-\Delta_{x'})^{-k}[(L_j h(x'))\, u(x',t') \det Z_x(x',t')].$$

It follows that $v_j \in C^0(V)$ for an appropriate choice of k and we may write, after an integration by parts,

$$r_j(x,t,t',\tau) = (\tau/\pi)^{m/2} \int v_j(x',t')(1-\Delta_{x'})^k [\chi(x')\mathrm{e}^{-\tau[Z(x,t)-Z(x',t')]^2}]\, \mathrm{d}x'.$$

Indeed, the convolution operator

$$(1-\Delta)^{-k} f(x) = \frac{1}{(2\pi)^m} \int \mathrm{e}^{ix\cdot\xi}(1+|\xi|^2)^{-k} \widehat{f}(\xi)\, \mathrm{d}\xi, \quad f \in \mathcal{S}(\mathbb{R}^m)$$

maps continuously $L^{2,s}(\mathbb{R}^m)$ onto $L^{2,s+2k}(\mathbb{R}^m)$ and the latter is contained in $L^\infty(\mathbb{R}^m) \cap C^0(\mathbb{R}^m)$ if $s + 2k > m/2$ by Sobolev's embedding theorem. Hence, $r_j(x,t,t',\tau)$ is continuous with respect to t' and converges to 0 uniformly for $|x| \leq R/2$, $|t'| \leq |t| \leq T$, as $\tau \to \infty$, since the derivatives in $(1-\Delta_{x'})^k$ produce powers of τ that are dominated by the exponential $\mathrm{e}^{-c\tau}$. Hence, $R_\tau u(x,t) \to 0$ uniformly as $\tau \to \infty$ and it is easy to see, by differentiating (II.11), that the same holds for the derivatives of any order with respect to x and t of $R_\tau u(x,t)$, as we wished to prove. □

Finally, it is enough to prove that (c) holds assuming that $u \in C^0\big(\{|t| \leq R\}, L^{2,k}_{\mathrm{loc}}(B_R)\big)$ for some integer k. Let us start with the case $k = 0$. We assume that $u \in C^0\big(\{|t| \leq R\}, L^2_{\mathrm{loc}}(B_{R'})\big)$ (with R' slightly larger that R) and we wish to prove that

$$\int_{|x|\leq R/4} |G_\tau u(x,t) - u(x,t)|^2\, \mathrm{d}x \to 0 \quad \text{uniformly in } |t| \leq T,$$

which certainly implies (c) in this case. Redefining u by zero off $B_R \times \mathbb{R}^n$ we may assume that $u(x,t) \in L^2(\mathbb{R}^m)$ for each fixed t, $|t| \le T$. Using once more (II.3′), we see that for any $x, x' \in \mathbb{R}^m$ and $t \in \mathbb{R}^n$

$$\begin{aligned}\mathcal{R}[Z(x,t) - Z(x',t)]^2 &= |x-x'|^2 - |\phi(x,t) - \phi(x',t)|^2 \\ &\ge (3/4)|x-x'|^2,\end{aligned}$$

so the exponential inside the integral that defines $G_\tau u$ has a bound $|e^{-\tau[Z(x,t)-Z(x',t')]^2}| \le e^{-3\tau|x-x'|^2/4}$. If we set

$$F_\tau(x) = \tau^{m/2} e^{-3\tau|x|^2/4}, \quad 0 < \tau < \infty,$$

we easily conclude for fixed $|t| \le R$ that

$$|G_\tau u(x,t)| \le C\,(F_\tau * |u|)\,(x,t)$$

where the convolution is performed in the x variable and t plays the role of a parameter. Since $\|F_\tau\|_{L^1} = \|F_1\|_{L^1} = C$, Young's inequality for convolution implies

$$\sup_{|t|\le T} \|G_\tau u(\cdot,t)\|_{L^2(\mathbb{R}^m)} \le C \sup_{|t|\le T} \|u(\cdot,t)\|_{L^2(\mathbb{R}^m)}. \qquad \text{(II.14)}$$

On the other hand, we proved in Section II.1 that if $u \in C_c^\infty(V)$ then $G_\tau u \to u$ uniformly in $U = B_{R/4} \times \{|t| < T\}$, which implies convergence in the mixed norm space $C^0(\{|t| \le T\}; L^2(B_{R/4}))$. So the operator $G_\tau|_U$ converges to the restriction operator $u \mapsto u|_U$, as $\tau \to \infty$, on a dense subset of $C^0(\{|t| \le T\}, L^2(B_R))$ and the family of operators $\{G_\tau|_U\}$ is equicontinuous because of (II.14). Thus, $G_\tau u|_U \to u|_U$ in the whole space $C^0(\{|t| \le T\}, L^2(B_R))$.

Assume now that $u \in C^0(\{|t| \le T\}, L^{2,1}(B_{R'}))$, $R' > R$. Introducing a cut-off function we may assume that $u \in C^0(\{|t| \le T\}, L^{2,1}(\mathbb{R}^m))$ without modifying u for $|x| < R$. Thus, for $|t| \le T$ fixed, we see that u, $(\partial u/\partial x_k)$ and $(\partial u/\partial t_j)$ are in $L^2(\mathbb{R}^m)$ for $1 \le k \le m$, $1 \le j \le n$. Since we are assuming that $\phi(x,t)$ is compactly supported, the coefficients of L_j and M_k are bounded, with bounded derivatives. In particular, $L_j u$ and $M_k u$ are in $L^2(\mathbb{R}^m)$ for $1 \le k \le m$, $1 \le j \le n$, uniformly in $|t| \le T$. To obtain the convergence result for $k = 1$ we will be able to reason as with the case $k = 0$ as soon as we prove an estimate analogous to (II.14) for the $L^{2,1}$ norm, i.e.,

$$\sup_{|t|\le T} \|G_\tau u(\cdot,t)\|_{L^{2,1}(\mathbb{R}^m)} \le C \sup_{|t||\le T} \|u(\cdot,t)\|_{L^{2,1}(\mathbb{R}^m)}. \qquad \text{(II.15)}$$

Any first-order derivative with respect to x is a linear combination with bounded coefficients of the M_k's, so it is enough to prove for $|t| \le T$, $1 \le k \le m$, $1 \le j \le n$, that

$$\|M_k G_\tau u(\cdot, t)\|_{L^2(\mathbb{R}^m)} \leq C \sup_{|t|\leq T} \|u(\cdot, t)\|_{L^{2,1}(\mathbb{R}^m)}. \tag{II.16}$$

Writing $M_k G_\tau = [M_k, G_\tau] + G_\tau M_k$ we are led to estimate $\|G_\tau M_k u\|_{L^2}$ and $\|[M_k, G_\tau]u\|_{L^2}$. By (II.14) we have $\|G_\tau M_k u\|_{L^2} \leq C\|M_k u\|_{L^2} \leq C'\|u\|_{L^{2,1}}$. Notice that an estimate like (II.14) holds as well with $[M_k, G_\tau]$ in the place of G_τ because G_τ and $[M_k, G_\tau]$ have very similar kernels, as (II.10) shows. Thus, $\|[M_k, G_\tau]u\|_{L^2} \leq C\|u\|_{L^{2,1}}$, which proves (II.16) and gives (II.15). This process can be continued to prove

$$\sup_{|t|\leq T} \|G_\tau u(\cdot, t)\|_{L^{2,k}(\mathbb{R}^m)} \leq C_k \sup_{|t||\leq T} \|u(\cdot, t)\|_{L^{2,k}(\mathbb{R}^m)}, \quad k = 1, 2\ldots \tag{II.17}$$

To deal with the case in which k' is a negative integer, i.e., $k' = -|k'| = -k$, we consider a slight modification of G_τ, namely, $G'_\tau u(x) = h(x)G_\tau u(x)$. Of course, $\{G_\tau u|_U\} = \{G'_\tau u|_U\}$ because $h(x) = 1$ for $|x| \leq R/2$, so this change will not affect our conclusions for $|x| \leq R/4$. The advantage of considering G'_τ is that for fixed t it becomes a formally symmetric operator in the x-variables, as soon as we use the pairing given by the complex measure $\mathrm{d}Z(x, t) = \det Z_x(x, t)\,\mathrm{d}x$. More precisely, for fixed t and $v, w \in C_c^\infty(\mathbb{R}^m)$ we have $\langle G'_\tau v, w\rangle = \langle v, G'_\tau w\rangle$ where we are using the notation $\langle a, b\rangle = \int a(x)b(x)\det Z_x(x, t)\,\mathrm{d}x$, when $a, b \in C^\infty(\mathbb{R}^m)$ and one of them has compact support. Thus,

$$\begin{aligned}
\|G'_\tau u(\cdot, t)\|_{L^{2,k'}(\mathbb{R}^m)} &\leq C \sup_{\substack{w\in C_c^\infty(\mathbb{R}^m)\\ \|w\|_{L^{2,k}}\leq 1}} |\langle G'_\tau u(\cdot, t), w\rangle| \\
&= C \sup_{\substack{w\in C_c^\infty(\mathbb{R}^m)\\ \|w\|_{L^{2,k}}\leq 1}} |\langle u(\cdot, t), G'_\tau w\rangle| \\
&\leq C \sup_{\substack{w\in C_c^\infty(\mathbb{R}^m)\\ \|w\|_{L^{2,k}}\leq 1}} \|u(\cdot, t)\|_{L^{2,k'}} \|G'_\tau w\|_{L^{2,k}} \\
&\leq C\|u(\cdot, t)\|_{L^{2,k'}},
\end{aligned} \tag{II.18}$$

where we have used (II.17) for the positive integer k in the last inequality. This extends (II.17) to all integers $k \in \mathbb{Z}$, proving the equicontinuity of G'_τ in all spaces $C^0\big(\{|t| \leq T\}, L^{2,k}(B_{R'})\big)$, $k \in \mathbb{Z}$, which together with the convergence of $G_\tau u|_U$ to $u|_U$ for the space of test functions $C_c^\infty(\overline{B_{R'}} \times \{|t| \leq T\})$ which is dense in any $C^0\big(\{|t| \leq T\}, L^{2,k}(B_{R'})\big)$ proves that $G_\tau u \to u$ in $C^0\big(\{|t| \leq T\}, L^{2,k}(B_{R/4})\big)$ for any $u \in C^0\big(\{|t| \leq T\}, L^{2,k}(B_{R'})\big)$. This proves (c) and concludes the proof of part (i) of Theorem II.1.1.

To prove part (ii) of the theorem—this is the fifth and final step of the proof—using the same method of proof, it will be enough to prove the equicontinuity of G_τ on the spaces

$$C^j\big(\{|t| \le T\}, C_b^k(\mathbb{R}^m)\big), \quad j, k = 0, 1, 2 \ldots,$$

where $C_b^k(\mathbb{R}^m)$ is the space of functions on $\mathbb{R}^m$ possessing continuous bounded derivatives of order $\le k$. For $j, k = 0$ this is easily achieved by noting that

$$|G_\tau u(x,t)| \le C\,(F_\tau * |u|)\,(x,t) \le C' \|u\|_{C^0\left(\{|t|\le T\}, C_b^0(\mathbb{R}^m)\right)}.$$

For $j, k \le 1$ one expresses the derivatives in terms of the vector fields L_j and M_k and reduces the equicontinuity for the norms of $C^j\big(\{|t| \le T\}, C_b^k(\mathbb{R}^m)\big)$ to the case $j = k = 0$ by introduction of the commutators $[G_\tau, L_j]$ and $[G_\tau, M_k]$, as was done before for Sobolev norms; iteration of this process gives the result for $k = 2, 3, \ldots$ This concludes the proof of Theorem II.1.1.

II.3 Convergence in standard functional spaces

As proved in Proposition II.2.1, $R_\tau u = G_\tau u - E_\tau u \to 0$ in $C^\infty(U)$, for any distribution u satisfying $\mathcal{L}u = 0$ in a larger open set V. This reduces the problem of the convergence $E_\tau u \to u$ in any space with coarser topology than C^∞-topology to the convergence of $G_\tau u \to u$ in the same space. Now, as the reader probably noticed in the proof of Theorem II.1.1, the operator G_τ is very close to convolution with a Gaussian in the x-variables with t playing the role of a parameter, and as such it is a very well-behaved approximation of the identity. Hence, loosely speaking, we may expect that the convergence $G_\tau u \to u$ on U holds in the topology of many functional spaces used in analysis, provided that u belongs to that space over the larger set V. In this section we deal with this question and the approach will always be the same: to prove convergence in a given space of distributions $X(U)$ we will first prove the equicontinuity of $\{G_\tau\}$ in the space $X(\mathbb{R}^N)$ and then try to apply the standard fact that under the hypotheses of equicontinuity it is enough to check the convergence on a convenient dense subset of $X(V)$. Usually the dense subset will be the space of test functions $\psi \in C_c^\infty(V)$, for which we know that $G_\tau\psi \to \psi$ in $C^\infty(\overline{U})$. Thus, this approach works if (i) $X(V)$ is a normal space of distributions (i.e., $C_c^\infty(V)$ is dense in $X(V)$), and (ii) $C^\infty(\overline{U}) \subset X(U)$ with continuous inclusion. We have already applied this principle in the proof of Theorem II.1.1 with $X(V) = C^0\big(\{|t| \le R\}, L^{2,k}(B_R)\big)$.

II.3.1 Convergence in L^p

The main result of this subsection is:

THEOREM II.3.1. *Let $\mathcal{L}$ be a locally integrable structure on Ω and assume that $\mathrm{d}Z_1, \dots, \mathrm{d}Z_m$ span $\mathcal{L}^\perp$ at every point of Ω. Then, for any $z \in \Omega$, there exist two open sets U and W, with $z \in U \subset \overline{U} \subset W \subset \Omega$, such that for any $u \in L^p_{\mathrm{loc}}(W)$, $1 \le p \le \infty$, satisfying $\mathcal{L}u = 0$,*

$$E_\tau u(x,t) \longrightarrow u(x,t) \quad \text{a.e. in } U \text{ as } \tau \to \infty. \tag{II.19}$$

In case p is finite, i.e., $1 \le p < \infty$, we also have

$$E_\tau u(x,t) \longrightarrow u(x,t) \quad \text{in } L^p(U) \text{ as } \tau \to \infty. \tag{II.20}$$

In (II.19) *and* (II.20) *we may replace the operator E_τ by a convenient sequence of polynomials in Z, $P_\ell(Z_1, \dots, Z_m)$.*

In the proof of Theorem II.3.1 we may assume from the start by shrinking W that $u \in L^p(W)$ and we will do so. We are also tacitly assuming that we are using special coordinates (x,t) adapted to a given set of local generators $\mathrm{d}Z_1, \dots, \mathrm{d}Z_m$ of $\mathcal{L}^\perp$ with linearly independent real parts so that $Z = x + i\phi(x,t)$, where $\phi(x,t)$ is smooth, real, has compact support and satisfies (II.3′). Once the special coordinates (x,t) are fixed, the operator E_τ referred to in (II.19) and (II.20) is defined precisely as in the proof of Theorem II.1.1.

We will also prove below theorems similar to Theorem II.3.1 for different norms and in all of them the first step will be to choose special local coordinates where Z has this special form where the operators E_τ and G_τ are defined and have good convergence properties. To avoid repetitions we will always assume that this step has already been carried out, even if not mentioned explicitly.

According to the considerations made at the beginning of the section, we need only prove that

$$G_\tau u \longrightarrow hu \quad \text{in} \quad L^p(W), \quad \tau \longrightarrow \infty, \quad u \in L^p(W). \tag{II.21}$$

For $1 \le p < \infty$, the space $C^0_c(W)$ is dense in $L^p(W)$ and (II.20) will be a consequence of

$$G_\tau u \longrightarrow hu \quad \text{uniformly}, \quad \tau \longrightarrow \infty, \quad u \in C^0_c(W)$$

(which we already know by Theorem II.1.1) and the uniform bound that we will prove later:

$$\|G_\tau u\|_p \le C\|u\|_p, \quad u \in L^p(\mathbb{R}^N), \quad \tau > 0, \tag{II.22}$$

where $\| \; \|_p$ denotes the L^p-norm.

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Let us set $W = B_x \times B_t$, where $B_x = \{|x| < R\}$ and $B_t = \{|t| < R\}$. Let $u \in L^p(W)$ and set $u^t(x) = u(x, t)$. Fubini's theorem guarantees that u^t is defined for a.e. t, it is measurable, and it belongs to $L^p(B_x)$. If, moreover, u satisfies $\mathcal{L}u = 0$, we know that u has a trace $T_t u$ and $B_t \ni t \mapsto T_t u \in \mathcal{D}'(B_x)$ is a smooth function. It will be useful to compare both types of restrictions of u to the slices $t = \text{const}$.

Lemma II.3.2. *If $u \in L^p(W)$, $1 \le p \le \infty$, and u is a solution of the system* (II.5) *then $T_t u = u^t$ for a.e. $t \in B_t$. In particular, $T_t u \in L^p(B_x)$ for a.e. $t \in B_t$.*

Proof. We take functions $\phi \in C_c^\infty(B_x)$ and $\psi \in C_c^\infty(B_t)$. We know that $t \mapsto \langle T_t u, \phi\rangle$ is a C^∞-function defined in B_t, $t \mapsto \langle u^t, \phi\rangle$ belongs to $L^p(B_t)$ and

$$\begin{aligned} \int \langle T_t u, \phi\rangle \psi(t)\, \mathrm{d}t &= \int \left(\int u(x, t)\phi(x)\, \mathrm{d}x \right) \psi(t)\, \mathrm{d}t \\ &= \int \langle u^t, \phi\rangle \psi(t)\, \mathrm{d}t. \end{aligned} \tag{II.23}$$

If we take $\psi(t) = \chi_j(t - t_0)$, $\chi_j(t) = j^n \chi(jt)$, $0 \le \chi \in C_c^\infty(\{|t| \le 1\})$, $\int \chi \mathrm{d}t = 1$, and let $j \to \infty$, the left-hand side of (II.23) converges for every $t \in B_t$ to $\langle T_t u, \phi\rangle$ while the right-hand side converges a.e. to $\langle u^t, \phi\rangle$. Hence, there is a null set $N(\phi) \subset B_t$ such that

$$\langle T_t u, \phi\rangle = \langle u^t, \phi\rangle, \qquad \phi \in C_c^\infty, \quad t \notin N(\phi).$$

If we apply the last identity to a dense sequence $\{\phi_k\} \subset C_c^\infty(B_x)$ and set $N = \bigcup N(\phi_n)$ we obtain that $T_t u = u^t$ as elements of $\mathcal{D}'(B_x)$ when t is not in the null set N. $\square$

Remark II.3.3. One cannot expect in general that, under the conditions of Lemma II.3.2, $T_t u \in L^p$ for all t. For instance, if $\Omega = (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$, $Z = x + it^2/2$, $L = \partial_t - it\partial_x$ is the Mizohata operator and $u(x, t) = 1/Z(x, t)$, it is simple to verify that $u \in L^p(\Omega)$ for $1 \le p < 3/2$, $Lu = 0$ in the sense of distributions and $T_t u \in C^\infty([-1, 1]) \subset L^\infty(-1, 1) \subset L^p(-1, 1)$ for $t \ne 0$ but for $t = 0$ we have $T_0 u = \mathrm{pv}(1/x) - i\pi\delta(x) \notin L^p(-1, 1)$.

We now prove Theorem II.3.1. Consider the maximal operator associated with $G_\tau u$:

$$G_\tau^* u(x, t) = \sup_{\tau \ge 1} |G_\tau u(x, t)|.$$

We claim that, for $u \in L^1(\Omega)$, there exists a constant $C > 0$ such that

$$G_\tau^* u(x, t) \le CM(h(x)T_t u(x)), \tag{II.24}$$

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for any t such that $T_t u \in L^1(B_x)$. Here

$$Mf(x,t) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(x',t)|\,\mathrm{d}x'$$

is the Hardy–Littlewood maximal operator acting in the x-variable, $B(x,r)$ is the ball of radius r centered at x, and $|B(x,r)|$ denotes its Lebesgue measure. In fact, $|G_\tau u(x,t)|$ can be estimated by

$$(\tau/\pi)^{m/2} \int_{\mathbb{R}^m} \mathrm{e}^{-\tau(|x-x'|^2-|\phi(x,t)-\phi(x',t)|^2)} |T_t u(x')||h(x')||\det Z_x(x',t)|\,\mathrm{d}x'$$

and this expression can be dominated by the maximal operator

$$\sup_{\tau\ge 1} F_\tau * |h\,T_t u \det Z_x| = C \sup_{\tau\ge 1} \tau^{m/2} \int_{\mathbb{R}^m} \mathrm{e}^{-3\tau|x-x'|^2/4} \\ |T_t u(x')||h(x')||\det Z_x(x',t)|\,\mathrm{d}x'$$

where

$$F_\tau(x) = C\tau^{m/2}\mathrm{e}^{-3\tau|x|^2/4}$$

and C is a constant. Hence,

$$G^*_\tau u(x,t) \le \sup_{\tau\ge 1} F_\tau * |h\,T_t u \det Z_x| \le CM(h(\cdot)T_t u(\cdot))(x).$$

The last inequality follows from the fact that $F_1(x) = C\mathrm{e}^{-3|x|^2/4}$ is radial decreasing and belongs to $L^1(\mathbb{R}^m)$ (see, for instance, [**S1**, page 62]). Thus, (II.24) is proved.

If $u \in C^0_c(W)$, we know that $G_\tau u(x,t) \to h(x)u(x,t)$, $\tau \to \infty$ uniformly. The standard properties of the maximal operator allow us to conclude that for any $t \in B_t$ such that $T_t u(x) \in L^1(B_x)$ there exists a subset $N_t \subset B_x$ with $|N_t| = 0$ such that

$$G_\tau u(x,t) \to h(x)u(x,t), \quad x \notin N_t.$$

Hence, if we choose $(x,t) \in U$ such that $T_t u \in L^1(B_x)$ and $x \notin N_t$, we get (recalling that $R_\tau u \to 0$ uniformly in U)

$$E_\tau u(x,t) \to h(x)u(x,t) = u(x,t) \quad \text{a.e. in } U$$

and therefore $E_\tau u(x,t) \to u(x,t)$ a.e. in U as we wished to prove.

We now prove (II.22). We observe that

$$|G_\tau u(x,t)| \le F_\tau * |h\,T_t u \det Z_x|$$

and then Young's inequality for convolution implies

$$\|G_\tau u(\cdot,t)\|_{L^p(\mathrm{d}x)} \le \|F_\tau\|_1 \|h\,T_t u \det Z_x\|_{L^p(\mathrm{d}x)} \le C\|T_t u\|_{L^p(\mathrm{d}x)},$$

since the L^1 norm of F_τ does not depend on τ and h det Z_x is bounded. Raising this inequality to the pth power and integrating with respect to t we obtain (II.22). Since $G_\tau u \to hu$ uniformly in W as $\tau \to \infty$ when u is continuous, the usual density argument shows that (II.21) holds for $1 \le p < \infty$. Thus, (II.19) and (II.20) have been proved. Finally, since $E_\tau u$ can be approximated in $C^\infty(U)$ by polynomials in Z for fixed τ, the proof is complete. □

It is obvious that (II.20) is, in general, false for $p = \infty$ because the uniform limit of a sequence of continuous functions, such as $E_\tau u(x, t)$, is continuous.

A simple consequence of Theorem II.3.1 is:

Corollary II.3.4. *Let $\mathcal{L}$ be a locally integrable structure over a C^∞ manifold U and let $u \in L^p_{\text{loc}}(U)$, $1 \le p \le \infty$, $v \in L^q_{\text{loc}}(U)$, $1/p + 1/q = 1$, be solutions of the system* (II.5). *Then the product $w = uv \in L^1_{\text{loc}}(U)$ also satisfies* (II.5).

Proof. By localization we may assume that U is the neighborhood where the conclusions of Theorem II.3.1 hold. Set $u_\tau = E_\tau u$, $w_\tau = u_\tau v$. Leibniz's rule shows that $\mathcal{L} w_\tau = 0$, as $u_\tau \in C^\infty(U)$. By Theorem II.3.1 and Hölder's inequality $w_\tau \to w$ in $L^1_{\text{loc}}(U)$, $\tau \to \infty$, showing that $\mathcal{L} w = 0$ in the sense of distributions. □

II.3.2 Convergence in Sobolev spaces

In this subsection we prove

Theorem II.3.5. *Let $\mathcal{L}$ be a locally integrable structure with first integrals $Z_1, \dots, Z_m$, defined in a neighborhood of the closure of $W = B_x \times B_t$. There exists a neighborhood $U \subset W$ of the origin such that for any $u \in L^{p,s}_{\text{loc}}(W)$, $1 < p < \infty$, $s \in \mathbb{R}$, satisfying $\mathcal{L} u = 0$,*

$$E_\tau u(x, t) \longrightarrow u(x, t) \ \text{ in } L^{p,s}_{\text{loc}}(U), \quad \tau \longrightarrow \infty. \tag{II.25}$$

As usual, we may replace the operator E_τ in (II.25) *by a convenient sequence of polynomials in Z, $P_\ell(Z_1, \dots, Z_m)$.*

We recall that for $1 \le p \le \infty$, $s \in \mathbb{R}$,

$$L^p_s(\mathbb{R}^N) = \{ f \in \mathcal{S}'(\mathbb{R}^N) : \|f\|_{p,s} \doteq \|\Lambda^s f\|_p < \infty \}$$

where $\Lambda^s f(x) = \mathcal{F}^{-1}[(1 + |\xi|^2)^{s/2} \mathcal{F} f(\xi)](x)$ and $\mathcal{F}$ denotes the Fourier transform in $\mathbb{R}^N$ (Λ^s is the Bessel potential and $\mathcal{S}'$ denotes the space of tempered distributions). For $k \in \mathbb{Z}_+$ and p in the range $1 < p < \infty$ the space $L^p_k(\mathbb{R}^N)$ is exactly the subspace of the functions in $L^p(\mathbb{R}^N)$ whose derivatives of

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order $\leq k$ in the sense of distributions belong to $L^p(\mathbb{R}^N)$. This space is equivalently normed by ([**S1**])

$$\|u\|_{L^p_k} = \sum_{|\alpha|\leq k} \|D^\alpha u\|_p. \tag{II.26}$$

The space $L^{p,s}_{\mathrm{loc}}(\Omega)$ is the subspace of $\mathcal{D}'(\Omega)$ of the distributions u such that $\psi u \in L^p_s(\mathbb{R}^N)$ for all test functions $\psi \in C^\infty_c(\Omega)$, equipped with the locally convex topology given by the seminorms $u \mapsto \|\psi u\|_{p,s}$, $\psi \in C^\infty_c(\Omega)$. Fix $p \in (1,\infty)$, $s \in \mathbb{R}$ and choose the open sets U and W as in Theorem II.1.1. The theorem will be proved if we show that

$$\lim_{\tau\to\infty} G_\tau v = h\, v \quad \text{in} \quad L^p_s(W), \quad \forall v \in C^\infty_c(W), \tag{II.27}$$

and there exists a positive constant C such that

$$\|G_\tau w\|_{p,s} \leq C\|w\|_{p,s} \quad \forall w \in L^p_s(\mathbb{R}^N). \tag{II.28}$$

Indeed, (II.27) and (II.28) imply as usual, by density and triangular approximation, that $\|G_\tau w - hw\|_{p,s} \to 0$ as $\tau \to \infty$ for any $w \in L^p_s(\mathbb{R}^N) \cap \mathcal{E}'(W)$—where $\mathcal{E}'(W)$ denotes the space of distributions compactly supported in W—which implies that $G_\tau w \to w$ in the topology of $L^{p,s}_{\mathrm{loc}}(U)$. We know that for $u \in C^\infty_c(U)$, $G_\tau u \to u$ in $C^\infty(U)$, thus (II.27) is clearly true and we need only worry about proving (II.28), which we prove first for a positive integer $s = k \in \mathbb{Z}_+$. The vector fields L_j and M_k form a basis of $\mathbb{C}T\mathbb{R}^n$ and we may express the derivatives D^α in (II.26) in terms of the vector fields L_j, $j = 1, \dots, n$, M_k, $k = 1, \dots, m$. This gives

$$\|G_\tau w\|_{L^p_k} \leq C \sum_{|\alpha_1|+|\alpha_2|\leq k} \|M^{\alpha_1} L^{\alpha_2} G_\tau w\|_p. \tag{II.29}$$

We write

$$L_j G_\tau w = G_\tau L_j w + [L_j, G_\tau]w,$$
$$M_k G_\tau w = G_\tau M_k w + [M_k, G_\tau]w.$$

As shown in Lemmas II.1.4 and II.1.6, the operators $[L_j, G_\tau]$ and $[M_k, G_\tau]$ are given by the same expression as G_τ with $h(x)$ replaced respectively by $L_j h(x)$ and $M_k h(x)$. Hence, the proof of Theorem II.3.1 gives bounds in L^p for the commutators that may be written as

$$\|[L_j, G_\tau]v\|_p + \|[M_k, G_\tau]v\|_p \leq C\|v\|_p, \quad v \in L^p(\mathbb{R}^N). \tag{II.30}$$

Thus, for $1 \le j \le n$, $1 \le k \le m$,

$$\begin{aligned} \|L_j G_\tau w\|_p + \|M_k G_\tau w\|_p &\le C(\|L_j w\|_p + \|M_k w\|_p + \|w\|_p) \\ &\le C(\|w\|_{p,1} + \|w\|_p) \\ &\le C\|w\|_{p,1}, \end{aligned} \tag{II.31}$$

where we have used (II.22) to estimate $G_\tau L_j w$ and $G_\tau M_k w$ in the first inequality. Thus, combining (II.26) for $u = G_\tau w$ and $k = 1$ with (II.31) we get (II.28) for $k = 1$. This reasoning can be iterated for any $s = k \in \mathbb{Z}_+$ and the theorem is proved for $s \in \mathbb{Z}_+$.

To prove (II.28) for nonintegral $s > 0$, we use interpolation of Sobolev spaces (on the subject of interpolation see, for instance, **[C1]** and **[C2]**). First we take $k \in \mathbb{Z}_+$ such that $0 < s < k$. The operator G_τ is of type $(p, p, 0, 0)$ and also of type (p, p, k, k), $k \in \mathbb{Z}_+$, that is, it verifies

$$\|G_\tau w\|_p \le C\|w\|_p, \quad w \in C_c^\infty(\mathbb{R}^N)$$

and

$$\|G_\tau w\|_{p,k} \le C\|w\|_{p,k}, \quad w \in C_c^\infty(\mathbb{R}^N).$$

By complex interpolation we obtain that G_τ is of type (p, p, s, s); that is, (II.28) holds for $0 < s < k$ and $w \in C_c^\infty(\mathbb{R}^N)$ and by density it also holds for $w \in L_s^p(\mathbb{R}^N)$. Finally, to prove (II.28) for $s < 0$, we invoke a slight variation of the duality argument that was used to extend (II.18) from positive integers to negative integers: we consider the modification of G_τ, $G'_\tau u(x) = h(x) G_\tau u(x)$ which is formally symmetric in the x-variables for fixed t for the pairing given by integration with respect to $\mathrm{d}Z(x,t) = \det Z_x(x,t)\,\mathrm{d}x$ and thus also symmetric in both variables x and t for the pairing given by integration with respect to $\mathrm{d}Z(x,t) \wedge \mathrm{d}t = \det Z_x(x,t)\,\mathrm{d}x\mathrm{d}t$. Since this is a nonsingular continuous pairing for the spaces $L_s^p(\mathbb{R}^N)$ and $L_{-s}^q(\mathbb{R}^N)$, $1/p + 1/q = 1$, it extends (II.28) to $s < 0$ as follows:

$$\begin{aligned} \|G'_\tau w\|_{L_s^p(\mathbb{R}^N)} &\le C \sup_{\substack{\psi \in C_c^\infty(\mathbb{R}^N) \\ \|\psi\|_{L_{-s}^q} \le 1}} |\langle G'_\tau w(\cdot, t), \psi\rangle| \\ &\le C \sup_{\substack{\psi \in C_c^\infty(\mathbb{R}^N) \\ \|\psi\|_{L_{-s}^q} \le 1}} |\langle w, G'_\tau \psi\rangle| \\ &\le C \sup_{\substack{\psi \in C_c^\infty(\mathbb{R}^N) \\ \|\psi\|_{L_{-s}^q} \le 1}} \|w\|_{L_s^p} \|G'_\tau \psi\|_{L_{-s}^q} \\ &\le C_s \|w\|_{L_s^p(\mathbb{R}^N)}, \end{aligned}$$

where in the last inequality we used (II.28) with q in the place of p and $-s > 0$ in the place of s. Thus, (II.28) is completely proved and the proof of Theorem II.3.5 is complete. □

II.3.3 Convergence in Hölder spaces

Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, convex set. The Hölder space $C^\alpha(\Omega)$ is defined as

$$C^\alpha(\Omega) = \{u \in C^k(\overline{\Omega}), \|u\|_\alpha < \infty\}$$

where

$$\|u\|_\alpha = |u|_\alpha + |u|_0,$$
$$|u|_0 = \sup_{x\in\overline{\Omega}} |u(x)|,$$
$$|u|_\alpha = \sup_{\substack{x,y\in\overline{\Omega},\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}, \quad 0<\alpha\leq 1,$$
$$|u|_\alpha = \sum_{\sigma\leq k} |D^\sigma u|_{\alpha-k}, \quad k<\alpha\leq k+1, \quad k\in\mathbb{Z}+, \quad u\in C^k(\Omega).$$

The spaces $C^\alpha(\mathbb{R}^N)$ are defined similarly. The approximation theorem is:

THEOREM II.3.6. *Let $\mathcal{L}$ be a locally integrable structure with first integrals $Z_1, \dots, Z_m$, defined in a neighborhood of the closure of $W = B_x \times B_t$. There exists a convex neighborhood $U \subset \Omega$ of the origin such that for any $u \in C^\beta(W)$, $\beta > 0$ satisfying $\mathcal{L}u = 0$ in a neighborhood of $\overline{W}$ and any $0 \leq \alpha < \beta$*

$$E_\tau u(x,t) \longrightarrow u(x,t) \text{ in } C^\alpha(U), \quad \tau \longrightarrow \infty. \tag{II.32}$$

As usual, we may replace the operator E_τ in (II.32) *by a convenient sequence of polynomials in Z, $P_\ell(Z_1, \dots, Z_m)$.*

PROOF. As always, since $C_c^\infty(W)$ is dense in $C_c^\beta(W)$ for the C^α norm, we need only prove

$$G_\tau u \longrightarrow u \quad \text{in} \quad C^\alpha(W), \quad u \in C_c^\infty(W),$$

and the inequality

$$\|G_\tau u\|_\alpha \leq C\|u\|_\alpha, \quad u \in C_c^\alpha(W).$$

It is obvious that $\|G_\tau u - u\|_\alpha \to 0$ when $\tau \to \infty$, $u \in C_c^\infty(W)$, because by Theorem II.1.1 $G_\tau u \to u$, $\tau \to \infty$ in $C^k(W)$ for every positive integer k. We may assume without loss of generality, as we always do, that $Z(x,t) = x + i\phi(x,t)$ is defined and satisfies (II.3′) throughout $\mathbb{R}^N$ and reduces to

$Z(x,t) \equiv x$ for (x,t) outside a compact set. We shall then prove

$$\|G_\tau u\|_\alpha \le C\|u\|_\alpha, \quad u \in C_c^\alpha(\mathbb{R}^N). \tag{II.33}$$

We assume first that $0 < \alpha < 1$. It will be useful to use the following well-known characterization of $C^\alpha(\mathbb{R}^N)$ ([**S2**, page 256]):

LEMMA II.3.7. *A function u belongs to $C^\alpha(\mathbb{R}^N)$, $0 < \alpha < 1$, if and only if there exist a sequence of functions $(u_k) \in C^1(\mathbb{R}^N)$, bounded and with bounded gradients, such that*

(i) $\|u_k\|_{L^\infty} \le K\,2^{-\alpha k}$, $k = 0, 1, \ldots$;
(ii) $\|\nabla u_k\|_{L^\infty} \le K\,2^{(1-\alpha)k}$, $k = 0, 1, \ldots$;
(iii) $u(z) = \sum_{k=0}^\infty u_k(z)$, $z \in \mathbb{R}^N$.

It also follows that the best constant K in (i) and (ii) above is proportional to $\|u\|_\alpha$. Such a sequence is usually called a sequence of best approximation for u. We start by writing $u = \sum u_k$ with (u_k) a sequence of best approximation for u. Then, $G_\tau u = \sum G_\tau u_k$ and we need to estimate the essential supremum of $G_\tau u_k$ and $\nabla G_\tau u_k$. Taking account of (II.22) with $p = \infty$ and (i) of Lemma II.3.7 we derive

$$\|G_\tau u_k\|_{L^\infty} \le C\|u_k\|_{L^\infty} \le CK2^{-\alpha k}, \quad k \in \mathbb{Z}_+. \tag{II.34}$$

In order to estimate $\nabla G_\tau u_k$ it is convenient to express any partial derivative in terms of the vector fields L_j and M_ℓ, $1 \le j \le n$, $1 \le \ell \le m$. Then, we are led to estimate $L_j G_\tau u_k$, $j = 1, \ldots, n$ and $M_\ell G_\tau u_k$, $\ell = 1, \ldots, m$. We may write $L_j G_\tau u_k = G_\tau L_j u_k + [L_j, G_\tau]u_k$ and recall that

$$\|[L_j, G_\tau]u_k\|_{L^\infty} \le C\|u_k\|_{L^\infty},$$

which follows from (II.30) with $p = \infty$. We get

$$\begin{aligned}\|L_j G_\tau u_k\|_{L^\infty} &\le C(\|L_j u_k\|_{L^\infty} + \|u_k\|_{L^\infty})\\ &\le C(\|\nabla u_k\|_{L^\infty} + \|u_k\|_{L^\infty}), \quad j = 1, \ldots, n, \quad k = 1, 2, \ldots\end{aligned}$$

Similar estimates are true for $M_\ell G_\tau u_k$, $\ell = 1, \ldots, m$, $k \in \mathbb{Z}_+$ and we obtain

$$\|\nabla G_\tau u_k\|_{L^\infty} \le C(\|u_k\|_{L^\infty} + \|\nabla u_k\|_{L^\infty}) \le C'K2^{(1-\alpha)k}, \quad k \in \mathbb{Z}_+. \tag{II.35}$$

Thus, (II.34), (II.35) and Lemma II.3.7 imply that (II.33) holds for $0 < \alpha < 1$.

Let us assume next that there is a positive integer k such that $\alpha = k + \eta$, $0 < \eta < 1$ and we wish to estimate

$$\|G_\tau u\|_\alpha \sim \sum_{|\sigma| \le k} \|D^\sigma G_\tau u\|_\eta \le C \sum_{|\sigma_1|+|\sigma_2| \le k} \|M^{\sigma_1} L^{\sigma_2} G_\tau u\|_\eta.$$

Using the commutation formulas of Lemmas II.1.4 and II.1.6 it is easy to prove (II.33) by induction on k, adapting the reasonings we used to deal with Sobolev norms of integral order in Section II.3.2; we leave the details to the reader. Finally, to prove (II.33) for $\alpha = k = 1, 2, \ldots,$ we observe that in this case $\|u\|_\alpha = \|u\|_k \sim \|u\|_{L^\infty_k}$ so (II.33) is a variation of the estimates already considered for Sobolev norms. This completes the proof of Theorem II.3.6. □

It is not possible to take $\alpha = \beta$ in Theorem II.3.6, as we will see next.

Example II.3.8. Consider in $\mathbb{R}^2$, where we denote the coordinates by (x, t), the structure $\mathcal{L}$ spanned by ∂_t with first integral $Z(x, t) = x$ and let $0 < \beta \leq 1$. Consider a function $u(x) \in C^\beta_c(\mathbb{R}^2)$ independent of t (so it satisfies $\mathcal{L}u = 0$) such that $u(x) = |x|^\beta$ for $|x| \leq 1$. If $w(x, t)$ is of class C^1 in a neighborhood of the origin, we have for $0 < \varepsilon < 1$ sufficiently small,

$$|u - w|_\beta \geq \frac{|u(\varepsilon) - w(\varepsilon, 0) - (u(0) - w(0, 0)|}{\varepsilon^\beta} \geq 1 - C\varepsilon^{1-\beta}$$

and the left-hand side is $\geq 1/2$ for ε small, showing that u cannot be approximated by continuously differentiable functions in the C^β topology.

II.3.4 Convergence in Hardy spaces

We recall that the real Hardy space $H^p(\mathbb{R}^N)$, $0 < p < \infty$, introduced by Stein and Weiss ([**SW**]), is equal to $L^p(\mathbb{R}^N)$ for $p > 1$, is properly contained in $L^1(\mathbb{R}^N)$ for $p = 1$, and is a space of not necessarily locally integrable distributions for $0 < p < 1$. For $p \leq 1$, $H^p(\mathbb{R}^N)$ is a substitute for $L^p(\mathbb{R}^N)$ ([**S2**]), as the latter is not a space of distributions and has trivial dual if $p < 1$; even for $p = 1$, $L^1(\mathbb{R}^N)$ does not behave as well as $L^p(\mathbb{R}^N)$, $1 < p < \infty$, for example on questions concerning the continuity of pseudo-differential operators. Let us choose a function $\Phi \in \mathcal{S}(\mathbb{R}^N)$, with $\int \Phi \mathrm{d}z \neq 0$ and write $\Phi_\varepsilon(z) = \varepsilon^{-N}\Phi(z/\varepsilon)$, $z \in \mathbb{R}^N$, and

$$M_\Phi f(z) = \sup_{0<\varepsilon<\infty} |(\Phi_\varepsilon * f)(z)|.$$

Then ([**S2**])

$$H^p(\mathbb{R}^N) = \{f \in \mathcal{S}'(\mathbb{R}^N) : \quad M_\Phi f \in L^p(\mathbb{R}^N)\}.$$

An obstacle to the localization of the elements of $H^p(\mathbb{R}^N)$, $0 < p \leq 1$, is that ψu may not belong to $H^p(\mathbb{R}^N)$ for $\psi \in C^\infty_c(\mathbb{R}^N)$ and $u \in H^p(\mathbb{R}^N)$. A way around this is the definition of localizable Hardy spaces $h^p(\mathbb{R}^N)$ ([**G**],[**S2**]) by means of the truncated maximal function

$$m_\Phi f(z) = \sup_{0<\varepsilon\leq 1} |(\Phi_\varepsilon * f)(z)|,$$
$$h^p(\mathbb{R}^N) = \{f \in \mathcal{S}'(\mathbb{R}^N) : \quad m_\Phi f \in L^p(\mathbb{R}^N)\}.$$

It turns out that if Φ is replaced in the definition of $h^p(\mathbb{R}^N)$ by any other function $\Phi \in \mathcal{S}(\mathbb{R})$ only required to satisfy $\int \Phi \neq 0$, this will not change the space $h^p(\mathbb{R}^N)$. It is also known that the space $h^p(\mathbb{R}^N)$ is stable under multiplication by test functions and also that $h^p(\mathbb{R}^N) = L^p(\mathbb{R}^N)$ for $1 < p < \infty$. For $0 < p \leq 1$, which we henceforth assume, $h^p(\mathbb{R}^N)$ is a metric space with the distance $d(f, g) = \int (m_\Phi(f-g)(z))^p \, dz$. If $\Omega \subset \mathbb{R}^N$ is an open set, the space $H^p_{\text{loc}}(\Omega)$ is the subspace of $\mathcal{D}'(\Omega)$ of the distributions u such that $\psi u \in h^p(\mathbb{R}^N)$ for all test functions $\psi \in C^\infty_c(\Omega)$. A sequence u_n converges to zero in $H^p_{\text{loc}}(\Omega)$ if $\psi u_n \to 0$ in $h^p(\mathbb{R}^N)$ for every $\psi \in C^\infty_c(\Omega)$. We have

THEOREM II.3.9. *Let $\mathcal{L}$ be a locally integrable structure with first integrals $Z_1, \ldots, Z_m$, defined in a neighborhood of the closure of $W = B_x \times B_t$. There exists a neighborhood $U \subset W$ of the origin such that for any $u \in H^p_{\text{loc}}(W)$, $0 < p < \infty$, satisfying $\mathcal{L}u = 0$,*

$$E_\tau u(x, t) \longrightarrow u(x, t) \text{ in } H^p_{\text{loc}}(U), \quad \tau \longrightarrow \infty. \tag{II.36}$$

As usual, we may replace the operator E_τ in (II.36) *by a convenient sequence of polynomials in Z, $P_\ell(Z_1, \ldots, Z_m)$.*

PROOF. Since $H^p_{\text{loc}}(W) = L^p_{\text{loc}}(W)$ for $p > 1$, Theorem II.3.9 follows from Theorem II.3.1 for these values of p and it is enough to assume that $0 < p \leq 1$. The space $C^\infty_c(W)$ is continuously included in $H^p_{\text{loc}}(W)$ and the theorem may be proved by showing once again that

$$\lim_{\tau\to\infty} G_\tau v = h\, v \quad \text{in} \quad h^p(\mathbb{R}^N), \quad \forall v \in C^\infty_c(W) \tag{II.37}$$
$$\|G_\tau w\|_{h^p} \leq C \|w\|_{h^p} \quad \forall w \in h^p(\mathbb{R}^N) \tag{II.38}$$

with the notation $\|w\|_{h^p} = (\int (m_\Phi w(z))^p \, dz)^{1/p}$, in spite of the fact that $w \to \|w\|_{h^p}$ is not a norm for $p < 1$. To prove (II.37) and (II.38) we use the atomic decomposition of h^p ([**G**],[**S2**]). An h^p atom, $p \leq 1$, is a bounded, compactly supported function $a(z)$ satisfying the following property: there exists a cube Q with sides parallel to the coordinate axes that contains the support of a and furthermore

(i) $|a(z)| \leq |Q|^{-1/p}$, a.e., with $|Q|$ denoting the Lebesgue measure of Q;
(ii) $\int z^\alpha\, a(z)\, \mathrm{d}z = 0$, $|\alpha| \leq N(1/p - 1)$, if the side length of Q happens to be less than 1.

Notice that if the support of a is contained in a cube Q such that (i) holds and the side of Q has length ≥ 1, then a is an atom, as condition (ii) is vacuous and only (i) is required in this case.

As always, (II.37) follows from the convergence $G_\tau v \to v$ in $C_c^\infty(\Omega)$, $v \in C_c^\infty(\Omega)$. So, to prove Theorem II.3.9, we need only show (II.38) and the density of $C_c^\infty(\mathbb{R}^N)$ in $h^p(\mathbb{R}^N)$. To prove the density, it is enough to approximate h^p atoms by smooth h^p atoms in the h^p norm. This is simply approximating a rough atom a by the convolution $a_\varepsilon = a * \psi_\varepsilon$, where $\psi_\varepsilon(z) = \varepsilon^{-N}\psi(z/\varepsilon)$, and $\psi \in C_c^\infty(\mathbb{R}^N)$ has integral equal to 1. Then, a_ε satisfies the vanishing moments condition (ii) because a does and satisfies (i) for a cube Q slightly larger than the one that worked for a, if $\varepsilon > 0$ is sufficiently small. Moreover, $a_\varepsilon \to a$ in the h^p 'norm' as $\varepsilon \to 0$. To check the last fact use Hölder's inequality to write

$$\begin{aligned}\int (m_\Phi(a - a_\varepsilon)(z))^p\, \mathrm{d}z &\leq |Q|^{1-p/2} \|m_\Phi(a - a_\varepsilon)\|_{L^2} \\ &\leq C|Q|^{1-p/2} \|M(a - a_\varepsilon)\|_{L^2} \\ &\leq C|Q|^{1-p/2} \|a - a_\varepsilon\|_{L^2}\end{aligned}$$

where we have majorized the maximal function $m_\Phi(a - a_\varepsilon)$ by the Hardy–Littlewood maximal function $M(a - a_\varepsilon)$ which is continuous in L^2.

Any $w \in h^p$ can be written as a convergent series in h^p, $w = \sum_k \lambda_k a_k$, where the a_k are atoms and λ_k are complex numbers such that $\sum_k |\lambda_k|^p \sim \|w\|_{h^p}$ ([**S2**]) (since atoms may be approximated by smooth atoms we may even assume that $a_k \in C_c^\infty(\mathbb{R}^N)$ for all k). Then, to prove (II.38) it is enough to verify that there is a constant $C > 0$ such that for all h^p atoms $a(z)$

$$\|G_\tau a\|_{h^p}^p = \int (m_\Phi G_\tau a(z))^p\, \mathrm{d}z \leq C, \quad \tau \geq 1. \tag{II.39}$$

Indeed,

$$\begin{aligned}\int \left(m_\Phi G_\tau \sum_k \lambda_k a_k\right)^p \mathrm{d}z &\leq \int \left(\sum_k |\lambda_k| m_\Phi G_\tau a_k\right)^p \mathrm{d}z \\ &\leq \sum_k |\lambda_k|^p \int (m_\Phi G_\tau a_k)^p\, \mathrm{d}z\end{aligned}$$

because $p \leq 1$. We assume without loss of generality that $\Phi \geq 0$ is supported in the unit ball (in fact, changing the function Φ by any other function in $\mathcal{S}(\mathbb{R}^N)$

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with nonvanishing integral will produce an equivalent 'norm' in $H^p(\mathbb{R}^N)$). We set $F(x) = \mathrm{e}^{-3|x|^2/4}$, $x \in \mathbb{R}^m$, $F_\sigma(x) = \sigma^{-m}F(s/\sigma)$ and we check that by the estimates of Section II.3.1 (see (II.24)):

$$\begin{aligned}|(\Phi_\varepsilon * G_\tau a)(x,t)| &\le C|\Phi_\varepsilon * (F_\sigma \overset{(x)}{*} a)(x,t)| \\ &= C|(\Phi_\varepsilon * a) \overset{(x)}{*} F_\sigma(x,t)|, \quad \sigma = \tau^{-1/2},\end{aligned}$$

where the symbol $\overset{(x)}{*}$ denotes convolution in the x-variable. Let $Q = Q_1 \times Q_2$, $Q_1 \subset \mathbb{C}^m$, $Q_2 \subset \mathbb{C}^n$, be a cube containing the support of a. Thus, invoking (i), we get

$$m_\Phi(G_\tau a)(x,t) \le C|Q|^{-1/p}\chi_{Q_2}(t). \tag{II.40}$$

Here and in the sequel, χ_A will denote the characteristic function of a measurable set A. Let Q_1^* (resp. Q_1^{**}) be the cube in $\mathbb{R}^m$ concentric with Q_1 having twice (resp. four times) the side length. Then (II.40) shows that

$$\int_{Q_1^{**}\times\mathbb{R}^n} |m_\Phi(G_\tau a)(x,t)|^p\,\mathrm{d}x\,\mathrm{d}t \le C, \tag{II.41}$$

with $C > 0$ independent of $0 < \varepsilon \le 1$, $\tau \ge 1$, $a(z)$ an atom. Thus, (II.39) will be proved as soon as we obtain

$$\int_{(\mathbb{R}^m\setminus Q_1^{**})\times\mathbb{R}^n} \sup_{0<\varepsilon\le 1} |\Phi_\varepsilon * (F_\sigma \overset{(x)}{*} a)(x,t)|^p\,\mathrm{d}x\,\mathrm{d}t \le C, \quad 0 < \sigma \le 1. \tag{II.42}$$

Assuming that $\Phi(x,t) = \Phi^1(x)\Phi^2(t)$, Φ_1 and Φ_2 supported in the unit ball of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, we are led to consider the convolution $\Phi_\varepsilon^1 \overset{(x)}{*} a \overset{(x)}{*} F_\sigma$. In order to simplify the notation we simply write $\Phi_\varepsilon^1 * a * F_\sigma$, letting t play the role of a parameter. Let us assume first that the side r of the cube Q is ≥ 1. Since Φ^1 is supported in the unit ball, $\Phi_\varepsilon^1 * a \doteq a^\varepsilon$, $0 < \varepsilon \le 1$, is supported in Q_1^*. Therefore, if $x \notin Q_1^{**}$, letting x_0 be the center of Q_1 and $C_L = \sup_{x\in\mathbb{R}^n} |x|^L F(x)$, we have

$$\begin{aligned}|(\Phi_\varepsilon^1 * a * F_\sigma)(x,t)| &\le \chi_{Q_2}(t)\left|\int a^\varepsilon(y,t)F_\sigma(x-y)\,\mathrm{d}y\right| \\ &\le CC_L\chi_{Q_2}(t)|Q|^{-1/p}|Q_1^*|\sigma^{-m}\left[\frac{|x-x_0|}{\sigma}\right]^{-L}\end{aligned}$$

where we have used that $|x - x_0| \sim |x - y|$ for $y \in Q_1^*$ and $x \notin Q_1^{**}$. Since $|Q_1^*| = (2r)^m \le (2|x - x_0|)^m$ and $\sigma^{L-m} \le 1$ if we take $L > m$, we obtain for a large integer $d = L - m$

$$|(\Phi_\varepsilon^1 * a * F_\sigma)(x,t)| \le C\chi_{Q_2}(t)|Q|^{-1/p}|x - x_0|^{-d}.$$

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Convolving with $\Phi^2_\varepsilon(t)$ gives, for $x \notin Q_1^{**}$ and $t \in \mathbb{R}^n$,

$$|\Phi_\varepsilon * (F_\sigma \overset{(x)}{*} a)(x,t)| \leq C|Q|^{-1/p}|x-x_0|^{-d}(\Phi^2_\varepsilon \overset{(t)}{*} \chi_{Q_2})(t) \\ \leq C|Q|^{-1/p}|x-x_0|^{-d}\chi_{Q_2^*}(t).$$

Choose $d = m+1$. If we take the supremum in $0 < \varepsilon \leq 1$, raise both sides to the pth power and integrate in $(\mathbb{R}^m \backslash Q_1^{**}) \times \mathbb{R}^n$, we obtain (II.42), under the assumption $r \geq 1$.

Let us assume now that $r < 1$, so $a(z)$ satisfies the moment conditions (ii). It is clear that these properties are inherited by $a^\varepsilon(z)$, i.e., $\int z^\alpha a^\varepsilon(z)\,\mathrm{d}z = 0$, $|\alpha| \leq N(1/p-1)$. We start by writing $F(x)$ as a convergent series in $\mathcal{S}(\mathbb{R}^m)$, $F(x) = \sum_k F^{(k)}(x)$ with $F^{(0)}$ supported in the unit ball $B = B(0,1)$ and each $F^{(k)}$ supported in some ball of radius 1. We aim at proving (II.42) with $F^{(k)}$ in the place of F. Using the vanishing of the moments of a

$$(a^\varepsilon \overset{(x)}{*} F_\sigma^{(k)})(x,t) = \chi_{Q_2^*}(t)\int a(y,t)\,G^{(k)}_{\sigma,\varepsilon}(x-y)\,\mathrm{d}y \\ = \int a(y,t)[G^{(k)}_{\sigma,\varepsilon}(x-y) - q_{x,\varepsilon}(y)]\,\mathrm{d}y \qquad \text{(II.43)}$$

where $G^{(k)}_{\sigma,\varepsilon} = \Phi^1_\varepsilon * F^{(k)}_\sigma$ and $q_{x,\varepsilon}(y)$ is the Taylor polynomial of degree d of the function $y \to G^{(k)}_{\sigma,\varepsilon}(x-y)$ expanded about x_0 and d is the integral part of $N(1/p-1)$. The usual estimates for the remainder of the Taylor expansion imply that the integrand in (II.43) is $\leq C|Q|^{-1/p}\sigma^{-(d+1+m)}\,r^{d+1}$. We assume first that $k=0$ so $F^{(0)}$ is supported in the unit ball. Since $|x-x_0| \leq C|x-y|$ when $y \in Q_1^*$ and $x \notin Q_1^{**}$, $|x-y| \leq \sigma$ on the support of $F^{(0)}_\sigma(x-y)$, and a is supported in the cube Q_1^* of measure $(2r)^m$ it follows that for any $0 < \epsilon \leq 1$ and $0 < \sigma \leq 1$

$$|(a^\varepsilon * F^{(0)}_\sigma)(x,t)|^p \leq C_0\chi_{Q_2}(t)\left(\frac{r}{|x-x_0|}\right)^{(d+m+1)p}, \quad x \notin Q_1^{**},$$

which after integration gives

$$\int_{(\mathbb{R}^m\backslash Q_1^{**})\times\mathbb{R}^n} \sup_{0<\varepsilon\leq 1} |\Phi_\varepsilon * (F^{(0)}_\sigma * a)(x,t)|^p\,\mathrm{d}x\,\mathrm{d}t \leq C_0. \qquad \text{(II.44)}$$

On the other hand, the proof of (II.41) shows that

$$\int_{Q_1^{**}\times\mathbb{R}^n} \sup_{0<\varepsilon\leq 1} |\Phi_\varepsilon * (F^{(0)}_\sigma * a)(x,t)|^p\,\mathrm{d}x\,\mathrm{d}t \leq C_0,$$

which combined with (II.44) gives

$$\int_{\mathbb{R}^m\times\mathbb{R}^n} \sup_{0<\varepsilon\leq 1} |\Phi_\varepsilon * (F^{(0)}_\sigma * a)(x,t)|^p\,\mathrm{d}x\,\mathrm{d}t \leq 2C_0. \qquad \text{(II.45)}$$

For other values of k we consider an appropriate translate $\tilde{F}^{(k)}$ of $F^{(k)}$ so that $\tilde{F}^{(k)}$ is supported in $B(0,1)$. If for any given σ we replace the atom a by a convenient translate $\tilde{a}$, which of course is also an atom, we may write $a^\varepsilon * F_\sigma^{(k)} = \tilde{a}^\varepsilon * \tilde{F}_\sigma^{(k)}$. Reasoning as before we get the analogue of (II.45):

$$\int_{\mathbb{R}^m \times \mathbb{R}^n} \sup_{0<\varepsilon\le 1} |\Phi_\varepsilon * (F_\sigma^{(k)} * a)(x,t)|^p \,\mathrm{d}x\,\mathrm{d}t \le C_k. \tag{II.46}$$

The proof also shows that there is a continuous seminorm p in $\mathcal{S}$ involving derivatives of order $\le d+1$ such that $C_k \le p(F^{(k)})$ and since the series $F = \sum_k F^{(k)}$ converges absolutely in $\mathcal{S}$ we see that $\sum_k C_k < \infty$. Estimates (II.46) imply (II.41) by subadditivity and the theorem is proved. □

II.4 Applications

In this section we discuss two typical applications of the Baouendi–Treves approximation formula. The first one deals with extensions of CR functions and the second with uniqueness of solutions of the equation $\mathcal{L}u = 0$ where $\mathcal{L}$ is a locally integrable structure. The principle that governs the first application is conceptually very simple: suppose that we know that a sequence of polynomials $P_\ell(\zeta)$, $\zeta \in \mathbb{C}^m$, converges uniformly in a compact set $K \subset \mathbb{C}^m$, then it converges uniformly in the holomorphic convex hull $\widehat{K}$ of K in $\mathbb{C}^m$. We recall that

$$\widehat{K} = \bigcap_{P\in\mathcal{P}} \{\zeta \in \mathbb{C}^m : \quad |P(\zeta)| \le \sup_K |P|\},$$

where $\mathcal{P}$ denotes the space of polynomials in m complex variables. Since on a ball that contains K any entire function, that is any holomorphic function defined throughout $\mathbb{C}^m$, can be uniformly approximated by the partial sums of its Taylor series, we also have

$$\widehat{K} = \{\zeta \in \mathbb{C}^m : \quad |f(\zeta)| \le \sup_K |f| \quad \text{for all entire functions } f\}.$$

Let $u \in C^0(W)$ satisfy $\mathcal{L}u = 0$ on W and let $K = Z(\overline{V})$ where $\overline{V} \subset U$ and U, W are the neighborhoods in the statement of Theorem II.1.1. We already noticed that we may write $u = \widehat{u} \circ Z$ on $\overline{V}$ where $\widehat{u} \in C^0(K)$ because u is constant on the fibers of Z in U. Now, we have a function $\widehat{U}(\zeta)$ defined on $\widehat{K}$ by $\widehat{U}(\zeta) = \lim_{\ell\to\infty} P_\ell(\zeta)$, $\zeta \in \widehat{K}$, which clearly extends $\widehat{u}$. Depending on the geometry of $Z(\overline{V})$, $\widehat{K}$ may have nonempty interior and on this open set the extension $\widehat{U}$ will be holomorphic because it is the uniform limits of polynomials in ζ. Composition with Z gives the required extension. When u

is not continuous but, say, belongs to L^p, things are technically more involved but essentially the same principle works.

This type of approach may also be seen at work in the following simple example. Consider the operator in $\mathbb{R}^2$

$$L = \frac{\partial}{\partial t} - 3it^2 \frac{\partial}{\partial x} \tag{a}$$

with first integral $Z(x, t) = x + it^3$. Indeed, it is easily verified that $LZ = 0$ and clearly $\mathrm{d}Z$ never vanishes. The operator L has real analytic coefficients and is elliptic off the x-axis but is not elliptic at $t = 0$, nevertheless it shares with elliptic vector fields with real analytic coefficients the following regularity property: if u is a C^1 solution of $Lu = 0$, then u is real-analytic ([**M**]). This is also true for distribution solutions (thus, (a) is analytic hypoelliptic) but to keep matters simple let us restrict ourselves to classical solutions. To prove the claim, it will be enough to prove that u is real-analytic at any point $(x, 0)$ of its domain, since for points (x, t) with $t \neq 0$ this follows from ellipticity. Let us prove, for instance, that u is real-analytic at the origin in case it is defined in a neighborhood of the origin. By Theorem II.1.1 we may find $\delta > 0$ such that for $|x| \leq \delta$ and $|t| \leq \delta$ the uniform limit $u(x, t) = \lim_{\ell\to\infty} P_\ell(x + it^3)$ holds for a certain sequence of polynomials P_ℓ, $\ell \in \mathbb{Z}_+$. This implies that the sequence $P_\ell(z) = P_\ell(x + iy)$ is a Cauchy sequence in the space $C^0(K)$ where $K = [-\delta, \delta] \times [-\delta^3, \delta^3]$. Hence, $\lim_{\ell\to\infty} P_\ell(z) \doteq \widehat{u}(z)$ is a continuous function on K which is a holomorphic function on $(-\delta, \delta) \times (-\delta^3, \delta^3)$ and we have that $u(x, t) = \widehat{u}(x + it^3)$ for $|x|, |t| \leq \delta$. Since $\widehat{u}$ is real-analytic in a neighborhood of the origin and so is $Z(x, t) = x + it^3$, it follows that u is real-analytic in a neighborhood of the origin as we wished to prove.

II.4.1 Extendability of CR functions

Consider the Heisenberg group

$$\mathbb{H}^n \simeq \mathbb{C}^n \times \mathbb{R} = \{(z, s) = (z_1, \dots, z_n, s) : \quad z \in \mathbb{C}^n, \ \ s \in \mathbb{R}\}$$

with the group law

$$(z, s) \cdot (w, s') = \left(z + w, s + s' + \Im \sum_{j=1}^{n} z_j \bar{w}_j \right).$$

Then $\mathbb{H}^n$ can be topologically identified with the boundary of the Siegel upper half-space

$$\mathbb{D}^{n+1} = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : \quad \Im z_{n+1} > \sum_{j=1}^{n} |z_j|^2\}$$

via the map

$$Z:(z_1,\ldots,z_n,t)\longmapsto(z_1,\ldots,z_n,t+i|z|^2). \tag{II.47}$$

This identification endows $\mathbb{H}^n$ with the CR structure transported from the boundary $\partial\mathbb{D}^{n+1}$ which possesses a standard CR structure as a smooth boundary of an open subset of $\mathbb{C}^{n+1}$ induced by the anti-holomorphic differentiations. A function $f\in C^1(\mathbb{H}^n)$ (or more generally a distribution) is a CR function (resp. CR distribution) if and only if it satisfies the overdetermined first-order linear system of equations

$$\tilde{L}_j f=\frac{\partial f}{\partial\bar{z}_j}-iz_j\frac{\partial f}{\partial s}=0,\quad j=1,\ldots,n. \tag{II.48}$$

Observe that the vector fields $\tilde{L}_j$ are left-invariant under the action of $\mathbb{H}^n$. The components of the map (II.47), that is, the functions $Z_1(z,s)=z_1,\ldots,Z_n(z,s)=z_n$, $W(z,s)=s+i|z|^2$ satisfy (II.48) and it is of interest to determine which solutions of (II.48) may be expressed as the composition of the map (II.47) with a holomorphic function defined in $\mathbb{D}^{n+1}$ and having a suitable trace in $\partial\mathbb{D}^{n+1}$. It is known ([**FS**]) that a function $f\in C^1(\mathbb{H}^n)$ is a CR function if and only if there exists a function $F\in C^1(\overline{\mathbb{D}}^{n+1})$ which is holomorphic in $\mathbb{D}^{n+1}$ and whose composition with the map (II.47) is equal to f. There is also a similar local result due to Hans Lewy ([**L1**]) which holds in the general set-up of CR structures of hypersurface type with nondegenerate Levi form which we now describe. Consider a hypersurface Ω in $\mathbb{C}^{n+1}$ with the CR structure $\mathcal{L}$ induced by the standard anti-holomorphic differentiations of $\mathbb{C}^{n+1}$. We may assume that, in a suitable neighborhood of the origin in $\mathbb{C}^{n+1}$, Ω is given by

$$t=\Phi(z_1,z_2,\ldots,z_n,s),\qquad z_i\in\mathbb{C},\quad s\in\mathbb{R},\quad i=1,\ldots,n$$

where

$$\Phi(z,s)=\sum_{i,j=1}^{n}\frac{\partial^2\Phi}{\partial z_j\partial\bar{z}_k}(0,0)z_j\bar{z}_k+O(|z|^3+|s||z|+s^2).$$

Then $\mathcal{L}$ is orthogonal to the differential of the functions

$$\begin{aligned}&Z_j(z,s)=z_j,\qquad j=1,\ldots,n,\qquad z=(z_1,\ldots,z_n)\\&W(z,s)=s+i\Phi(z,s),\end{aligned}$$

and generated by the vector fields

$$L_j=\frac{\partial}{\partial\bar{z}_j}-i\Phi_{\bar{z}_j}(z,s)[1+i\Phi_s(z,s)]^{-1}\frac{\partial}{\partial s},\quad j=1,\ldots,n. \tag{II.49}$$

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Using z_j and $w = s + it$ as a system of coordinates, the Levi form at $(0, \mathrm{d}s)$ is represented by the matrix

$$\frac{\partial^2 \Phi}{\partial z_k \partial \bar{z}_j}(0, 0).$$

The aforementioned result of Hans Lewy asserts that, when the Levi form of $\mathcal{L}$ at $(0, \mathrm{d}s)$ has a positive eigenvalue, there exists a neighborhood V of the origin in $\mathbb{C}^{n+1}$ such that every continuous function satisfying

$$L_j u = 0 \tag{II.50}$$

in $Z^{-1}(\Omega \cap V)$, $Z = (z_1, \ldots, z_n, s + i\Phi(z, s))$, can written as

$$u = F \circ Z$$

where F is a continuous function defined in $\{(z, w) \in V,\ t \geq \Phi(z, s)\}$ and holomorphic in $V^+ = \{(z, w) \in V,\ t > \Phi(z, s)\}$.

We now return to the Heisenberg group $\mathbb{H}^n$ and recall that the (global) holomorphic Hardy space $\mathcal{H}^p(\mathbb{D}^{n+1})$, $0 < p < \infty$, is the set of functions F, holomorphic in $\mathbb{D}^{n+1}$, which satisfy

$$\sup_{0<\rho<\infty} \int_{\mathbb{C}\times\mathbb{R}} |F(z, s + i(|z|^2 + \rho))|^p \, \mathrm{d}m(z) \, \mathrm{d}s < \infty.$$

Here $\mathrm{d}m$ is the Lebesgue measure on $\mathbb{C}^n$, $\mathrm{d}s$ is the Lebesgue measure on the real line and it turns out that the pullback of the product measure $\mathrm{d}m \times \mathrm{d}s$ is the Haar measure on $\mathbb{H}^n$. If $F \in \mathcal{H}^p(\mathbb{D}^{n+1})$, F has a pointwise boundary value f at almost every point of $\partial\mathbb{D}^{n+1}$ given by the normal limit which exists also in L^p norm and, of course, f is a CR distribution. We now prove an analogue of Lewy's local extension result within the framework of local L^p spaces, $1 \leq p < \infty$.

THEOREM II.4.1. *Let Ω be a smooth hypersurface of $\mathbb{C}^{n+1}$ passing through the origin and assume that the Levi form has a nonzero eigenvalue. Then, for any $1 \leq p < \infty$ and $f \in L^p_{loc}(\Omega)$ which is a CR distribution in a neighborhood of the origin, there exists an open set $V \ni 0$ of $\mathbb{C}^{n+1}$ and a holomorphic function F in $L^p(V^+)$ (V^+ denotes the portion of V lying on the 'convex' side of Ω) such that f is the trace of F.*

PROOF. In view of the hypothesis we may assume V^+ is given by $t = \Im z_{n+1} > \Phi(z, s)$ with

$$\Phi(z, s) = |z_1|^2 + \sum_{j=2}^{n} \epsilon_j |z_j|^2 + O(|z|^3 + |s||z| + s^2) \tag{II.51}$$

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where each ϵ_j may assume the values $+1$, -1, or 0. We will assume initially that the remainder terms vanish identically because the proof is very simple in this case. Hence, we assume that

$$\Phi(z,s) = \Phi(z) = |z_1|^2 + \sum_{j=2}^{n} \epsilon_j |z_j|^2. \qquad \text{(model case)}$$

Since f is a CR function, it follows that $f \circ Z$ satisfies the overdetermined system (II.50) where the vector fields L_j are given by (II.49). By Theorem II.3.1 there is a sequence of polynomials $P_\ell(Z)$, $Z = (z_1, ..., z_n, s + i\Phi(z,s))$ that converges to $f \circ Z$ in L^p norm in a neighborhood of the origin in $\mathbb{C}^n_z \times \mathbb{R}_s$. We may assume that the closure of the Cartesian product of the polydisk $\Delta(0, 2\sqrt{a})$ of radius $0 < a \le 1$ times the interval $(-a, a)$ is contained in that neighborhood. Let us write $z' = (z_2, \ldots, z_n)$. Then, for each z' and t fixed, the set

$$\{z_1 : \quad (z_1, z', s + it) \in V^+\}$$

is a disk centered at the origin of radius $R(z',t) = (t - \sum_{j=2}^{n} \epsilon_j |z_j|^2)^{1/2}$ if $(t - \sum_{j=2}^{n} \epsilon_j |z_j|^2) \ge 0$ and empty if the latter quantity is negative. We will denote this (possibly empty) disk by $D(z',t)$. Given an entire function u defined on $\mathbb{C}^{n+1}$ (actually we will only use that u is harmonic in the first variable), we wish to estimate the L^p norm of u on

$$V_a^+ = \{(z_1, z', s + it) \in V^+ : \quad |z_j| \le a,\ j = 2, \ldots, n,\ |s|, |t| \le a\}$$

in terms of the L^p norm of the restriction of u to the boundary of V^+. As the disks $D(z', r)$ sweep V^+, their boundaries sweep the boundary of V^+, which suggests the use of Poisson's formula. A change to polar coordinates $re^{i\theta}$ in the variable (x_1, y_1) allows us to express the integral

$$I = \int_{-a}^{a} \mathrm{d}s \int_{-a}^{a} \mathrm{d}t \int_{\Delta'(0,a)} \mathrm{d}x' \,\mathrm{d}y' \int_{D(z',t)} |u(x_1 + iy_1, z', s, t)|^p \mathrm{d}x_1 \mathrm{d}y_1$$

as

$$I = \int_{-a}^{a} \mathrm{d}s \int_{-a}^{a} \mathrm{d}t \int_{\Delta'(0,a)} \mathrm{d}x' \,\mathrm{d}y' \int_0^{2\pi} \mathrm{d}\theta \int_0^{R(z',t)} |u(re^{i\theta}, z', s, t)|^p \, r\mathrm{d}r.$$

It is a well-known consequence of Poisson's formula and Young's inequality for convolution that

$$\int_0^{R(z',t)} \int_0^{2\pi} |u(re^{i\theta}, z', s, t)|^p \,\mathrm{d}\theta \, r\mathrm{d}r \le \frac{R(z',t)^2}{2} \int_0^{2\pi} |u(R(z',t)e^{i\theta}, z', s, t)|^p \,\mathrm{d}\theta.$$

A more geometric way of writing this inequality for any disk D is

$$\int_D |u|^p \,\mathrm{d}A \le \frac{\operatorname{diam}(D)}{4} \int_{\partial D} |u|^p \,\mathrm{d}\sigma \qquad \text{(II.52)}$$

where $\mathrm{d}A$ is the element of area and $\mathrm{d}\sigma$ indicates arc length. Hence,

$$I \le \int_{-a}^{a} \mathrm{d}s \int_{\Delta'(0,a)} \mathrm{d}x'\,\mathrm{d}y' \int_{\alpha(z')}^{a} \mathrm{d}t\, \frac{R(z',t)^2}{2} \int_0^{2\pi} |u(R(z',t)\mathrm{e}^{i\theta}, z', s, t)|^p\,\mathrm{d}\theta,$$

where, for a given z', $\alpha(z')$ indicates the value of t below which the disk $D(z', t)$ becomes empty (if this ever happens) or $-a$, whichever is larger. Now the substitution $\tau = R(z', t)$ in the integral with respect to t (so that $t = \Phi(\tau, z')$) yields, assuming a is sufficiently small,

$$\begin{aligned} I &\le \int_{-a}^{a} \mathrm{d}s \int_{\Delta'(0,a)} \mathrm{d}x'\,\mathrm{d}y' \int_{-2\sqrt{a}}^{2\sqrt{a}} |\tau|^3 \mathrm{d}\tau \int_0^{2\pi} |u(\tau\mathrm{e}^{i\theta}, z', s, \Phi(\tau, z'))|^p\,\mathrm{d}\theta \\ &\le \int_{-a}^{a} \mathrm{d}s \int_{\Delta'(0,a)} \mathrm{d}x'\,\mathrm{d}y' \int_{-2\sqrt{a}}^{2\sqrt{a}} |\tau| \mathrm{d}\tau \int_0^{2\pi} |u(\tau\mathrm{e}^{i\theta}, z', s, \Phi(\tau, z'))|^p\,\mathrm{d}\theta \\ &\le 2\int_{\Delta(0,2\sqrt{a})\times(-a,a)} |u\circ Z|^p\,\mathrm{d}x\mathrm{d}y\mathrm{d}s. \end{aligned}$$

Thus, we have proved that

$$\int_{V_a^+} |u|^p\,\mathrm{d}x\mathrm{d}y\mathrm{d}s\mathrm{d}t \le 2\int_{\Delta(0,2\sqrt{a})\times(-a,a)} |(u\circ Z)(z,s)|^p\,\mathrm{d}x\mathrm{d}y\mathrm{d}s \tag{II.53}$$

and applying this to $u = P_\ell - P_{\ell'}$ we conclude that the sequence P_ℓ converges in $L^p(V_a^+)$ to a holomorphic function F that has a trace $F/\partial V_a^+$ such that $F/\partial V_a^+ \circ Z = f \circ Z$ and this implies that $F/\partial V_a^+ = f$, as we wished to prove (it follows from Cauchy's formula that L^p-convergence implies local uniform convergence). To deal with a general Φ given by (II.51) we may reason exactly in the same way, except that now the domains of $\mathbb{C}$

$$\tilde{D}(z', s, t) = \{z_1 : \quad (z_1, z', s + it) \in V^+\}$$

will no longer be round disks centered at the origin. However, they are simply connected and may be regarded as smooth perturbations of a disk $D(z', t)$ of radius $R(z', t)$ which can be mapped by a Riemann map $z_1 \mapsto \Psi(z_1; z', s, t)$ onto $D(z', t)$. Thus, we will be able to reason as in the proof of (II.52) as soon as we prove the following substitute for (II.52):

$$\int_{\tilde{D}(z',s,t)} |u|^p\,\mathrm{d}A \le C\,\mathrm{diam}\,(\tilde{D}(z', s, t)) \int_{\partial\tilde{D}(z',s,t)} |u|^p\,\mathrm{d}\sigma$$

where $C > 0$ is independent of (z', s, t) in a neighborhood of the origin and u is any harmonic function defined in $\tilde{D}(z', s, t)$ and continuous in its closure. To simplify the notation we omit any reference to the variables (z', s) that play the role of parameters and write $z = x + iy$ instead of $z_1 = x_1 + y_1$. Thus, we are led to consider the class $\mathcal{F}_\epsilon$ of smooth functions $\phi(x, y)$ in $\mathbb{R}^2$ whose Taylor series at the origin is $\phi(x, y) \sim a + bx + cy + x^2 + y^2 + O(|z|^3)$, when

$z \to 0$, where $|a|+|b|+|c| < \epsilon$ and such that $|D^\alpha \phi(x, y)| \le C_\alpha$ (here $\epsilon > 0$ is a conveniently chosen small number and (C_α) is a given fixed sequence of positive constants). We will need to study the sublevel sets in a fixed small neighborhood of the origin,

$$\tilde{D}(t) = \{z = x + iy: \quad |z| < r, \ \phi(x, y) < t\},$$

for an arbitrary $\phi \in \mathcal{F}_\epsilon$. Observe that any $\phi \in \mathcal{F}_\epsilon$ has a small local minimum m at a point $z_0 = (x_0, y_0)$ located close to the origin for small ϵ. It follows that

$$m + 2^{-1}|z - z_0|^2 \le \phi(x, y) \le m + 2|z - z_0|^2,$$

in a neighborhood of the origin and thus

$$D(z_0, \sqrt{t/2}) \subset \tilde{D}(m+t) \subset D(z_0, \sqrt{2t}).$$

We see that $\tilde{D}(m+t)$ is empty for $t \le 0$ and contained between concentric disks of radius comparable to $\sqrt{t}$ if t is positive and small. Furthermore, the implicit function theorem shows that, in the latter case, $\tilde{D}(m+t)$ has a smooth boundary made up of a simple closed curve contained in the annulus $t/2 < |z - z_0|^2 < 2t$.

LEMMA II.4.2. *There exist $t_0, r_0 > 0$ such that for all $0 < t \le t_0$ and $\phi \in \mathcal{F}_\epsilon$, $\tilde{D}(m+t)$ is a relatively compact simply connected open subset of the disk $D(0, r_0)$. Furthermore, there exists $C > 0$ such that for every harmonic function u defined in a neighborhood of the closure of $\tilde{D}(m+t)$ and any $1 \le p < \infty$, the following a priori inequality holds:*

$$\int_{\tilde{D}(m+t)} |u|^p \, \mathrm{d}A \le C \operatorname{diam}(\tilde{D}(m+t)) \int_{\partial \tilde{D}(m+t)} |u|^p \, \mathrm{d}\sigma. \tag{II.54}$$

PROOF. After a translation, we may assume that $z_0 = 0$. For small $t > 0$, the level curve $\phi(x, y) = m + t$, which is implicitly given in polar coordinates by $r^2(A(\theta) + rB(r, \theta)) = t$ where $A(\theta) = \alpha \cos^2\theta + 2\beta \sin\theta \cos\theta + \gamma \sin^2\theta$ and all derivatives of B with respect to x and y are bounded, may also be explicitly expressed by $r = r(\theta, t)$. Observe that if ϵ is small, α and γ are close to 1 and β is close to zero. Implicit differentiation shows that

$$r' = \frac{\partial r}{\partial \theta} = -\frac{rA_\theta + r^2 B_\theta}{2A + 3rB + r^2 B_r} = O(\sqrt{t}), \quad t \to 0.$$

Differentiating further the expression above we conclude that the higher-order derivatives $r^{(n)}$, $n = 1, 2, \ldots$, are also $O(\sqrt{t})$ as $t \to 0$. Consider a dilation of $\tilde{D}(m+t)$, $\mathcal{D}_t = (1/\sqrt{t})\tilde{D}(m+t)$, whose boundary is given by $R_t(\theta) = r(\theta, t)/\sqrt{t} = A^{-1/2}(\theta) + O(\sqrt{t})$. Observe that we also have

$$\mathrm{d}^n R_t/\mathrm{d}\theta^n = \mathrm{d}^n A^{-1/2}/\mathrm{d}\theta^n + O(\sqrt{t}) \quad \text{for } n \geq 1.$$

Since (II.54) is invariant under dilations of the domain, it will be enough to prove it for the dilate $\mathcal{D}_t$ that converges in C^∞ to the domain $\mathcal{D}_0$ with equation $R < A^{-1/2}(\theta)$ as $t \to 0$. To do so it is enough to show that, for small t, the derivative F_t' of the Riemann map F_t from $\mathcal{D}_t$ to the unit disk satisfies $1/C \leq |F_t'| \leq C$. Indeed, if u is harmonic in $\mathcal{D}_t$ and continuous up to the boundary, so will be $v = u \circ F_t^{-1}$ on the unit disk, and starting from (II.52) applied to v, the change of variables $w = F_t(z)$ will give

$$\int_{\mathcal{D}_t} |u|^p \,\mathrm{d}A \leq C \int_{\partial\mathcal{D}_t} |u|^p \,\mathrm{d}\sigma. \tag{II.55}$$

Notice that if we introduce the factor $\operatorname{diam}(\mathcal{D}_t)$ on the right-hand side of (II.55) the inequality remains valid because $2/\sqrt{2} \leq \operatorname{diam}(\mathcal{D}_t) \leq 2\sqrt{2}$. Hence, the proof of (II.54) will be finished as soon as we prove

Lemma II.4.3. *There exist $t_0 > 0$ and $C > 0$ such that for $0 \leq t \leq t_0$ the Riemann map F_t from $\mathcal{D}_t$ to the unit disk D satisfies $1/C \leq |F_t'| \leq C$.*

Proof. Let u be the solution of the Dirichlet problem

$$\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & \text{on } \mathcal{D}_t, \\ u|_{\partial\mathcal{D}_t} = u(R_t(\theta)\mathrm{e}^{i\theta}) = \log(R_t(\theta)), & 0 \leq \theta \leq 2\pi. \end{cases} \tag{II.56}$$

Let v be the harmonic conjugate of u in $\mathcal{D}_t$ (say, normalized by $v(0) = 0$) and set $f_t = u + iv$. Then a Riemann map from $\mathcal{D}_t$ onto the unit disk $D = D(0, 1)$ is (*cf.* the proof of theorem 3.3 in [**F**])

$$F_t(z) = z\mathrm{e}^{-f_t(z)}.$$

Thus, $F_t' = \mathrm{e}^{-f_t(z)}(1 - zf_t'(z))$ and $|F_t'| = \mathrm{e}^{-u(z)}|1 - zf_t'(z)|$ which implies, by the maximum principle, that

$$C^{-1} \inf_{\mathcal{D}_t} |1 - zf_t'(z)| \leq |F_t'| \leq C \sup_{\mathcal{D}_t} |1 - zf_t'(z)|,$$

with $C > 0$ independent of t, for small t. Indeed, $\log(R_t(\theta))$ converges to $-(1/2)\log A(\theta) = -(1/2)\log[\alpha\cos^2\theta + 2\beta\sin\theta\cos\theta + \gamma\sin^2\theta]$ as $t \to 0$ and the domain $\mathcal{D}_0$ is close to the unit disk for small ϵ. Therefore, to conclude the proof, we need only show that $|f_t'| \leq 1/2$ for small t and ϵ. Since $f_t' = u_x - iu_y$ we must show that the derivatives of u are uniformly small in $\mathcal{D}_t$. The domains $\mathcal{D}_t$ change with t and the analysis may be simplified by mapping $\mathcal{D}_t \cup \partial\mathcal{D}_t$

onto the fixed domain $\mathcal{D}_0 \cup \partial\mathcal{D}_o$ by a diffeomorphism (of manifolds with boundaries) Φ_t such that all derivatives of Φ_t and Φ_t^{-1} are bounded uniformly with bounds that do not depend on $t \in [0, t_0]$. Such Φ_t are easily constructed. Then, $U_t = u \circ \Phi_t^{-1}$ is the solution of a Dirichlet problem on $\mathcal{D}_0$ with respect to an elliptic second-order differential operator $P_t(x, y, D_x, D_y)U_t = 0$ and in particular satisfies the boundary condition $U_t|_{\partial\mathcal{D}_0} = (\log|\Phi_t^{-1}|)|_{\partial\mathcal{D}_0}$. The coefficients of $P_t(x, y, D_x, D_y)$ depend continuously on $t \in [0, t_0]$ as well as their derivatives. The usual regularity theory of smooth elliptic boundary value problems implies that there exists a positive integer $N > 0$ with the following property: given $\rho > 0$ there exists $\delta > 0$ such that any function U that satisfies the equation $P_t(x, y, D_x.D_y)U = 0$ for some $t \in [0, t_0]$, and has in addition all tangential derivatives at the boundary bounded up to order N by δ, will satisfy the estimate $|\nabla U(x, y)| \leq \rho$. Since $\mathcal{D}_0$ is close to the unit disk for small ϵ, it follows that $(\log|\Phi_t^{-1}|)|_{\partial\mathcal{D}_0}$ will have small tangential derivatives up to any fixed order, and thus $U_t = u \circ \Phi_t^{-1}$ will have uniformly small gradient. The chain rule now implies that $u = U \circ \Phi_t$ has small gradient, uniformly in $(x, y) \in \mathcal{D}_t$ and $t \in [0, t_0)$, proving that $|f_t'| = |\nabla u| \leq 1/2$ for small t and ϵ. □

Since Lemma II.4.3 implies (II.54), Lemma II.4.2 is proved. □

As we pointed out, the control of the L^p norm of u on the sublevel sets $\tilde{D}(m+t)$ in terms of the L^p norm of u on their boundaries $\partial\tilde{D}(m+t)$ given by Lemma II.4.2 is all that is needed in order to extend the proof carried out in the model case to the general case. The proof of Theorem II.4.1 is then complete. □

Remark II.4.4. Stronger results than Theorem II.4.1 are known. In fact, it is possible to sweep V^+ with suitable translates of Ω so that the L^p norm of the restriction of F to those translates is uniformly bounded ([**Ro**]). Theorem II.4.1 then follows from an application of Fubini's theorem.

II.4.2 Propagation of zeros of homogeneous solutions

Given a locally integrable structure $\mathcal{L}$ in a manifold Ω, and a solution u of $\mathcal{L}u = 0$ a natural question is: what additional conditions must the solution u satisfy in order to conclude that u vanishes identically? The local version of the question is: given $p \in \Omega$, and a neighborhood V of p, what conditions guarantee that there exists a neighborhood $p \in U \subset V$ on which u vanishes identically? A natural additional condition would be to require that u vanish in some subset of V. In a small neighborhood of p, $\mathcal{L}u = 0$ may be expressed as an overdetermined system of equations (II.5). To get some insight, let us consider the simplest case of a single vector field $L = A\partial_x + B\partial_y$, $|A| + |B| > 0$, defined in an open set $\Omega \subset \mathbb{R}^2$. Since the constant functions $u = C$ always satisfy $Lu = 0$ it is apparent that some additional condition is needed; for instance, requiring that u vanishes at p certainly rules out the nonzero constants, but for most vector fields this is not enough (there exist, however, vector fields whose only homogeneous solutions are the constant functions [**N1**]). If $L = \overline{\partial}$ is the Cauchy–Riemann operator of Example II.1.2, one could require that u vanishes at p to *infinite order* which would imply that u vanishes throughout any connected open set U that contains p. However, this condition will not be enough for the vector field $L = \partial_x$ since a smooth function $u(y)$ independent of x could vanish to infinite order at p and yet not vanish identically in any neighborhood U of p. A better condition for $L = \partial_x$ would be then to require that u vanishes on the curve $\Sigma = \{(p_1, y)\}$, $p = (p_1, p_2)$. So requiring that u vanishes on Σ, that is a submanifold of Ω of codimension one, works for both $\overline{\partial}$ and ∂_x but it does not work for ∂_y (show this). The main point is that ∂_y is tangent to $\{(p_1, y)\}$ while the two previous vector fields are transversal to any vertical line (for a complex vector field $L = X + iY$ with real part X and imaginary part Y, L transversal to Σ means that at least one of the two vectors X and Y is transversal). This suggests that we should look at the case where u vanishes on a submanifold Σ of codimension one to which L is transversal. Note that if the structure $\mathcal{L}$ of rank $n = 1$ generated by L is locally integrable, the corank m of $L^{\perp}$ must be one, so we have $N = 2$, $m = 1$, and $n = 1$. Elaboration of this type of consideration for the case of a locally integrable structure $\mathcal{L}$ of rank n and corank m defined in a manifold of dimension $N = n + m$ leads to the following definition:

Definition II.4.5. *Let $\Sigma \subset \Omega$ be an embedded submanifold. We say that Σ is maximally real with respect to $\mathcal{L}$ if*

(i) the dimension of Σ is equal to m;
(ii) for every $p \in \Sigma$, any nonvanishing section L of $\mathcal{L}$ defined in a neighborhood of p is transversal to Σ at p.

If local coordinates $\{x_1, \dots, x_m, t_1, \dots, t_n\}$ vanishing at p are chosen according to Corollary I.10.2, then $\mathcal{L}^{\perp}$ is generated in a neighborhood of $x = 0$, $t = 0$, by $\mathrm{d}Z_1, \dots, \mathrm{d}Z_m$, where $Z_j(x, t) = x_j + i\phi_j(x, t)$, $\phi_j(0, 0) = 0$, $(\partial\phi_j/\partial x_k(0, 0)) = 0$, $1 \le j, k, \le m$, and the vectors $L_1, \dots, L_n$ become $L_j = \partial/\partial t_j$, $j = 1, \dots, n$ at the origin. If Σ is maximally real, the vectors $\partial_{t_j}|_0$ are transversal to Σ at the origin, so by the implicit function theorem we may find locally defined functions $\tau_j(x)$ such that $\Sigma = \{(x, \tau(x)\}$, where $\tau(x) = (\tau_1(x), \dots, \tau_n(x))$. If we perform the change of coordinates $x' = x$, $t' = t - \tau(x)$ the expression of Z in the new coordinates is $Z'(x', t') = x' + i\phi(x', t' + \tau(x')) = x' + i\phi'(x', t')$ and now Σ is given by $t' = 0$. In other words, if Σ is maximally real, we may always assume that the set of coordinates (x, t) of Corollary I.10.2 are such that not only Z has the form $Z(x, t) = x + i\phi$ with ϕ real, $\phi(0, 0) = 0$, $\mathrm{d}_x\phi(0, 0) = 0$ but also that Σ is given locally by $\Sigma = \{(x, 0)\}$. In particular, if u is a distribution solution of $\mathcal{L}u = 0$ we may always consider its restriction to Σ, $u|_\Sigma$, which is just the trace $u(x, 0)$ which we have seen to exist from considerations on the wave front set of u.

THEOREM II.4.6. *Let $\mathcal{L}$ be a locally integrable structure on the manifold Ω and let $\Sigma \subset \Omega$ be an embedded submanifold maximally real with respect to $\mathcal{L}$. If $u \in \mathcal{D}'(\Omega)$ satisfies*

(i) *$\mathcal{L}u = 0$ in Ω;*
(ii) *$u|_\Sigma = 0$;*

then u vanishes identically in a neighborhood V of Σ.

PROOF. It is enough to see that any point $p \in \Sigma$ is contained in a neighborhood U on which u vanishes identically. According to our previous remarks, given $p \in \Sigma$ we may assume that the special coordinates of Corollary I.10.2 that were used to prove Theorem II.1.1 are such that Σ is given by $\Sigma = \{(x, 0)\}$ and $p = (0, 0)$. We may find open sets $0 \in U \subset W$ as in Theorem II.1.1 so that W is contained in the coordinate neighborhood and u is approximated in U by $E_\tau u$ in the sense of $\mathcal{D}'(U)$. However, the formula that defines $E_\tau u$ right after (II.5) shows that $E_\tau u(x, t) = 0$ because $u(x, 0)$ vanishes on $\Sigma \cap W$. Thus, $u \equiv 0$ on U. □

COROLLARY II.4.7. *Let $\mathcal{L}$ be a locally integrable structure on a manifold Ω and let $u \in \mathcal{D}'(\Omega)$ satisfy $\mathcal{L}u = 0$ in Ω. Let L be a local section of $\mathcal{L}$, let $X = \Re L$. Assume that γ is an integral curve of X joining the points p and $q \in \Omega$. Then $p \in \operatorname{supp} u \implies q \in \operatorname{supp} u$.*

PROOF. If X vanishes at p then $p = q$ and there is nothing to prove. We may assume that $\gamma : [0, 1] \to \Omega$ is a nonconstant solution of $\gamma'(s) = X \circ \gamma(s)$, $0 \le s \le 1$, with $\gamma(0) = q$ and $\gamma(1) = p$, so X does not vanish in a neighborhood of γ. Denote by $K = \operatorname{supp} u$ the support of u and let us assume for the sake of a contradiction that $p \in K$ and $q \notin K$. Replacing p by the first point $\gamma(s)$ such that $\gamma(s) \in K$ we may assume that p and q are as close as we wish and all points in γ between q and p are not in K. We may find a local set

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of generators of $\mathcal{L}$, $L \doteq L_1, L_2, \ldots, L_n$ such that in appropriate coordinates (x, t), $|x| < 1$, $|t| < 2$, that rectify the flow of $X_1 \doteq X$ we have

(i)
$$X_1 = \Re L_1 = \frac{\partial}{\partial t_1}$$
$$X_j = \Re L_j = \frac{\partial}{\partial t_j} + \sum_{k=1}^{m} \lambda_{jk} \frac{\partial}{\partial x_k}, \quad j = 2, \ldots, n$$
and $p = (0, 0)$;

(ii) $\gamma(s) = (s - 1, 0, \ldots, 0)$, $q = \gamma(0) = (-1, 0, \ldots, 0)$;

(iii) for some $a > 0$ the embedded closed m-ball given by $|x| \leq a$, $t' = 0$, $t_1 = -1$ does not meet K (here $t' = (t_2, \ldots, t_n)$). Since it is an embedded submanifold with boundary we may denote this m-ball as $\Sigma_0 \cup \partial\Sigma_0$, where Σ_0 is the corresponding open m-ball.

Consider now the one-parameter family of embedded submanifolds Σ_σ (without their boundaries) given by the equations

$$t_1 = \sigma - 1 - \sigma \frac{|x|^2}{a^2}, \quad t_2 = \cdots = t_n = 0, \quad |x| < a, \quad 0 \leq \sigma \leq 1.$$

Since $\Sigma_0 \cap K = \emptyset$ and $(0, 0) \in \Sigma_1 \cap K$ there is a largest $\sigma_0 \in (0, 1]$ such that $\Sigma_\sigma \cap K = \emptyset$ for $0 \leq \sigma < \sigma_0$. Note that the submanifolds Σ_σ are all maximally real with respect to $\mathcal{L}$. Indeed, the vector fields X_j, $1 \leq j \leq n$, are transversal to any Σ_σ. This is clear for $j \geq 2$ because Σ_σ is contained in the slice $t_2 = \cdots = t_n = 0$ and it is also obvious for $j = 1$ because $(\partial/\partial t_1)$ is never tangent to Σ_σ. Hence, the trace $u|_{\Sigma_\sigma}$ is well-defined and furthermore $u|_{\Sigma_\sigma} = 0$ for $0 < \sigma < \sigma_0$ and, since $\sigma \mapsto u|_{\Sigma_\sigma}$ depends continuously on σ, we conclude that

$$u|_{\Sigma_{\sigma_0}} = 0. \tag{A}$$

We claim that

$$\Sigma_{\sigma_0} \cap K \neq \emptyset. \tag{B}$$

Indeed, since $\operatorname{dist}(\Sigma_{\sigma_0}, K) = 0$, this is certainly true if we replace Σ_{σ_0} by its closure $\overline{\Sigma_{\sigma_0}}$ which amounts to adding to Σ_{σ_0} its boundary points $\partial\Sigma_{\sigma_0}$. But, for any $\sigma \in [0, 1]$, $\partial\Sigma_\sigma$ is given by $|x| = a$, $t_1 = -1$, $t_2 = \cdots = t_n = 0$, so (iii) shows that $\partial\Sigma_\sigma \cap K = \emptyset$. Hence, $\Sigma_{\sigma_0} \cap K = \overline{\Sigma_{\sigma_0}} \cap K \neq \emptyset$. However, applying Theorem II.4.6 to Σ_{σ_0}, (A) implies that u vanishes in a neighborhood of Σ_{σ_0} in $|x| < a$, $|t| < 2$. This contradicts (B). $\square$

Let Ω be a manifold and consider a collection $D = \{X\}$ of locally defined, smooth, real vector fields X. In Chapter III, the notion of *orbit of D* is

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defined. Suppose now that $\mathcal{L}$ is a locally integrable structure and we consider the collection $D_{\mathcal{L}} = \{\Re L\}$ of all vector fields that are real parts of local sections of $\mathcal{L}$. In this case the orbits of $D_{\mathcal{L}}$ are simply called the orbits of $\mathcal{L}$. In the language of orbits, Corollary II.4.7 implies that if an orbit of $\mathcal{L}$ intersects the support K of a solution u of the equation $\mathcal{L}u = 0$ it must be entirely contained in K. This is equivalent to saying that K is a union of orbits of $\mathcal{L}$. Thus, Corollary II.4.7 gives an alternative proof of Theorem III.2.1. The proof presented in Chapter III follows in a remarkable simple way—thanks to the use of a criterion of Bony about flow-invariant sets—from a related uniqueness result that we now describe. An embedded submanifold of Ω of codimension 1 will be called a hypersurface. A hypersurface $\Sigma \subset \Omega$ is noncharacteristic with respect to $\mathcal{L}$ at $p \in \Sigma$ if there exists a local section L of $\mathcal{L}$ defined in a neighborhood of p that is transversal to Σ at p (which means, changing L by iL if necessary, that $X = \Re L$ is transversal to Σ at p). Notice that if u is a solution of $\mathcal{L}u = 0$ defined in a neighborhood U of p, the trace $u|_{\Sigma \cap U}$ is defined because u satisfies the equation $Lu = 0$ for any local section of $\mathcal{L}$, so choosing L transversal to Σ we see that the wave front set of u does not contain Σ's conormal directions.

Definition II.4.8. *Let $\mathcal{L}$ be a formally integrable structure in the manifold Ω. We say that $\mathcal{L}$ has the Uniqueness in the Cauchy Problem property for noncharacteristic hypersurfaces if and only if the following holds: for every hypersurface Σ, every point $p \in \Sigma$ such that Σ is noncharacteristic at p and every distribution solution u of $\mathcal{L}u = 0$ defined in a neighborhood U of p,*

$$u|_{U\cap\Sigma} = 0 \implies u \textit{ vanishes in a neighborhood of } p.$$

Corollary II.4.9. *The Uniqueness in the Cauchy Problem property for noncharacteristic hypersurfaces holds for every locally integrable structure $\mathcal{L}$.*

Proof. Let Σ be a noncharacteristic hypersurface at p. As usual, we denote by N the dimension of the manifold Ω, by n the rank of $\mathcal{L}$ and set $m = N - n$. In appropriate local coordinates (x, t) we may assume that $\mathcal{L}^{\perp}$ is generated by $\mathrm{d}Z_1, \dots, \mathrm{d}Z_m$, $Z = x + i\phi(x, t)$, $\phi(0, 0) = 0$, $\mathrm{d}_x\phi(0, 0) = 0$, $p = (0, 0)$. Hence, $\mathcal{L}$ is spanned at $(0, 0)$ by

$$\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n},$$

and since $\mathcal{L}$ is transversal to Σ at $p = (0, 0)$, the implicit function theorem gives a local representation of Σ as $t_1 = t_1(t', x)$, $t' = (t_2, \dots, t_n)$, after renumbering the t-coordinates if necessary. Let Σ_1 be given by $t_1 = t_1(0, x)$, $t' = 0$. Then, Σ_1 is a maximally real submanifold contained in Σ that contains

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$p = (0, 0)$. Consider now a neighborhood U of $p = (0, 0)$ and $u \in \mathcal{D}'(U)$ such that $\mathcal{L}u = 0$ and $u|_{U\cap\Sigma} = 0$. Since $\Sigma_1 \subset \Sigma$ we also have that $u|_{U\cap\Sigma_1} = 0$ and it follows from Theorem II.4.6 that u vanishes in a neighborhood of p. $\square$

EXAMPLE II.4.10. P. Cohen ([**Co**]) (see also [**Zu**] and the references therein) constructed smooth functions $u(x, y)$ and $a(x, y)$ defined on $\mathbb{R}^2$ such that

(1) $Lu(x, y) = \dfrac{\partial u}{\partial y} + a(x, y)\dfrac{\partial u}{\partial x} = 0$;
(2) $u(x, y) = a(x, y) = 0$ for all $y \leq 0$;
(3) $\operatorname{supp} u = \operatorname{supp} a = \{(x, y) : \quad y \geq 0\}$.

Thus, the formally integrable structure $\mathcal{L}$ spanned by the vector field L fails to have the Uniqueness in the Cauchy Problem property for the noncharacteristic curve $\Sigma = \{t = 0\}$ and, by Corollary II.4.9, cannot be locally integrable in any open set that intersects the x-axis. The construction of $a(x, y)$ shows that $a(x, y)$ is real-analytic for $y \neq 0$, so for any point $p = (x, y)$ with $y \neq 0$ we may find a function Z defined in a neighborhood of p such that $LZ = 0$ and $\mathrm{d}Z(p) \neq 0$. On the other hand, if Z is a smooth function defined in a neighborhood of $p = (x, 0)$ such that $LZ = 0$ we must have that $\mathrm{d}Z(p) = 0$, otherwise $\mathcal{L}$ would be locally integrable in some open set that intersects the x-axis, a contradiction. A nonlocally integrable vector field was first exhibited by Nirenberg ([**N1**]) who used a completely different method to construct a vector field whose only homogeneous solutions are contant.

II.4.3 An extension

In the applications to uniqueness we have seen so far, the 'initial' maximally real manifold $t = 0$ is in the interior of the domain where the solution u of $\mathcal{L}u = 0$ is defined. This is quite convenient because in this case the trace $u(\cdot, t)$ exists and $t \mapsto u(\cdot, t)$ is a continuous function of t valued in the space of distributions. However, in the study of one-sided Cauchy problems or boundary values of solutions, it is desirable to consider the case where the solution is not defined in a neighborhood of the 'initial' manifold. We will say that a set $\Gamma \subset \mathbb{R}^n\backslash\{0\}$ is a cone (or a cone with vertex at the origin to be explicit) if $t \in \Gamma \Longleftrightarrow \rho t \in \Gamma\ \forall\ 0 < \rho < \infty$. A set $\Gamma_T \subset \mathbb{R}^n\backslash\{0\}$, $0 < T$, will be called a truncated cone if there exists a cone Γ such that $\Gamma_T = \Gamma \cap \{|t| < T\}$. An open truncated cone is a truncated cone which is an open set. Notice that the origin is in the closure of Γ and Γ_T but it does not belong to them. A cone Γ' is said to be a proper subcone of Γ if $\overline{\Gamma'} \cap \{|x| = 1\}$ is a compact subset of Γ. This is, for instance, the case if Γ and Γ' are circular cones with the same axis and Γ' has a smaller aperture than Γ. If Γ' is a proper subcone of Γ and $T' < T$ we say that $\Gamma'_{T'}$ is a proper truncated subcone of the truncated cone Γ_T. When $n = 1$, a truncated cone is an interval of the form $(0, T)$ or $(-T, 0)$ or the union of both. If $W \subset \mathbb{R}^m$ is an open set and $\Gamma_T \subset \mathbb{R}^n$ is an open truncated cone, the set $W \times \Gamma_T \subset \mathbb{R}^m \times \mathbb{R}^n$ is usually called a wedge with edge W.

Consider a locally integrable structure $\mathcal{L}$ of rank n in an N-manifold and assume that the standard coordinates (x, t) used in the proof of Theorem II.1.1 had been chosen in a neighborhood of the origin. Let $B_x \subset \mathbb{R}^m$, $m = N - n$, be a ball centered at the origin, $\Gamma_T \subset \mathbb{R}^n$ a truncated open cone, and assume that u is a distribution satisfying the system (II.5) in $B_x \times \Gamma_T$. Under this circumstances we can assert that the trace $u|_{B_x \times \{t\}} = T_t u(x) = u(x, t)$ is defined and depends smoothly on $t \in \Gamma_T$ as a map valued in $\mathcal{D}'(B_x)$, but $u(x, 0)$ might not be defined. On the other hand, we may *assume* that $\lim_{t\to 0} T_t u \doteq bu$ exists in $\mathcal{D}'(B_x)$ as $t \to 0$.

If $n = N - m = 1$, $B_x \times \{0\}$ divides $\Omega = B_x \times (-T, T)$ into two components $\Omega^+ = \{(x, t) \in \Omega : t > 0\}$ and $\Omega^- = \{(x, t) \in \Omega : t < 0\}$ and in this case we consider distributions u that satisfy the system (II.5) in Ω^+ and such that $\lim_{t\searrow 0} T_t u = bu$ exists. In other words, we assume that $u \in C^0(\Gamma_T \cup \{0\}, \mathcal{D}'(B_x))$ (resp. $u \in C^0([0, T), \mathcal{D}'(B_x))$ for $n = 1$). We see that $E_\tau u$ can still be defined by

$$E_\tau u(x, t) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} \mathrm{e}^{-\tau[Z(x,t)-Z(x',0)]^2} u(x', 0) h(x') \det Z_x(x', 0)\, \mathrm{d}x'$$

as soon as we interpret $u(x', 0)$ as $bu(x')$. For a given $t \in \Gamma_T$ and $0 < \epsilon < 1$ consider

$$R^\epsilon_\tau u(x, t) = G_\tau u(x, t) - E^\epsilon_\tau u(x, t)$$

where $E^\epsilon_\tau u$ is given by

$$E^\epsilon_\tau u(x, t) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} \mathrm{e}^{-\tau[Z(x,t)-Z(x',0)]^2} u(x', \epsilon t) h(x') \det Z_x(x', 0)\, \mathrm{d}x'$$

and

$$G_\tau u(x, t) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} \mathrm{e}^{-\tau[Z(x,t)-Z(x',t)]^2} u(x', t) h(x') \det Z_x(x', t)\, \mathrm{d}x'.$$

As in the proof of Theorem II.1.1, the remainder $R^\epsilon_\tau u$ is given by

$$R^\epsilon_\tau u(x, t) = \int_{[\epsilon t, t]} \sum_{j=1}^m r_j(x, t, t', \tau)\, \mathrm{d}t'_j,$$

where

$$r_j(x, t, t', \tau) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} \mathrm{e}^{-\tau[Z(x,t)-Z(x',t')]^2} u(x', t') L_j h(x') \det Z_x(x', t')\, \mathrm{d}x'.$$

Letting $\epsilon \to 0$ we obtain

$$R_\tau u = G_\tau u - E_\tau u,$$

with R_τ given by

$$R_\tau u(x,t) = \int_{[0,t]} \sum_{j=1}^{m} r_j(x,t,t',\tau)\,\mathrm{d}t'_j,$$

$$r_j(x,t,t',\tau) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} \mathrm{e}^{-\tau[Z(x,t)-Z(x',t')]^2} u(x',t') L_j h(x') \det Z_x(x',t')\,\mathrm{d}x'.$$

The proof of Theorem II.1.1 now shows that there is a ball $B'_x = B'_x(0,\delta)$ and proper subcone $\Gamma'_\rho \subset \Gamma_T$ such that $R_\tau u \to 0$ uniformly in $B'_x \times \Gamma'_\rho$ as $\tau \to \infty$. Indeed, we can find a fixed k such that $v_j(x,t) = (1-\Delta_x)^{-k}[(L_j h(x))\,u(x,t)$ $\det Z_x(x,t))]$ is continuous in $B_x \times \Gamma'_\rho$, since the distributions $x \to L_j h(x)\,u(x,t)$ $\det Z_x(x,t)$ lie in a bounded set of some Sobolev space when t ranges over a compact subset of $\Gamma_T \cup \{0\}$ because $u \in C^0(\Gamma_T \cup \{0\}, \mathcal{D}'(B_x))$. Since the continuity of $\Gamma_T \cup \{0\} \ni t \to T_t u(x) \in \mathcal{D}'(B_x)$ implies the continuity of $\Gamma_T \cup \{0\} \ni t \to D_x^\alpha T_t u(x) = T_t D_x^\alpha u(x) \in \mathcal{D}'(B_x)$ and equation (II.5) allows us to express the derivatives of u with respect to t as a linear combination with smooth coefficients of derivatives of u with respect to x for $t \neq 0$, we conclude that actually $u \in C^\infty(\Gamma_T \cup \{0\}, \mathcal{D}'(B_x))$. The derivatives of $R_\tau u$ can be estimated in the same fashion and we obtain

Corollary II.4.11. *Let $u \in C^0(\Gamma_T \cup \{0\}, \mathcal{D}'(B_x))$ (resp. $u \in C^0([0,T), \mathcal{D}'(B_x))$ for $n = 1$) be a distribution satisfying the system* (II.5) *in $\Omega = B_x \times \Gamma_T$ (resp. in $\Omega^+ = B_x \times (0,T)$ for $n=1$). There exist $\delta > 0$, and a proper subcone $\Gamma'_\rho \subset \Gamma_T$ (resp. a number $\rho > 0$ for $n = 1$) such that for all multi-indexes $\alpha \in \mathbb{Z}_+^m$ and $\beta \in \mathbb{Z}_+^n$*

$$D_x^\alpha D_t^\beta R_\tau u(x,t) \longrightarrow 0 \quad \textit{uniformly on } B_x(0,\delta) \times (\Gamma_\rho \cup \{0\})$$

(resp. on $B_x(0,\delta) \times [0,\rho)$ for $n = 1$).

Corollary II.4.11 reduces the study of the approximation of u by E_τ to the problem of approximating u by $G_\tau u$. As an illustration, we sketch the proof of a version of the approximation for wedges. Consider a wedge $W = B_x \times \Gamma_T$—where $B_x \subset \mathbb{R}^m$ is a ball centered at the origin and $\Gamma_T \subset \mathbb{R}^n$ is an open truncated cone—and a locally integrable structure $\mathcal{L}$ with first integrals $Z_1 = x_1 + i\phi_1(x,t), \ldots, Z_m = x_m + i\phi_m(x,t)$, $\phi(0,0) = \mathrm{d}_x\phi(0,0) = 0$, defined in a neighborhood of the closure of W. Let $u \in C^0(\Gamma_T \cup \{0\}, L^p_{\mathrm{loc}}(B_x))$, $1 \le p < \infty$ satisfy $\mathcal{L}u = 0$ and we wish to approximate u by polynomials in Z in the topology of $C^0(\Gamma'_\rho \cup \{0\}, L^p_{loc}(B_x(\delta))$, where Γ'_ρ is a proper subcone of Γ of height ρ, $B_x(\delta) \subset B_x$ is a ball of radius δ and $\rho, \delta > 0$ are small.

Shrinking B_x we may assume that $u(\cdot, t) \in L^p$ and by Corollary II.4.11 it will be enough to approximate u by $G_\tau u$ in the norm

$$\sup_{t \in \Gamma_T} \|u(\cdot, t) - G_\tau u(\cdot, t)\|_{L^p(\mathbb{R}^m)}.$$

By the proof of Theorem II.3.1 we know that the norm of G_τ as an operator on $L^p(\mathbb{R}^m)$ (depending on t as a parameter) may be bounded by a constant independent of $t \in \Gamma_T$. Thus, it is enough to check that G_τ converges strongly to the identity on a dense subset of $C^0(\Gamma_T \cup \{0\}, L^p(B_x))$. This is indeed the case, because if $v(x, t)$ is continuous and supported in $(\Gamma_T \cup \{0\}) \times B'_x$ where B'_x is a ball concentric with B_x and of smaller radius, we know by the proof of Theorem II.1.1 that $G_\tau v(x, t) \to v(x, t)$ uniformly on $\Gamma_T \times B_x$ and this implies convergence in the norm of $C^0(\Gamma_T \cup \{0\}, L^p(B_x))$. This proves

THEOREM II.4.12. *Let $\mathcal{L}$ be a locally integrable structure with first integrals $Z_1, \dots, Z_m$, defined in a neighborhood of the closure of $W = B_x \times \Gamma_T$. There exist a ball $B'_x \subset B_x$ and a proper truncated subcone Γ'_ρ of Γ_T such that for any $u \in C^0(\Gamma_T \cup \{0\}, L^p(B_x))$, $1 \le p < \infty$, satisfying $\mathcal{L}u = 0$*

$$E_\tau u(x, t) \longrightarrow u(x, t) \text{ in } C^0(\Gamma'_\rho \cup \{0\}, L^p(B'_x)), \quad \tau \longrightarrow \infty. \tag{II.57}$$

As usual, we may replace the operator E_τ in (II.57) *by a convenient sequence of polynomials in Z, $P_\ell(Z_1, \dots, Z_m)$.* □

Notes

The approximation formula of Section II.1 for classical solutions was first proved by Baouendi and Treves in [**BT1**], building upon their previous work ([**BT2**]) that dealt with a corank one system of real-analytic vector fields. For distribution solutions, the proof in [**BT1**] relied on a local representation formula proved under a supplementary hypothesis on the locally integrable structure. This representation formula, which is of independent interest and states that any distribution solution u of $\mathcal{L}u = 0$ may be written as $u = P(x, D)v$, where v is a classical solution of $\mathcal{L}v = 0$ and $P(x, D)$ is a differential operator that commutes with the local generators L_j, $1 \le j \le n$, of $\mathcal{L}$. This representation formula was proved in general by Treves in [**T4**], who also stated and proved the approximation formula for distribution solutions in all generality. Metivier studied the case of a nonlinear first-order analytic single equation and proved an approximation formula for solutions of class C^2, obtaining as a consequence uniqueness in the Cauchy problem ([**Met**]).

The convergence in L^p of the approximation formula for solutions in L^p is an unpublished observation of S. Chanillo and S. Berhanu; the proofs presented here for L^p as well as for other functional spaces follow [**HMa1**].

It was soon realized by researchers in several complex variables theory that the approximation formula, although formulated in the rather general context of locally integrable structures, could be applied with success to deal with classical questions and it was used early as a tool in the problem of extending CR functions ([**BP**],[**W1**], [**BT3**]) and other matters like the study of the Radó property for CR functions ([**RS**]) (see also [**HT1**] for the Radó property for solutions of locally solvable vector fields).

Because the approximation is obtained through the operator E_τ that depends linearly on the trace of the solution on a maximally real submanifold, it is hardly surprising that it would have consequences for uniqueness questions. One remarkable feature is that it applies directly to distribution solutions in sharp contrast with other methods, like Carleman's estimates, which were devised to deal with functions rather than with less regular distributions. Before the definition of orbits by Sussmann in 1973 ([**Su**]), the propagation of zeros had been observed for some operators with real-analytic coefficients ([**Z**]) using as propagators Nagano's leaves ([**Na**]), which coincide with Sussmann's orbits in the real-analytic set-up. The theorem stating that the support of a solution is a union of Sussmann's orbits was initially stated and proved in [**T4**]. Another early application to uniqueness is [**BT4**]. Nowadays, the use of the approximation formula is so standard that probably there is no point in keeping track of its use in the literature. Anyway, we mention [**BH3**] as a recent uniqueness result that takes advantage of the approximation formula. Another application outside the scope of the theory of holomorphic functions is its use in the study of removable singularities for solutions of locally solvable vector fields ([**HT2**], [**HT3**], [**HT4**]).