13

Hyperbolic reductions

This chapter discusses several methods for the construction of *symmetric hyperbolic evolution systems* out of the conformal Einstein field equations. Once suitable evolution systems have been obtained, the methods of Chapter 12 allow, in turn, one to make statements about the existence of solutions to the equations. Direct inspection of the conformal field equations reveals that these are overdetermined – there are more equations than unknowns, even if the symmetries of the various tensorial and spinorial fields are taken into account. Thus, the process of hyperbolic reduction for the conformal field equations necessarily requires discarding some of the equations. The discarded equations are then treated as constraints. It is a remarkable structural property of the conformal field equations that these constraints satisfy a system of evolution equations – a so-called ***subsidiary evolution system*** – from where it can be concluded that the constraint equations will be satisfied if they hold at some initial hypersurface and the evolution equations are imposed. This construction is called the ***propagation of the constraints***. The solution of the evolution system together with the propagation of the constraints yields the required solution of the conformal Einstein field equations.

In this chapter, two different procedures for the hyperbolic reduction of the conformal Einstein field equations are considered. The first method, based on the notion of ***gauge source functions***, exploits the fact that certain derivatives of the conformal fields are not directly determined by the equations and, thus, can be freely specified. In the spinorial formulation of the equations, once the required gauge source functions have been specified, the irreducible decomposition of the various zero quantities leads to the required evolution equations. The equations obtained by this procedure include the conformal factor as an unknown.

The second hyperbolic reduction procedure presented in this chapter exploits the properties of congruences of conformal geodesics to construct ***conformal Gaussian gauge systems***. As discussed in Chapter 5, the connection coefficients and components of the Schouten tensor with respect to a frame which is

Weyl propagated along the congruence satisfy certain relations which lead to a particularly simple system of equations in which the evolution of all the geometric unknowns, save for the components of the rescaled Weyl spinor, are either fixed by the gauge or given by transport equations along the congruence. Moreover, as a consequence of the properties of the conformal geodesics one gains an a priori knowledge of the location of the conformal boundary; see Proposition 5.1. Despite these attractive features, this method is less flexible than the one based on the use of gauge functions and may not be readily extended to non-vacuum situations.

13.1 A model problem: the Maxwell equations on a fixed background

To illustrate the various aspects of the construction of evolution equations for the conformal Einstein field equations, it is convenient to analyse the analogous problem for the Maxwell equations on a fixed background.

In the remainder of this section, let $\mathcal{U}$ denote an open region of a spacetime $(\mathcal{M}, \boldsymbol{g})$. It will be assumed that $\mathcal{U}$ is covered by a non-singular congruence of curves with tangent vector $\boldsymbol{\tau}$ satisfying the normalisation condition $\boldsymbol{g}(\boldsymbol{\tau}, \boldsymbol{\tau}) = 2$. The vector $\boldsymbol{\tau}$ does not need to be hypersurface orthogonal. Let $\tau^{AA'}$ denote the spinorial counterpart of τ^a. As discussed in Section 4.2.5, the spinor $\tau^{AA'}$ gives rise to a Hermitian structure, and, accordingly, one can introduce a space spinor formalism. Let $\{\epsilon_{\boldsymbol{A}}{}^A\}$ denote a spin basis such that

$$\tau^{\boldsymbol{AA'}} = \epsilon_{\boldsymbol{0}}{}^{\boldsymbol{A}} \epsilon_{\boldsymbol{0'}}{}^{\boldsymbol{A'}} + \epsilon_{\boldsymbol{1}}{}^{\boldsymbol{A}} \epsilon_{\boldsymbol{1'}}{}^{\boldsymbol{A'}}, \tag{13.1}$$

and with $\{\boldsymbol{e}_{\boldsymbol{AA'}}\}$ its associated null frame. At every point $p \in \mathcal{U}$ a basis of the subspace $\langle \boldsymbol{\tau} \rangle^{\perp}|_p \subset T|_p(\mathcal{U})$ orthogonal to $\boldsymbol{\tau}$ is given by $\boldsymbol{e}_{\boldsymbol{AB}} = \tau_{(\boldsymbol{B}}{}^{\boldsymbol{A'}} \boldsymbol{e}_{\boldsymbol{A})\boldsymbol{A'}}$. In terms of local coordinates $x = (x^\mu)$ in $\mathcal{U}$ one writes

$$\boldsymbol{e}_{\boldsymbol{AB}} = e_{\boldsymbol{AB}}{}^{\mu} \boldsymbol{\partial}_\mu. \tag{13.2}$$

In principle, it is possible for the frame vectors $\boldsymbol{e}_{\boldsymbol{AB}}$ to have components with respect to the time coordinate. The frame components $e_{\boldsymbol{AB}}{}^{\mu}$ satisfy the reality conditions

$$e_{\boldsymbol{01}}{}^{\mu} = \overline{e_{\boldsymbol{01}}{}^{\mu}}, \qquad e_{\boldsymbol{00}}{}^{\mu} = -\overline{e_{\boldsymbol{11}}{}^{\mu}}. \tag{13.3}$$

All spinorial objects will be expressed with respect to the spin basis $\{\epsilon_{\boldsymbol{A}}{}^A\}$. In particular, the spinorial Maxwell Equation (9.15) is written as

$$\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A'}} \varphi_{\boldsymbol{BQ}} = 0. \tag{13.4}$$

In what follows, it will be convenient to introduce the ***zero quantity***

$$\omega_{\boldsymbol{A'B}} \equiv \nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A'}} \varphi_{\boldsymbol{BQ}},$$

so that (13.4) can be expressed as $\omega_{\boldsymbol{A'B}} = 0$. Here and in the remainder of this chapter, zero quantities such as $\omega_{\boldsymbol{A'B}}$ serve as convenient bookkeeping devices to denote the various field equations.

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

13.1.1 Space spinor description of the Maxwell equations and hyperbolic reductions

The space spinor version of Equation (13.4) leads to a decomposition into evolution and constraint equations. Following the discussion of Chapter 4 one considers the *unprimed zero quantity* $\omega_{\boldsymbol{BA}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A}'}\omega_{\boldsymbol{A}'\boldsymbol{A}}$. One then has that

$$\begin{aligned}\omega_{\boldsymbol{BA}} = \nabla^{\boldsymbol{Q}}{}_{\boldsymbol{B}}\varphi_{\boldsymbol{AQ}} &= \frac{1}{2}\epsilon^{\boldsymbol{Q}}{}_{\boldsymbol{A}}\mathcal{P}\varphi_{\boldsymbol{BQ}} + \mathcal{D}^{\boldsymbol{Q}}{}_{\boldsymbol{A}}\varphi_{\boldsymbol{BQ}} \\ &= -\frac{1}{2}\mathcal{P}\varphi_{\boldsymbol{AB}} + \mathcal{D}^{\boldsymbol{Q}}{}_{\boldsymbol{A}}\varphi_{\boldsymbol{BQ}},\end{aligned}$$

where $\mathcal{P}$ is the covariant directional derivative along $\boldsymbol{\tau}$, $\mathcal{D}_{\boldsymbol{AB}}$ is the Sen covariant derivative implied by $\nabla_{\boldsymbol{AA}'}$ and $\nabla_{\boldsymbol{AB}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A}'}\nabla_{\boldsymbol{AA}'}$. In the above expressions, the decomposition

$$\nabla_{\boldsymbol{AB}} = \frac{1}{2}\epsilon_{\boldsymbol{AB}}\mathcal{P} + \mathcal{D}_{\boldsymbol{AB}} \tag{13.5}$$

has been used; see Section 4.3.1. The spinor $\omega_{\boldsymbol{BA}}$ can, in turn, be decomposed in irreducible parts as

$$\omega_{\boldsymbol{BA}} = \frac{1}{2}\epsilon_{\boldsymbol{BA}}\omega + \omega_{(\boldsymbol{AB})},$$

with

$$\omega \equiv \omega_{\boldsymbol{Q}}{}^{\boldsymbol{Q}} = \mathcal{D}^{\boldsymbol{PQ}}\varphi_{\boldsymbol{PQ}}, \qquad \omega_{(\boldsymbol{AB})} = -\frac{1}{2}\mathcal{P}\varphi_{\boldsymbol{AB}} + \mathcal{D}^{\boldsymbol{Q}}{}_{(\boldsymbol{A}}\varphi_{\boldsymbol{B})\boldsymbol{Q}}.$$

Thus, the Maxwell Equations (13.4) imply the equations

$$\omega = \mathcal{D}^{\boldsymbol{PQ}}\varphi_{\boldsymbol{PQ}} = 0, \tag{13.6a}$$

$$-2\omega_{(\boldsymbol{AB})} = \mathcal{P}\varphi_{\boldsymbol{AB}} - \mathcal{D}^{\boldsymbol{Q}}{}_{(\boldsymbol{A}}\varphi_{\boldsymbol{B})\boldsymbol{Q}} = 0. \tag{13.6b}$$

The decomposition of the spinorial Maxwell equation given by (13.6a) and (13.6b) shows that Equation (13.4) is overdetermined. Equation (13.6a) will be interpreted as a constraint equation on the orthogonal subspaces of the distribution generated by the vector field $\boldsymbol{\tau}$, while (13.6b) will be regarded as suitable evolution equations for the symmetric spinorial field $\varphi_{\boldsymbol{AB}}$.

13.1.2 The symmetric hyperbolicity of the Maxwell evolution equations

To apply the theory of Chapter 12 one needs to verify that the evolution Equations (13.6b) give rise to a symmetric hyperbolic system for the independent components of $\varphi_{\boldsymbol{AB}}$. One considers the slightly modified version

$$\binom{2}{\boldsymbol{A}+\boldsymbol{B}}\left(\mathcal{P}\varphi_{\boldsymbol{AB}} - \mathcal{D}^{\boldsymbol{Q}}{}_{(\boldsymbol{A}}\varphi_{\boldsymbol{B})\boldsymbol{Q}}\right) = 0, \tag{13.7}$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

where the binomial coefficient in front of the equation has been included to make the expression manifestly symmetric hyperbolic. The principal part of Equation (13.7) can be written as

$$\binom{2}{\boldsymbol{A}+\boldsymbol{B}}\left(\tau^{\mu}\partial_{\mu}\varphi_{\boldsymbol{AB}} - e^{\boldsymbol{Q}}{}_{(\boldsymbol{A}}{}^{\mu}\partial_{|\mu|}\varphi_{\boldsymbol{B})\boldsymbol{Q}}\right).$$

As a result of the symmetry of $\varphi_{\boldsymbol{AB}}$, the above principal part contains three independent expressions. These can be arranged in the matricial expression

$$\mathbf{A}^{\mu}\partial_{\mu}\boldsymbol{\varphi} \equiv \begin{pmatrix} \tau^{\mu}+e_{\boldsymbol{10}}{}^{\mu} & -e_{\boldsymbol{00}}{}^{\mu} & 0 \\ e_{\boldsymbol{11}}{}^{\mu} & 2\tau^{\mu} & e_{\boldsymbol{00}}{}^{\mu} \\ 0 & e_{\boldsymbol{11}}{}^{\mu} & \tau^{\mu}-e_{\boldsymbol{01}}{}^{\mu} \end{pmatrix} \partial_{\mu} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{pmatrix},$$

with

$$\varphi_0 \equiv \varphi_{\boldsymbol{00}}, \qquad \varphi_1 \equiv \varphi_{\boldsymbol{01}}, \qquad \varphi_2 \equiv \varphi_{\boldsymbol{11}}.$$

Thus, making use of the reality conditions (13.3), it follows that the matrices $\mathbf{A}^{\mu}$ are Hermitian. Moreover, the matrix

$$\mathbf{A}^{\mu}\tau_{\mu} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

clearly is positive definite. Thus, Equation (13.7) implies a symmetric hyperbolic system for the independent components of $\varphi_{\boldsymbol{AB}}$. Finally, a direct computation shows that given an arbitrary covector ξ_{μ},

$$\begin{aligned} \det(\mathbf{A}^{\mu}\xi_{\mu}) &= 2(\tau^{\mu}\xi_{\mu})\big(\tau^{\nu}\tau^{\lambda} + e_{\boldsymbol{00}}{}^{\nu}e_{\boldsymbol{11}}{}^{\lambda} - e_{\boldsymbol{01}}{}^{\nu}e_{\boldsymbol{10}}{}^{\lambda}\big)\xi_{\nu}\xi_{\lambda} \\ &= 4(\tau^{\mu}\xi_{\mu})(g^{\nu\lambda}\xi_{\nu}\xi_{\lambda}), \end{aligned}$$

where in the last line Equation (4.14) for the 1+3 decomposition of the spacetime metric has been used. Thus, $\boldsymbol{g}$-null hypersurfaces are characteristics of Equation (13.7) – these types of characteristics are often called ***physical characteristics***. By contrast, the factor $(\tau^{\mu}\xi_{\mu})$ is associated to ***gauge characteristics***.

For completeness, it is observed that the principal part of the constraint equation is given, explicitly, by

$$e_{\boldsymbol{00}}{}^{\mu}\partial_{\mu}\varphi_0 + e_{\boldsymbol{01}}{}^{\mu}\partial_{\mu}\varphi_1 + e_{\boldsymbol{11}}{}^{\mu}\partial_{\mu}\varphi_2,$$

so that, in general, it will contain derivatives in the time direction. More generally, if the vector $\boldsymbol{\tau}$ is not hypersurface orthogonal, then the constraint equation $\omega = 0$ will not be intrinsic to the leaves of a foliation.

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

13.1.3 The subsidiary system for the spinorial Maxwell equations

The hyperbolic reduction for the Maxwell equations discussed in Section 13.1.1 splits Equation (13.4) into three evolution equations and one constraint equation. Thus, if one wants to obtain a solution to Equation (13.4) through a Cauchy initial value problem, one uses, in first instance, the theory of Chapter 12 to show the existence of a unique solution to the evolution equations. In a second stage, one has to show that if the constraint equation is satisfied initially, then, by virtue of the evolution equations, it must be satisfied also at later times. This last argument requires the construction of a suitable hyperbolic evolution equation for ω.

To obtain an equation for the zero quantity ω one considers the expression $\nabla^{\boldsymbol{AA}'}\omega_{\boldsymbol{A}'\boldsymbol{A}}$. Using that $\omega_{\boldsymbol{A}'\boldsymbol{A}} = -\tau^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}\omega_{\boldsymbol{QA}}$ one has that

$$\begin{aligned}\nabla^{\boldsymbol{AA}'}\omega_{\boldsymbol{A}'\boldsymbol{A}} &= -\nabla^{\boldsymbol{AA}'}\big(\tau^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}\omega_{\boldsymbol{QA}}\big)\\ &= \nabla^{\boldsymbol{AQ}}\omega_{\boldsymbol{QA}} - (\nabla^{\boldsymbol{AA}'}\tau^{\boldsymbol{Q}}{}_{\boldsymbol{A}'})\omega_{\boldsymbol{QA}}.\end{aligned}$$

Now, using Equation (4.17), a calculation yields

$$\nabla^{\boldsymbol{AA}'}\tau^{\boldsymbol{Q}}{}_{\boldsymbol{A}'} = -\sqrt{2}\chi^{\boldsymbol{A}}{}_{\boldsymbol{P}}{}^{\boldsymbol{PQ}}, \tag{13.8}$$

so that

$$\nabla^{\boldsymbol{AA}'}\omega_{\boldsymbol{A}'\boldsymbol{A}} = \nabla^{\boldsymbol{AQ}}\omega_{\boldsymbol{QA}} + \sqrt{2}\chi^{\boldsymbol{A}}{}_{\boldsymbol{P}}{}^{\boldsymbol{PQ}}\omega_{\boldsymbol{QA}}.$$

Thus, the split (13.5) leads to the expression

$$\mathcal{P}\omega + 2\mathcal{D}^{\boldsymbol{AB}}\omega_{(\boldsymbol{AB})} + 2\sqrt{2}\chi^{\boldsymbol{A}}{}_{\boldsymbol{P}}{}^{\boldsymbol{PQ}}\omega_{\boldsymbol{QA}} = 2\nabla^{\boldsymbol{AA}'}\omega_{\boldsymbol{A}'\boldsymbol{A}}.$$

If the evolution equations hold – that is, $\omega_{(\boldsymbol{AB})} = 0$ – then $\omega_{\boldsymbol{AB}} = \frac{1}{2}\epsilon_{\boldsymbol{AB}}\omega$ and one obtains

$$\mathcal{P}\omega + \sqrt{2}\chi_{\boldsymbol{AB}}{}^{\boldsymbol{AB}}\omega = 2\nabla^{\boldsymbol{AA}'}\omega_{\boldsymbol{A}'\boldsymbol{A}}.$$

The next step is to evaluate $\nabla^{\boldsymbol{AA}'}\omega_{\boldsymbol{A}'\boldsymbol{A}}$ in an alternative manner. Using the definition of the zero quantity one has that

$$\nabla^{\boldsymbol{AA}'}\omega_{\boldsymbol{A}'\boldsymbol{A}} = \nabla^{\boldsymbol{AA}'}\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}\varphi_{\boldsymbol{AQ}}.$$

From the commutator

$$\nabla_{\boldsymbol{AA}'}\nabla_{\boldsymbol{BB}'}\varphi_{\boldsymbol{CD}} - \nabla_{\boldsymbol{BB}'}\nabla_{\boldsymbol{AA}'}\varphi_{\boldsymbol{CD}} = -R^{\boldsymbol{P}}{}_{\boldsymbol{CAA}'\boldsymbol{BB}'}\varphi_{\boldsymbol{PD}} - R^{\boldsymbol{P}}{}_{\boldsymbol{DAA}'\boldsymbol{BB}'}\varphi_{\boldsymbol{CP}},$$

suitably contracting indices one obtains

$$\nabla^{\boldsymbol{AA}'}\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}\varphi_{\boldsymbol{AQ}} = -2R^{\boldsymbol{P}}{}_{\boldsymbol{A}}{}^{\boldsymbol{AA}'\boldsymbol{Q}}{}_{\boldsymbol{A}'}\varphi_{\boldsymbol{PQ}} - 2R^{\boldsymbol{P}}{}_{\boldsymbol{Q}}{}^{\boldsymbol{AA}'\boldsymbol{Q}}{}_{\boldsymbol{A}'}\varphi_{\boldsymbol{AP}}.$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

Thus, combining the above equation with the decomposition

$$R_{\boldsymbol{A}\boldsymbol{B}\boldsymbol{C}\boldsymbol{C}'\boldsymbol{D}\boldsymbol{D}'} = \Psi_{\boldsymbol{A}\boldsymbol{B}\boldsymbol{C}\boldsymbol{D}}\epsilon_{\boldsymbol{C}'\boldsymbol{D}'} + L_{\boldsymbol{B}\boldsymbol{C}'\boldsymbol{D}\boldsymbol{D}'}\epsilon_{\boldsymbol{C}\boldsymbol{A}} - L_{\boldsymbol{B}\boldsymbol{D}'\boldsymbol{C}\boldsymbol{C}'}\epsilon_{\boldsymbol{D}\boldsymbol{A}}, \tag{13.9}$$

where $\Psi_{\boldsymbol{A}\boldsymbol{B}\boldsymbol{C}\boldsymbol{D}}$ and $L_{\boldsymbol{B}\boldsymbol{C}'\boldsymbol{D}\boldsymbol{D}'}$ denote, respectively, the spinorial counterparts of the Weyl and Schouten tensors, one concludes that $\nabla^{\boldsymbol{A}\boldsymbol{A}'}\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}\varphi_{\boldsymbol{A}\boldsymbol{Q}} = 0$. Hence, the evolution equation for ω takes the form

$$\mathcal{P}\omega + \sqrt{2}\chi_{\boldsymbol{A}\boldsymbol{B}}{}^{\boldsymbol{A}\boldsymbol{B}}\omega = 0 \qquad \text{if } \omega_{(\boldsymbol{A}\boldsymbol{B})} = 0.$$

The form of this equation implies, in together with Corollary 12.1, that if $\omega = 0$ on some spacelike hypersurface $\mathcal{S}_\star$ in $\mathcal{U}$, then $\omega = 0$ on lens-shaped domains having $\mathcal{S}_\star$ as base.

13.2 Hyperbolic reductions using gauge source functions

In this section hyperbolic reduction procedures for the conformal Einstein field equations based on the notion of ***gauge source functions*** are considered. Gauge source functions naturally arise in the analysis of frame formulations of the conformal Einstein field equations written in terms of the Levi-Civita connection $\boldsymbol{\nabla}$ of an unphysical metric $\boldsymbol{g}$. The present analysis will be restricted to the spinorial version of the conformal field equations: Equations (8.36a) and (8.36b) or, alternatively, Equations (8.38a) and (8.38b).

Basic set up and assumptions

As in the analysis of the Maxwell equations in Section 13.1, all the calculations will be performed in an open subset $\mathcal{U} \subset \mathcal{M}$ of an unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$ which is conformally related to a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ satisfying the Einstein field equations. On $\mathcal{U}$ one considers some local coordinates $x = (x^\mu)$ and an arbitrary frame $\{\boldsymbol{c}_{\boldsymbol{a}}\}$ which may or may not be a coordinate frame. Let $\{\boldsymbol{\alpha}^{\boldsymbol{a}}\}$ denote the dual coframe so that $\langle \boldsymbol{\alpha}^{\boldsymbol{a}}, \boldsymbol{c}_{\boldsymbol{b}}\rangle = \delta_{\boldsymbol{b}}{}^{\boldsymbol{a}}$. In what follows, let $\boldsymbol{\nabla}$ denote the Levi-Civita covariant derivative of the metric $\boldsymbol{g}$.

It will be assumed that $\mathcal{U}$ is covered by a non-singular congruence of curves with tangent vector $\boldsymbol{\tau}$ satisfying the normalisation condition $\boldsymbol{g}(\boldsymbol{\tau}, \boldsymbol{\tau}) = 2$. The vector $\boldsymbol{\tau}$ does not need to be hypersurface orthogonal. Let $\tau^{AA'}$ denote the spinorial counterpart of τ^a. In what follows, only spin bases $\{\epsilon_{\boldsymbol{A}}{}^A\}$ satisfying condition (13.1) will be considered. All spinors will be expressed in components with respect to this spin basis.

Let $\{\boldsymbol{e}_{\boldsymbol{A}\boldsymbol{A}'}\}$ and $\{\boldsymbol{\omega}^{\boldsymbol{A}\boldsymbol{A}'}\}$ denote, respectively, the null frame and coframe associated to the spin basis $\{\epsilon_{\boldsymbol{A}}{}^A\}$. By definition, one has that $\langle \boldsymbol{\omega}^{\boldsymbol{A}\boldsymbol{A}'}, \boldsymbol{e}_{\boldsymbol{B}\boldsymbol{B}'}\rangle = \epsilon_{\boldsymbol{B}}{}^{\boldsymbol{A}}\epsilon_{\boldsymbol{B}'}{}^{\boldsymbol{A}'}$. At every point $p \in \mathcal{U}$ a basis of $\langle \boldsymbol{\tau}\rangle^\perp|_p$, the subspace of $T|_p(\mathcal{U})$ orthogonal to $\boldsymbol{\tau}$ is given by $\boldsymbol{e}_{\boldsymbol{A}\boldsymbol{B}} = \tau_{(\boldsymbol{B}}{}^{\boldsymbol{A}'}\boldsymbol{e}_{\boldsymbol{A})\boldsymbol{A}'}$. The *spatial* frame can be expanded in terms of the vectors $\boldsymbol{c}_{\boldsymbol{a}}$ as $\boldsymbol{e}_{\boldsymbol{A}\boldsymbol{B}} = e_{\boldsymbol{A}\boldsymbol{B}}{}^{\boldsymbol{a}}\boldsymbol{c}_{\boldsymbol{a}}$. If the basis $\{\boldsymbol{c}_{\boldsymbol{a}}\}$ is a coordinate basis, the last expression reduces to the one given in Equation (13.2).

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

A model equation

The general strategy behind the procedure of hyperbolic reduction using gauge functions is best understood through a *model equation.*

In Section 12.1.3 it has been shown that spinorial equations of the form

$$\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}\varphi_{\boldsymbol{Q}\boldsymbol{B}'\boldsymbol{C}\cdots\boldsymbol{D}} = F_{\boldsymbol{A}'\boldsymbol{B}'\boldsymbol{C}\cdots\boldsymbol{D}} \tag{13.10}$$

imply a symmetric hyperbolic system for the components of the field $\varphi_{\boldsymbol{Q}\boldsymbol{B}'\boldsymbol{C}\cdots\boldsymbol{D}}$ which is not assumed to have any special symmetries. This equation is now contrasted with the equation

$$\nabla_{\boldsymbol{A}\boldsymbol{A}'}\varphi_{\boldsymbol{B}\boldsymbol{B}'\boldsymbol{C}\cdots\boldsymbol{D}} - \nabla_{\boldsymbol{B}\boldsymbol{B}'}\varphi_{\boldsymbol{A}\boldsymbol{A}'\boldsymbol{C}\cdots\boldsymbol{D}} = F_{\boldsymbol{A}\boldsymbol{A}'\boldsymbol{B}\boldsymbol{B}'\boldsymbol{C}\cdots\boldsymbol{D}}. \tag{13.11}$$

Exploiting the antisymmetry in the pairs $_{\boldsymbol{A}\boldsymbol{A}'}$ and $_{\boldsymbol{B}\boldsymbol{B}'}$ it follows that

$$\nabla^{\boldsymbol{Q}}{}_{(\boldsymbol{A}'}\varphi_{|\boldsymbol{Q}|\boldsymbol{B}')\boldsymbol{C}\cdots\boldsymbol{D}} = \frac{1}{2}F^{\boldsymbol{Q}}{}_{\boldsymbol{A}'\boldsymbol{Q}\boldsymbol{B}'\boldsymbol{C}\cdots\boldsymbol{D}}. \tag{13.12}$$

Thus, while Equation (13.10) determines the full derivative $\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}\varphi_{\boldsymbol{Q}\boldsymbol{B}'\boldsymbol{C}\cdots\boldsymbol{D}}$, Equation (13.12) determines only its symmetric part. More precisely, writing

$$\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}\varphi_{\boldsymbol{Q}\boldsymbol{B}'\boldsymbol{C}\cdots\boldsymbol{D}} = \nabla^{\boldsymbol{Q}}{}_{(\boldsymbol{A}'}\varphi_{|\boldsymbol{Q}|\boldsymbol{B}')\boldsymbol{C}\cdots\boldsymbol{D}} - \frac{1}{2}\epsilon_{\boldsymbol{A}'\boldsymbol{B}'}\nabla^{\boldsymbol{Q}\boldsymbol{Q}'}\varphi_{\boldsymbol{Q}\boldsymbol{Q}'\boldsymbol{C}\cdots\boldsymbol{D}}, \tag{13.13}$$

one has that the first term in the right-hand side is determined by Equation (13.12), while the *divergence* $\nabla^{\boldsymbol{Q}\boldsymbol{Q}'}\varphi_{\boldsymbol{Q}\boldsymbol{Q}'\boldsymbol{C}\cdots\boldsymbol{D}}$ remains unspecified. Thus, in the absence of other equations providing information about this term, the latter observation suggests completing Equation (13.13) by setting

$$\nabla^{\boldsymbol{Q}\boldsymbol{Q}'}\varphi_{\boldsymbol{Q}\boldsymbol{Q}'\boldsymbol{C}\cdots\boldsymbol{D}} = f_{\boldsymbol{C}\cdots\boldsymbol{D}}(x),$$

where $f_{\boldsymbol{C}\cdots\boldsymbol{D}} \in \mathcal{X}(\mathcal{M})$ are *smooth freely specifiable functions of the coordinates.* In what follows, functions of this type will be known as ***gauge source functions***. Thus, from (13.13) one obtains the equation

$$\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}\varphi_{\boldsymbol{Q}\boldsymbol{A}\boldsymbol{C}\cdots\boldsymbol{D}} = \frac{1}{2}F^{\boldsymbol{Q}}{}_{\boldsymbol{A}'\boldsymbol{Q}\boldsymbol{B}'\boldsymbol{A}\boldsymbol{C}\cdots\boldsymbol{D}} - \frac{1}{2}\epsilon_{\boldsymbol{A}'\boldsymbol{B}'}f_{\boldsymbol{A}\boldsymbol{C}\cdots\boldsymbol{D}}(x),$$

for which one can extract a symmetric hyperbolic evolution system for the components of $\varphi_{AA'C\cdots D}$; see the discussion of Section 12.1.3. In particular, the characteristics of this evolution system are null hypersurfaces of the spacetime metric $\boldsymbol{g}$.

As will be seen in the following subsections, several of the conformal Einstein field equations admit an analysis similar to that of Equation (13.11). A detailed discussion of the resulting evolution equations exploits the particular symmetries of the field appearing in the principal part.

13.2.1 Coordinate gauge source functions

The purpose of this subsection is to analyse the evolution equations arising from the *no-torsion condition* in the frame and spinor formulations of the conformal field equations; see Equations (8.31a), (8.35a), (8.44a) and (8.53a). This leads to the first class of gauge source functions that will be considered in this chapter: the ***coordinate gauge source functions***. Following the general discussion of Chapter 8, the no-torsion condition will be regarded as a differential condition on the coefficients of the frame $\{\boldsymbol{e}_{\boldsymbol{AA'}}\}$. Thus, the ultimate purpose of this section is to derive a symmetric hyperbolic subsystem for these quantities.

In Section 8.3.2 an expression for the spinorial counterpart of the torsion tensor $\Sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{BB'}}$ in terms of the spinorial connection coefficients $\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{BB'}}$ has been given; see Equation (8.35a). In what follows, it is more convenient to make use of an expression involving the reduced spin connection coefficients. Using the relation

$$\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{BB'}} = \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{B}}\epsilon_{\boldsymbol{B'}}{}^{\boldsymbol{C'}} + \bar{\Gamma}_{\boldsymbol{A'A}}{}^{\boldsymbol{C'}}{}_{\boldsymbol{B'}}\epsilon_{\boldsymbol{B}}{}^{\boldsymbol{C}},$$

– compare Equation (3.33) – it can be seen that

$$\begin{aligned}\Sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{BB'}}\boldsymbol{e}_{\boldsymbol{QQ'}} = [\boldsymbol{e}_{\boldsymbol{BB'}}, \boldsymbol{e}_{\boldsymbol{AA'}}] - \Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{A}}\boldsymbol{e}_{\boldsymbol{QA'}} - \bar{\Gamma}_{\boldsymbol{BB'}}{}^{\boldsymbol{Q'}}{}_{\boldsymbol{A'}}\boldsymbol{e}_{\boldsymbol{AQ'}}\\ + \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B}}\boldsymbol{e}_{\boldsymbol{QB'}} + \bar{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{Q'}}{}_{\boldsymbol{B'}}\boldsymbol{e}_{\boldsymbol{BQ'}}.\end{aligned} \tag{13.14}$$

Using the frame $\{\boldsymbol{c}_{\boldsymbol{a}}\}$ one can write

$$\boldsymbol{e}_{\boldsymbol{AA'}} = e_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}\boldsymbol{c}_{\boldsymbol{a}},$$

so that for fixed frame spinorial indices ${}_{\boldsymbol{AA'}}$, the coefficients $e_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}$ have the natural interpretation of the components of $\boldsymbol{e}_{\boldsymbol{AA'}}$ with respect to $\boldsymbol{c}_{\boldsymbol{a}}$. However, there is an alternative interpretation: for fixed frame index ${}_{\boldsymbol{a}}$, the coefficients $e_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}$ correspond to the components of the covectors $\boldsymbol{\alpha}^{\boldsymbol{a}}$ with respect to the coframe $\boldsymbol{\omega}^{\boldsymbol{AA'}}$. That is, one has

$$\boldsymbol{\alpha}^{\boldsymbol{a}} = e_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}\boldsymbol{\omega}^{\boldsymbol{AA'}},$$

from where it follows that $e_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}\omega^{\boldsymbol{AA'}}{}_{\boldsymbol{b}} = \delta_{\boldsymbol{b}}{}^{\boldsymbol{a}}$. In view of this interpretation, it is convenient to define

$$\nabla_{\boldsymbol{CC'}}e_{\boldsymbol{BB'}}{}^{\boldsymbol{a}} \equiv e_{\boldsymbol{CC'}}{}^{\boldsymbol{b}}\boldsymbol{c}_{\boldsymbol{b}}(e_{\boldsymbol{BB'}}{}^{\boldsymbol{a}}) - \Gamma_{\boldsymbol{CC'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B}}e_{\boldsymbol{QB'}}{}^{\boldsymbol{a}} - \bar{\Gamma}_{\boldsymbol{CC'}}{}^{\boldsymbol{Q'}}{}_{\boldsymbol{B'}}e_{\boldsymbol{BQ'}}{}^{\boldsymbol{a}}, \tag{13.15}$$

so that $\nabla_{\boldsymbol{CC'}}\boldsymbol{\alpha}^{\boldsymbol{a}} = \big(\nabla_{\boldsymbol{CC'}}e_{\boldsymbol{BB'}}{}^{\boldsymbol{a}}\big)\boldsymbol{\omega}^{\boldsymbol{BB'}}$. Expression (13.15) corresponds to the formula one would use to compute the covariant derivative of $e_{\boldsymbol{BB'}}{}^{\boldsymbol{a}}$ if it were the components of a tensor – which, of course, it is not.

Intuition into this general discussion is gained by considering the particular case of a coordinate frame for which $\boldsymbol{e}_{\boldsymbol{AA'}} = e_{\boldsymbol{AA'}}{}^{\mu}\boldsymbol{\partial}_{\mu}$ so that

$$\boldsymbol{e}_{\boldsymbol{AA'}}(x^{\nu}) = e_{\boldsymbol{AA'}}{}^{\mu}\partial_{\mu}(x^{\nu}) = e_{\boldsymbol{AA'}}{}^{\mu}\delta_{\mu}{}^{\nu} = e_{\boldsymbol{AA'}}{}^{\nu}.$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

Moreover, writing $\boldsymbol{\omega}^{\boldsymbol{AA}'} = \omega^{\boldsymbol{AA}'}{}_{\mu}\mathbf{d}x^{\mu}$ one has that

$$\mathbf{d}x^{\mu} = e_{\boldsymbol{AA}'}{}^{\mu}\boldsymbol{\omega}^{\boldsymbol{AA}'}.$$

That is, for fixed coordinate index ${}^{\mu}$, the coefficients $e_{\boldsymbol{AA}'}{}^{\mu}$ are the components of the coordinate differential $\mathbf{d}x^{\mu}$ with respect to the coframe $\boldsymbol{\omega}^{\boldsymbol{AA}'}$.

Returning to the general discussion, using the identity

$$[f\boldsymbol{v}, \boldsymbol{u}] = f[\boldsymbol{v}, \boldsymbol{u}] - \boldsymbol{u}(f)\boldsymbol{v}$$

for $\boldsymbol{v}$, $\boldsymbol{u} \in T(\mathcal{M})$ and $f \in \mathcal{X}(\mathcal{M})$, together with expression (13.15) one can rewrite Equation (13.14) as

$$\Sigma_{\boldsymbol{AA}'}{}^{\boldsymbol{QQ}'}{}_{\boldsymbol{BB}'}e_{\boldsymbol{QQ}'}{}^{\boldsymbol{c}} = \nabla_{\boldsymbol{BB}'}e_{\boldsymbol{AA}'}{}^{\boldsymbol{c}} - \nabla_{\boldsymbol{AA}'}e_{\boldsymbol{BB}'}{}^{\boldsymbol{c}} - e_{\boldsymbol{AA}'}{}^{\boldsymbol{a}}e_{\boldsymbol{BB}'}{}^{\boldsymbol{b}}C_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}, \quad (13.16)$$

where $C_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ are the ***commutation coefficients*** defined by

$$[\boldsymbol{c}_{\boldsymbol{a}}, \boldsymbol{c}_{\boldsymbol{b}}] = C_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{c}_{\boldsymbol{c}}.$$

In the case of a coordinate frame one obtains the simpler expression

$$\Sigma_{\boldsymbol{AA}'}{}^{\boldsymbol{QQ}'}{}_{\boldsymbol{BB}'}e_{\boldsymbol{QQ}'}{}^{\mu} = \nabla_{\boldsymbol{BB}'}e_{\boldsymbol{AA}'}{}^{\mu} - \nabla_{\boldsymbol{AA}'}e_{\boldsymbol{BB}'}{}^{\mu},$$

as $[\boldsymbol{\partial}_{\mu}, \boldsymbol{\partial}_{\nu}] = 0$.

A final simplification is obtained by exploiting the antisymmetry of Equation (13.16). Contracting the indices ${}_{\boldsymbol{A}'}$ and ${}_{\boldsymbol{B}'}$ and symmetrising in ${}_{\boldsymbol{AB}}$ one concludes that

$$\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q}'}e_{\boldsymbol{B})\boldsymbol{Q}'}{}^{\boldsymbol{a}} + \frac{1}{2}e_{\boldsymbol{A}}{}^{\boldsymbol{Q}'\boldsymbol{b}}e_{\boldsymbol{BQ}'}{}^{\boldsymbol{c}}C_{\boldsymbol{b}}{}^{\boldsymbol{a}}{}_{\boldsymbol{c}} = \Sigma_{\boldsymbol{AB}}{}^{\boldsymbol{a}}, \quad (13.17)$$

with

$$\Sigma_{\boldsymbol{AB}}{}^{\boldsymbol{a}} \equiv \frac{1}{2}\Sigma_{\boldsymbol{A}}{}^{\boldsymbol{Q}'\boldsymbol{CC}'}{}_{\boldsymbol{BQ}'}e_{\boldsymbol{CC}'}{}^{\boldsymbol{a}}.$$

As the frame $\boldsymbol{e}_{\boldsymbol{AA}'}$ is Hermitian, that is, $\overline{\boldsymbol{e}_{\boldsymbol{AA}'}} = \boldsymbol{e}_{\boldsymbol{AA}'}$, one has that (13.17) is completely equivalent to Equation (13.16). Moreover, if $\Sigma_{\boldsymbol{AB}}{}^{\boldsymbol{a}} = 0$, then $\Sigma_{\boldsymbol{AA}'}{}^{\boldsymbol{CC}'}{}_{\boldsymbol{BB}'} = 0$ and the connection is torsion free.

The structure of Equation (13.17) is similar to that of the model Equation (13.12), suggesting that by introducing a gauge source function one will obtain a symmetric hyperbolic system for the frame coefficients $e_{\boldsymbol{AA}'}{}^{\boldsymbol{a}}$. Now, Equation (13.17) does not impose restrictions on the divergences $\nabla^{\boldsymbol{QQ}'}e_{\boldsymbol{QQ}'}{}^{\boldsymbol{a}}$ so that one can set

$$\nabla^{\boldsymbol{QQ}'}e_{\boldsymbol{QQ}'}{}^{\boldsymbol{a}} = F^{\boldsymbol{a}}(x), \quad (13.18)$$

where the ***coordinate gauge source functions*** $F^{\boldsymbol{a}}(x)$ are smooth functions of the coordinates $x = (x^{\mu})$. In the case of a coordinate frame the above expression reduces to

$$\nabla^{\boldsymbol{QQ}'}\nabla_{\boldsymbol{QQ}'}x^{\mu} = F^{\mu}(x), \quad (13.19)$$

the so-called ***generalised wave coordinates condition***.

Combining the identity

$$\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q}'} e_{\boldsymbol{B})\boldsymbol{Q}'}{}^{\boldsymbol{a}} = \nabla_{\boldsymbol{A}}{}^{\boldsymbol{Q}'} e_{\boldsymbol{B}\boldsymbol{Q}'}{}^{\boldsymbol{a}} + \frac{1}{2}\epsilon_{\boldsymbol{A}\boldsymbol{B}} \nabla^{\boldsymbol{P}\boldsymbol{P}'} e_{\boldsymbol{P}\boldsymbol{P}'}{}^{\boldsymbol{a}}$$

with Equations (13.17) and (13.18) one finally obtains, for $\Sigma_{\boldsymbol{A}\boldsymbol{B}}{}^{\boldsymbol{a}} = 0$, the equation

$$\nabla_{\boldsymbol{A}}{}^{\boldsymbol{Q}'} e_{\boldsymbol{B}\boldsymbol{Q}'}{}^{\boldsymbol{a}} + \frac{1}{2}\epsilon_{\boldsymbol{A}\boldsymbol{B}} F^{\boldsymbol{a}}(x) + \frac{1}{2} e_{\boldsymbol{A}}{}^{\boldsymbol{Q}'\boldsymbol{b}} e_{\boldsymbol{B}\boldsymbol{Q}'}{}^{\boldsymbol{c}} C_{\boldsymbol{b}}{}^{\boldsymbol{a}}{}_{\boldsymbol{c}} = 0,$$

from which a symmetric hyperbolic system for the frame components of $e_{\boldsymbol{B}\boldsymbol{Q}'}{}^{\boldsymbol{a}}$ can be deduced.

Geometric interpretation

The generalised wave coordinate condition (13.19) shows that a particular choice of coordinate gauge is, implicitly, a choice of coordinates. Equation (13.19) can always be solved locally by choosing some coordinates $x = (x^0, x^\alpha)$ on some fiduciary surface $\mathcal{S}_\star$. If this surface is described by the condition $x^0 = 0$, then it is also natural to require that

$$\frac{\partial x^\alpha}{\partial x^0} = 0, \qquad \text{on } \mathcal{S}_\star.$$

Moreover, one needs the coordinate differentials $\mathbf{d}x^\mu$ to be linearly independent on $\mathcal{S}_\star$. These conditions ensure the existence of a solution to Equation (13.19) close to $\mathcal{S}_\star$.

Conversely, given a particular coordinate choice on a spacetime $(\mathcal{M}, \boldsymbol{g})$, one can use Equation (13.19) to compute the coordinate gauge source function $F^\mu(x)$ associated with the coordinates. Thus, local coordinates and coordinate gauge source functions are in a one-to-one correspondence.

Construction of coordinates in perturbations of spacetimes

The discussion of the previous subsection can be applied to the construction of coordinates in spacetimes $(\mathcal{M}, \boldsymbol{g})$ which are *perturbations* of a certain *exact background spacetime* $(\mathring{\mathcal{M}}, \mathring{\boldsymbol{g}})$. In this situation, one would expect the spacetime manifolds $\mathcal{M}$ and $\mathring{\mathcal{M}}$ to be diffeomorphic to each other so that coordinates in the background spacetime could be used as coordinates in the perturbed spacetime. This does not mean that the spacetimes $(\mathcal{M}, \boldsymbol{g})$ and $(\mathring{\mathcal{M}}, \mathring{\boldsymbol{g}})$ are isometric! The intuition expressed in this paragraph will now be formalised.

In what follows, assume that one has two spacetimes $(\mathcal{M}, \boldsymbol{g})$ and $(\mathring{\mathcal{M}}, \mathring{\boldsymbol{g}})$ such that the manifolds $\mathcal{M}$ and $\mathring{\mathcal{M}}$ are diffeomorphic. Let $\varphi : \mathcal{M} \to \mathring{\mathcal{M}}$ denote a diffeomorphism between them. This choice is clearly not unique. The subsequent discussion will single out a particular type of diffeomorphism between $\mathcal{M}$ and $\mathring{\mathcal{M}}$.

Let $x = (x^\mu)$ and $\mathring{x} = (\mathring{x}^\mu)$ denote, respectively, local coordinates on $\mathcal{M}$ and $\mathring{\mathcal{M}}$. In terms of these local coordinates the diffeomorphism φ is given by $\mathring{x}^\mu = \mathring{x}^\mu(x)$ and its inverse by $x^\mu = x^\mu(\mathring{x})$. On $\mathring{\mathcal{M}}$ consider a frame $\{\mathring{\boldsymbol{c}}_{\boldsymbol{a}}\}$ and its

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

dual coframe $\{\mathring{\boldsymbol{\alpha}}^{\boldsymbol{a}}\}$. The frame is not necessarily assumed to be $\mathring{\boldsymbol{g}}$-orthonormal. From this frame and coframe one can introduce a frame $\{\boldsymbol{c}_a\}$ and a coframe $\{\boldsymbol{\alpha}^{\boldsymbol{a}}\}$ on $\mathcal{M}$ using, respectively, the push-forward and the pull-back implied by $\varphi : \mathcal{M} \to \mathring{\mathcal{M}}$. More precisely,

$$\mathring{\boldsymbol{c}}_{\boldsymbol{a}} = (\varphi)_* \boldsymbol{c}_{\boldsymbol{a}}, \qquad \mathring{\boldsymbol{\alpha}}^{\boldsymbol{a}} = (\varphi^{-1})^* \boldsymbol{\alpha}^{\boldsymbol{a}}.$$

Thus, writing

$$\boldsymbol{\alpha}^{\boldsymbol{a}} = \alpha^{\boldsymbol{a}}{}_\mu \mathbf{d}x^\mu, \qquad \mathring{\boldsymbol{\alpha}}^{\boldsymbol{a}} = \mathring{\alpha}^{\boldsymbol{a}}{}_\mu \mathbf{d}\mathring{x}^\mu,$$

one concludes that

$$\alpha^{\boldsymbol{a}}{}_\mu = \mathring{\alpha}^{\boldsymbol{a}}{}_\nu \frac{\partial \mathring{x}^\nu}{\partial x^\mu}.$$

Now, observing that $\langle \boldsymbol{\alpha}^{\boldsymbol{a}}, \boldsymbol{e}_{\boldsymbol{b}} \rangle = \delta_{\boldsymbol{b}}{}^{\boldsymbol{a}}$, it follows that $\nabla_{\boldsymbol{c}} \boldsymbol{\alpha}^{\boldsymbol{a}} = (\nabla_{\boldsymbol{c}} e_{\boldsymbol{b}}{}^{\boldsymbol{a}}) \boldsymbol{\alpha}^{\boldsymbol{b}}$ and, consequently,

$$\nabla^{\boldsymbol{b}} e_{\boldsymbol{b}}{}^{\boldsymbol{a}} = \eta^{\boldsymbol{cd}} \langle \nabla_{\boldsymbol{c}} \boldsymbol{\alpha}^{\boldsymbol{a}}, \boldsymbol{e}_{\boldsymbol{d}} \rangle = e_{\boldsymbol{b}}{}^\mu \nabla^{\boldsymbol{b}} \alpha^{\boldsymbol{a}}{}_\mu = \nabla^\mu \alpha^{\boldsymbol{a}}{}_\mu.$$

The above expression can be used to write the divergence $\nabla^{\boldsymbol{QQ}'} e_{\boldsymbol{QQ}'}{}^{\boldsymbol{a}}$ appearing in Equation (13.18) in terms of quantities associated to the diffeomorphism $\varphi : \mathcal{M} \to \mathring{\mathcal{M}}$.

Treating the coordinates $\mathring{x} = (\mathring{x}^\mu)$ as scalars and recalling that $\mathring{\alpha}^{\boldsymbol{a}}{}_\nu = \langle \mathring{\boldsymbol{\alpha}}^{\boldsymbol{a}}, \boldsymbol{\partial}/\boldsymbol{\partial}\mathring{x}^\nu \rangle$ so that the coefficients $\mathring{\alpha}^{\boldsymbol{a}}{}_\nu$ are also scalars, one finds that

$$\begin{aligned}
\nabla_\nu \alpha^{\boldsymbol{a}}{}_\mu &= \mathring{\alpha}^{\boldsymbol{a}}{}_\lambda \nabla_\nu \left(\frac{\partial \mathring{x}^\lambda}{\partial x^\mu} \right) + \frac{\partial \mathring{\alpha}^{\boldsymbol{a}}{}_\lambda}{\partial x^\nu} \frac{\partial \mathring{x}^\lambda}{\partial x^\mu} \\
&= \mathring{\alpha}^{\boldsymbol{a}}{}_\lambda \nabla_\nu \nabla_\mu \mathring{x}^\lambda + \mathring{\nabla}_\rho \mathring{\alpha}^{\boldsymbol{a}}{}_\lambda \frac{\partial \mathring{x}^\rho}{\partial x^\nu} \frac{\partial \mathring{x}^\lambda}{\partial x^\mu},
\end{aligned}$$

where in the last equality the chain rule has been used. Consequently, one has

$$\nabla^\mu \alpha^{\boldsymbol{a}}{}_\mu = \mathring{\alpha}^{\boldsymbol{a}}{}_\lambda \nabla^\mu \nabla_\mu \mathring{x}^\lambda + g^{\mu\nu} \mathring{\nabla}_\rho \mathring{\alpha}^{\boldsymbol{a}}{}_\lambda \frac{\partial \mathring{x}^\rho}{\partial x^\nu} \frac{\partial \mathring{x}^\lambda}{\partial x^\mu} = F^{\boldsymbol{a}}(x),$$

or, more suggestively,

$$\nabla^\mu \nabla_\mu \mathring{x}^\sigma + \mathring{c}_{\boldsymbol{a}}{}^\sigma \left(g^{\mu\nu} \mathring{\nabla}_\rho \mathring{\alpha}^{\boldsymbol{a}}{}_\lambda \frac{\partial \mathring{x}^\rho}{\partial x^\nu} \frac{\partial \mathring{x}^\lambda}{\partial x^\mu} - F^{\boldsymbol{a}}(x) \right) = 0.$$

So far, the diffeomorphism $\varphi : \mathcal{M} \to \mathring{\mathcal{M}}$ has been kept completely general. However, if one sets

$$g^{\mu\nu} \mathring{\nabla}_\rho \mathring{\alpha}^{\boldsymbol{a}}{}_\lambda \frac{\partial \mathring{x}^\rho}{\partial x^\nu} \frac{\partial \mathring{x}^\lambda}{\partial x^\mu} = F^{\boldsymbol{a}}(x), \tag{13.20}$$

one finds that

$$\nabla^\mu \nabla_\mu \mathring{x}^\sigma = 0.$$

That is, under condition (13.20), the diffeomorphism $\varphi : \mathcal{M} \to \mathring{\mathcal{M}}$ given by $\mathring{x}^\mu = \mathring{x}^\mu(x)$ is a ***wave map***. Wave maps can be regarded as a generalisation of the geodesic equation. Further discussion on this notion, which plays an important role in modern research in PDE theory and geometric analysis, can be found in the review by Tataru (2004).

Now, it is convenient to regard the manifolds $\mathcal{M}$ and $\mathring{\mathcal{M}}$ as being the same and let $\mathring{x}^\mu = \mathring{x}^\mu(x)$ be the ***identity map*** so that $\partial\mathring{x}^\rho/\partial x^\nu = \delta_\nu{}^\rho$. *This amounts to saying that the coordinates* $\mathring{x} = (\mathring{x}^\mu)$ *are used as coordinates of the perturbed spacetime* $(\mathcal{M}, \boldsymbol{g})$. In this case condition (13.20) reduces to

$$\mathring{\nabla}^{\boldsymbol{b}}\mathring{\alpha}^{\boldsymbol{a}}{}_{\boldsymbol{b}} = F^{\boldsymbol{a}}(x).$$

If in the reference spacetime one has $\mathring{\boldsymbol{\omega}}^{\boldsymbol{a}} = \mathring{\boldsymbol{\alpha}}^{\boldsymbol{a}}$ so that $\mathring{\alpha}^{\boldsymbol{a}}{}_{\boldsymbol{b}} \equiv \langle \mathring{\boldsymbol{\alpha}}^{\boldsymbol{a}}, \mathring{\boldsymbol{c}}_{\boldsymbol{b}} \rangle = \delta_{\boldsymbol{b}}{}^{\boldsymbol{a}}$, then

$$\mathring{\nabla}^{\boldsymbol{b}}\mathring{\alpha}^{\boldsymbol{a}}{}_{\boldsymbol{b}} = -\eta^{\boldsymbol{bc}}\mathring{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{a}}{}_{\boldsymbol{c}}.$$

Accordingly, the coordinate gauge source function $F^{\boldsymbol{a}}(x)$ can be expressed in terms of the connection of the background spacetime via

$$F^{\boldsymbol{a}}(x) = -\eta^{\boldsymbol{bc}}\mathring{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{a}}{}_{\boldsymbol{c}},$$

or, in spinorial terms

$$F^{\boldsymbol{a}}(x) = -\epsilon^{\boldsymbol{AB}}\epsilon^{\boldsymbol{A'B'}}e_{\boldsymbol{AA'}}{}^{\boldsymbol{b}}e_{\boldsymbol{BB'}}{}^{\boldsymbol{c}}\mathring{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{a}}{}_{\boldsymbol{c}}.$$

Space spinor decomposition of the equation for the frame coefficients

The space spinor decomposition of Equation (13.17) provides a systematic approach to the extraction of the required symmetric hyperbolic system. Accordingly, one considers the space spinor split of the frame fields given by

$$e_{\boldsymbol{AA'}}{}^{\boldsymbol{a}} = \frac{1}{2}\tau_{\boldsymbol{AA'}}e^{\boldsymbol{a}} - \tau^{\boldsymbol{Q}}{}_{\boldsymbol{A'}}e_{\boldsymbol{AQ}}{}^{\boldsymbol{a}}$$

with

$$e^{\boldsymbol{a}} \equiv \tau^{\boldsymbol{AA'}}e_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}, \qquad e_{\boldsymbol{AB}}{}^{\boldsymbol{a}} \equiv \tau_{(\boldsymbol{A}}{}^{\boldsymbol{A'}}e_{\boldsymbol{B})\boldsymbol{A'}}{}^{\boldsymbol{a}}.$$

Alternatively, one can write

$$\tau_{\boldsymbol{B}}{}^{\boldsymbol{Q'}}e_{\boldsymbol{AQ'}}{}^{\boldsymbol{a}} = \frac{1}{2}\epsilon_{\boldsymbol{AB}}e^{\boldsymbol{a}} + e_{\boldsymbol{AB}}{}^{\boldsymbol{a}}.$$

Using

$$\nabla_{\boldsymbol{AB}}\tau_{\boldsymbol{CD'}} = -\sqrt{2}\tau^{\boldsymbol{D}}{}_{\boldsymbol{D'}}\chi_{\boldsymbol{ABCD}},$$

– compare Equation (4.17) – together with the decomposition of $\nabla_{\boldsymbol{AB}}$ given in Equation (13.5), it follows from Equation (13.18) that

$$\mathcal{P}e^{\boldsymbol{a}} + 2\mathcal{D}^{\boldsymbol{PQ}}e_{\boldsymbol{PQ}}{}^{\boldsymbol{a}} + \sqrt{2}e^{\boldsymbol{a}}\chi_{\boldsymbol{PQ}}{}^{\boldsymbol{PQ}} + 2\sqrt{2}e_{\boldsymbol{PQ}}{}^{\boldsymbol{a}}\chi^{\boldsymbol{P}}{}_{\boldsymbol{C}}{}^{\boldsymbol{CQ}} - 2F^{\boldsymbol{a}}(x) = 0. \tag{13.21}$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

A similar computation for Equation (13.17) yields

$$\begin{aligned}\Sigma_{\boldsymbol{AB}}{}^{\boldsymbol{a}} &= \frac{1}{2}\mathcal{P}e_{\boldsymbol{AB}}{}^{\boldsymbol{a}} - \frac{1}{2}\mathcal{D}_{\boldsymbol{AB}}e^{\boldsymbol{a}} + \mathcal{D}_{(\boldsymbol{A}}{}^{\boldsymbol{Q}}e_{\boldsymbol{B})\boldsymbol{Q}}{}^{\boldsymbol{a}} - \frac{1}{\sqrt{2}}e^{\boldsymbol{a}}\chi_{(\boldsymbol{A}|\boldsymbol{Q}|}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B})} \\ &\quad + \sqrt{2}e_{\boldsymbol{P}(\boldsymbol{A}}{}^{\boldsymbol{a}}\chi_{\boldsymbol{B})\boldsymbol{Q}}{}^{\boldsymbol{QP}} - \frac{1}{2}(e^{\boldsymbol{b}}e_{\boldsymbol{AB}}{}^{\boldsymbol{c}} + e_{\boldsymbol{AQ}}{}^{\boldsymbol{b}}e_{\boldsymbol{B}}{}^{\boldsymbol{Qc}})C_{\boldsymbol{b}}{}^{\boldsymbol{a}}{}_{\boldsymbol{c}}.\end{aligned}$$

A further independent equation can be obtained from the Hermitian conjugate

$$\Sigma^{+}_{\boldsymbol{AB}}{}^{\boldsymbol{a}} \equiv \tau_{\boldsymbol{A}}{}^{\boldsymbol{A}'}\tau_{\boldsymbol{B}}{}^{\boldsymbol{B}'}\bar{\Sigma}_{\boldsymbol{A}'\boldsymbol{B}'}{}^{\boldsymbol{a}}.$$

Exploiting the identity

$$\tau_{\boldsymbol{A}}{}^{\boldsymbol{A}'}\tau_{\boldsymbol{B}}{}^{\boldsymbol{B}'}\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}e_{\boldsymbol{QB}'}{}^{\boldsymbol{a}} = \nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A}}\big(\tau_{\boldsymbol{B}}{}^{\boldsymbol{B}'}e_{\boldsymbol{QB}'}{}^{\boldsymbol{a}}\big) - e_{\boldsymbol{QB}'}{}^{\boldsymbol{a}}\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A}}\tau_{\boldsymbol{B}}{}^{\boldsymbol{B}'},$$

one arrives at

$$\begin{aligned}\Sigma^{+}_{\boldsymbol{AB}}{}^{\boldsymbol{a}} &= -\frac{1}{2}\mathcal{P}e_{\boldsymbol{AB}}{}^{\boldsymbol{a}} + \frac{1}{2}\mathcal{D}_{\boldsymbol{AB}}e^{\boldsymbol{a}} + \mathcal{D}_{(\boldsymbol{A}}{}^{\boldsymbol{Q}}e_{\boldsymbol{B})\boldsymbol{Q}}{}^{\boldsymbol{a}} + \frac{1}{\sqrt{2}}e^{\boldsymbol{a}}\chi^{\boldsymbol{Q}}{}_{(\boldsymbol{AB})\boldsymbol{Q}} \\ &\quad - \sqrt{2}e_{\boldsymbol{PQ}}{}^{\boldsymbol{a}}\chi^{\boldsymbol{P}}{}_{(\boldsymbol{AB})}{}^{\boldsymbol{Q}} - \frac{1}{2}(e^{\boldsymbol{b}}e_{\boldsymbol{AB}}{}^{\boldsymbol{c}} + e_{\boldsymbol{AQ}}{}^{\boldsymbol{b}}e_{\boldsymbol{B}}{}^{\boldsymbol{Qc}})C_{\boldsymbol{b}}{}^{\boldsymbol{a}}{}_{\boldsymbol{c}}.\end{aligned}$$

The required evolution equation complementing (13.21) is then obtained from

$$\Sigma_{\boldsymbol{AB}}{}^{\boldsymbol{a}} - \Sigma^{+}_{\boldsymbol{AB}}{}^{\boldsymbol{a}} = 0,$$

where

$$\begin{aligned}\Sigma_{\boldsymbol{AB}}{}^{\boldsymbol{a}} - \Sigma^{+}_{\boldsymbol{AB}}{}^{\boldsymbol{a}} &= \mathcal{P}e_{\boldsymbol{AB}}{}^{\boldsymbol{a}} - \mathcal{D}_{\boldsymbol{AB}}e^{\boldsymbol{a}} - \frac{1}{\sqrt{2}}e^{\boldsymbol{a}}\big(\chi_{(\boldsymbol{A}|\boldsymbol{Q}|}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B})} + \chi_{\boldsymbol{Q}(\boldsymbol{AB})}{}^{\boldsymbol{Q}}\big) \\ &\quad + \sqrt{2}e_{\boldsymbol{P}(\boldsymbol{A}}{}^{\boldsymbol{a}}\chi_{\boldsymbol{B})\boldsymbol{Q}}{}^{\boldsymbol{QP}} + \sqrt{2}e_{\boldsymbol{PQ}}{}^{\boldsymbol{a}}\chi^{\boldsymbol{P}}{}_{(\boldsymbol{AB})}{}^{\boldsymbol{Q}} \\ &\quad - e^{\boldsymbol{c}}e_{\boldsymbol{AB}}{}^{\boldsymbol{b}}C_{\boldsymbol{b}}{}^{\boldsymbol{a}}{}_{\boldsymbol{c}}. \end{aligned} \tag{13.22}$$

A direct inspection shows that Equations (13.21) and (13.22) imply, for fixed frame index ${}^{\boldsymbol{a}}$, a symmetric hyperbolic system of four equations for $e^{\boldsymbol{a}}$ and the independent components of $e_{\boldsymbol{AB}}{}^{\boldsymbol{a}}$. A further computation shows that the characteristic polynomial of the system is given by

$$-4(\tau^{\mu}\xi_{\mu})^2(g^{\lambda\rho}\xi_{\lambda}\xi_{\rho}).$$

As a by-product of the analysis one obtains the constraint equations implied by (13.17) from

$$\Sigma_{\boldsymbol{AB}}{}^{\boldsymbol{a}} + \Sigma^{+}_{\boldsymbol{AB}}{}^{\boldsymbol{a}} = 0,$$

where

$$\begin{aligned}\Sigma_{\boldsymbol{AB}}{}^{\boldsymbol{a}} + \Sigma^{+}_{\boldsymbol{AB}}{}^{\boldsymbol{a}} &= 2\mathcal{D}^{\boldsymbol{Q}}{}_{(\boldsymbol{A}}e_{\boldsymbol{B})\boldsymbol{Q}}{}^{\boldsymbol{a}} + \frac{1}{\sqrt{2}}e^{\boldsymbol{a}}\big(\chi_{(\boldsymbol{A}|\boldsymbol{Q}|}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B})} + \chi^{\boldsymbol{Q}}{}_{(\boldsymbol{AB})\boldsymbol{Q}}\big) \\ &\quad + \sqrt{2}e_{\boldsymbol{P}(\boldsymbol{A}}{}^{\boldsymbol{a}}\chi_{\boldsymbol{B})\boldsymbol{Q}}{}^{\boldsymbol{QP}} - \sqrt{2}e_{\boldsymbol{PQ}}{}^{\boldsymbol{a}}\chi^{\boldsymbol{P}}{}_{(\boldsymbol{AB})}{}^{\boldsymbol{Q}} \\ &\quad - \big(e^{\boldsymbol{b}}e_{\boldsymbol{AB}}{}^{\boldsymbol{c}} + e_{\boldsymbol{AQ}}{}^{\boldsymbol{b}}e_{\boldsymbol{B}}{}^{\boldsymbol{Qc}}\big)C_{\boldsymbol{b}}{}^{\boldsymbol{a}}{}_{\boldsymbol{c}}.\end{aligned}$$

Expanding the principal part of this constraint equation, one finds it contains derivatives in the time direction.

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

13.2.2 Frame gauge source functions

After having analysed the gauge source conditions arising from the no-torsion condition, one can now consider the gauge source functions associated to the Ricci identity – that is, the condition requiring that the geometric and the algebraic curvatures coincide. As with the no-torsion condition, the equality between the two expressions for the curvature is part of the frame and spinorial formulations of the conformal field equations; compare Equations (8.31b), (8.35b), (8.44b) and (8.53b).

Rather than working with the full expressions for the curvature spinors, in the subsequent discussion it will be convenient to make use of the reduced spinorial counterpart of the Riemann tensor in terms of the reduced connection coefficients:

$$\begin{aligned} R_{\boldsymbol{ABCC'DD'}} + \Sigma_{\boldsymbol{CC'}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{DD'}}\Gamma_{\boldsymbol{QQ'AB}} &= \nabla_{\boldsymbol{DD'}}\Gamma_{\boldsymbol{CC'AB}} - \nabla_{\boldsymbol{CC'}}\Gamma_{\boldsymbol{DD'AB}} \\ &\quad - \Gamma_{\boldsymbol{DD'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{A}}\Gamma_{\boldsymbol{CC'QB}} - \Gamma_{\boldsymbol{CC'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{A}}\Gamma_{\boldsymbol{DD'QB}}, \end{aligned} \tag{13.23}$$

where the definition

$$\begin{aligned} \nabla_{\boldsymbol{DD'}}(\Gamma_{\boldsymbol{CC'AB}}) \equiv \boldsymbol{e}_{\boldsymbol{DD'}}(\Gamma_{\boldsymbol{CC'AB}}) &- \Gamma_{\boldsymbol{DD'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{C}}\Gamma_{\boldsymbol{QC'AB}} \\ &- \bar{\Gamma}_{\boldsymbol{DD'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{C}}\Gamma_{\boldsymbol{QC'AB}} - \Gamma_{\boldsymbol{DD'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B}}\Gamma_{\boldsymbol{CC'AQ}} \end{aligned}$$

has been used in order to obtain a more concise expression; see Section 8.3.2 for further details. This last expression is formally the same as the one for the covariant derivative of a spinor field with the same index structure as $\Gamma_{\boldsymbol{CC'AB}}$. Equation (13.23) is encoded in the zero quantity

$$\Xi_{\boldsymbol{ABCC'DD'}} \equiv R_{\boldsymbol{ABCC'DD'}} - \rho_{\boldsymbol{ABCC'DD'}},$$

where $R_{\boldsymbol{ABCC'DD'}}$ and $\rho_{\boldsymbol{ABCC'DD'}}$ denote, respectively, the geometric and algebraic curvatures. One has the symmetries

$$\Xi_{\boldsymbol{ABCC'DD'}} = \Xi_{(\boldsymbol{AB})\boldsymbol{CC'DD'}} = -\Xi_{\boldsymbol{ABDD'CC'}}.$$

Exploiting the antisymmetry of Equation (13.23) one obtains the pair of equations

$$\nabla_{(\boldsymbol{C}}{}^{\boldsymbol{Q'}}\Gamma_{\boldsymbol{D})\boldsymbol{Q'AB}} + \Gamma_{(\boldsymbol{C}}{}^{\boldsymbol{Q'Q}}{}_{|\boldsymbol{A}|}\Gamma_{\boldsymbol{D})\boldsymbol{Q'QB}} = R_{\boldsymbol{ABCD}} + \Sigma_{\boldsymbol{C}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{D}}\Gamma_{\boldsymbol{QQ'AB}}, \tag{13.24a}$$

$$\nabla^{\boldsymbol{P}}{}_{(\boldsymbol{C'}}\Gamma_{|\boldsymbol{P}|\boldsymbol{D'})\boldsymbol{AB}} + \Gamma^{\boldsymbol{P}}{}_{(\boldsymbol{C'}}{}^{\boldsymbol{Q}}{}_{|\boldsymbol{A}}\Gamma_{\boldsymbol{P}|\boldsymbol{D'})\boldsymbol{QB}} = R_{\boldsymbol{ABC'D'}} + \Sigma_{\boldsymbol{C'}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{D'}}\Gamma_{\boldsymbol{QQ'AB}}, \tag{13.24b}$$

where

$$R_{\boldsymbol{ABCD}} \equiv \frac{1}{2}R_{\boldsymbol{ABCQ'D}}{}^{\boldsymbol{Q'}}, \qquad R_{\boldsymbol{ABC'D'}} \equiv \frac{1}{2}R_{\boldsymbol{ABQC'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{D'}},$$
$$\Sigma_{\boldsymbol{C}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{D}} \equiv \frac{1}{2}\Sigma_{\boldsymbol{CP'}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{D}}{}^{\boldsymbol{P'}}, \qquad \Sigma_{\boldsymbol{C'}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{D'}} \equiv \frac{1}{2}\Sigma_{\boldsymbol{PC'}}{}^{\boldsymbol{QQ'P}}{}_{\boldsymbol{D'}}.$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

As the field $\Gamma_{\boldsymbol{AA'BC}}$ is not Hermitian, two reduced equations are necessary to encode the content of (13.23) – contrast this with the analysis of the no-torsion Equation (13.14).

From the structure of Equations (13.24a) and (13.24b) one concludes that the derivative $\nabla^{\boldsymbol{QQ'}}\Gamma_{\boldsymbol{QQ'AB}}$ is not determined by the equations. Accordingly, one can set

$$\nabla^{\boldsymbol{QQ'}}\Gamma_{\boldsymbol{QQ'AB}} = F_{\boldsymbol{AB}}(x), \tag{13.25}$$

where $F_{\boldsymbol{AB}} = F_{(\boldsymbol{AB})}$ are smooth arbitrary functions of the coordinates – the ***frame gauge source functions***.

Geometric interpretation

To gain intuition on the role played by the frame gauge source functions recall that $\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} = \epsilon^{\boldsymbol{B}}{}_B\nabla_{\boldsymbol{AA'}}\epsilon_{\boldsymbol{C}}{}^B$; see Equation (3.32). Equation (13.25) can be rewritten as

$$\epsilon^{\boldsymbol{A}}{}_B\nabla^{\boldsymbol{PP'}}\nabla_{\boldsymbol{PP'}}\epsilon_{\boldsymbol{B}}{}^B + \nabla^{\boldsymbol{PP'}}\epsilon^{\boldsymbol{A}}{}_B\nabla_{\boldsymbol{PP'}}\epsilon_{\boldsymbol{B}}{}^B = F^{\boldsymbol{A}}{}_{\boldsymbol{B}}(x). \tag{13.26}$$

This is to be read as a quasilinear wave equation for the spin frame $\{\epsilon_{\boldsymbol{B}}{}^B\}$. Using the symmetry of $F_{\boldsymbol{AB}}$ and the wave Equation (13.26) one obtains

$$\nabla^{\boldsymbol{PP'}}\nabla_{\boldsymbol{PP'}}\big(\epsilon_{\boldsymbol{B}}{}^B\epsilon^{\boldsymbol{A}}{}_B\big) = 0,$$

so that by choosing

$$\epsilon_{\boldsymbol{B}}{}^B\epsilon^{\boldsymbol{A}}{}_B = \delta_{\boldsymbol{B}}{}^{\boldsymbol{A}}, \qquad \nabla_{\boldsymbol{PP'}}\big(\epsilon_{\boldsymbol{B}}{}^B\epsilon^{\boldsymbol{A}}{}_B\big) = 0,$$

on some fiduciary hypersurface $\mathcal{S}_\star$ one obtains a spin frame which is normalised at later times.

Space spinor decomposition of the equation for the spin connection coefficients

To obtain a suitable space spinor decomposition of Equations (13.24a), (13.24b) and (13.25), one defines

$$\Gamma_{\boldsymbol{ABCD}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\Gamma_{\boldsymbol{AA'CD}}$$

and considers the split

$$\Gamma_{\boldsymbol{ABCD}} = \frac{1}{2}\epsilon_{\boldsymbol{AB}}\Gamma_{\boldsymbol{CD}} + \Gamma_{(\boldsymbol{AB})\boldsymbol{CD}}, \qquad \Gamma_{\boldsymbol{CD}} \equiv \Gamma_{\boldsymbol{Q}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{CD}}.$$

Now, from

$$\begin{aligned}\nabla^{\boldsymbol{QQ'}}\Gamma_{\boldsymbol{QQ'AB}} &= -\nabla^{\boldsymbol{QQ'}}\big(\tau^{\boldsymbol{P}}{}_{\boldsymbol{Q'}}\Gamma_{\boldsymbol{QPAB}}\big)\\ &= \tau^{\boldsymbol{SQ'}}\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{Q'}}\Gamma_{\boldsymbol{QSAB}} - (\nabla^{\boldsymbol{QQ'}}\tau^{\boldsymbol{S}}{}_{\boldsymbol{Q'}})\Gamma_{\boldsymbol{QSAB}}\\ &= \nabla^{\boldsymbol{PQ}}\Gamma_{\boldsymbol{PQAB}} + \sqrt{2}\chi^{\boldsymbol{P}}{}_{\boldsymbol{R}}{}^{\boldsymbol{QR}}\Gamma_{\boldsymbol{PQAB}},\end{aligned}$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

it follows, using the split of $\nabla_{\boldsymbol{AB}}$, that

$$\mathcal{P}\Gamma_{\boldsymbol{AB}} + 2\mathcal{D}^{\boldsymbol{PQ}}\Gamma_{(\boldsymbol{PQ})\boldsymbol{AB}} + 2\sqrt{2}\chi^{\boldsymbol{P}}{}_{\boldsymbol{R}}{}^{\boldsymbol{QR}}\Gamma_{\boldsymbol{PQAB}} = 2F_{\boldsymbol{AB}}(x). \tag{13.27}$$

In view of its symmetries, the zero quantity $\Xi_{\boldsymbol{ABCC'EE'}}$ is decomposed as

$$\Xi_{\boldsymbol{ABCC'EE'}} = \Xi_{\boldsymbol{ABCE}}\epsilon_{\boldsymbol{C'E'}} + \Xi_{\boldsymbol{ABC'E'}}\epsilon_{\boldsymbol{CE}},$$

with

$$\Xi_{\boldsymbol{ABCE}} \equiv \frac{1}{2}\Xi_{\boldsymbol{ABCQ'E}}{}^{\boldsymbol{Q'}}, \qquad \Xi_{\boldsymbol{ABC'E'}} \equiv \frac{1}{2}\Xi_{\boldsymbol{ABQC'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{E'}}.$$

In terms of space spinors the latter decomposition can be rewritten as

$$\Xi_{\boldsymbol{ABCDEF}} = \Xi_{\boldsymbol{ABCE}}\epsilon_{\boldsymbol{DF}} + \Xi^{*}_{\boldsymbol{ABDF}}\epsilon_{\boldsymbol{CE}},$$

where

$$\Xi_{\boldsymbol{ABCDEF}} \equiv \tau_{\boldsymbol{D}}{}^{\boldsymbol{C'}}\tau_{\boldsymbol{F}}{}^{\boldsymbol{E'}}\Xi_{\boldsymbol{ABCC'EE'}},$$
$$\Xi_{\boldsymbol{ABDF}} \equiv \tau_{\boldsymbol{D}}{}^{\boldsymbol{C'}}\tau_{\boldsymbol{F}}{}^{\boldsymbol{E'}}\Xi_{\boldsymbol{ABC'E'}}, \qquad \Xi^{*}_{\boldsymbol{ABDF}} \equiv \tau_{\boldsymbol{D}}{}^{\boldsymbol{C'}}\tau_{\boldsymbol{F}}{}^{\boldsymbol{E'}}\Xi_{\boldsymbol{ABC'E'}}.$$

To expand $\Xi_{\boldsymbol{ABCE}}$ and $\Xi^{*}_{\boldsymbol{ABDF}}$ it is observed that

$$\begin{aligned}
\nabla_{(\boldsymbol{C}}{}^{\boldsymbol{Q'}}\Gamma_{\boldsymbol{D})\boldsymbol{Q'AB}} &= -\nabla_{(\boldsymbol{C}}{}^{\boldsymbol{Q'}}\left(\Gamma_{\boldsymbol{D})\boldsymbol{SAB}}\tau^{\boldsymbol{S}}{}_{\boldsymbol{Q'}}\right) \\
&= -\tau^{\boldsymbol{S}}{}_{\boldsymbol{Q'}}\nabla_{(\boldsymbol{C}}{}^{\boldsymbol{Q'}}\Gamma_{\boldsymbol{D})\boldsymbol{SAB}} - \nabla_{(\boldsymbol{C}}{}^{\boldsymbol{Q'}}\tau^{\boldsymbol{S}}{}_{|\boldsymbol{Q'}|}\Gamma_{\boldsymbol{D})\boldsymbol{SAB}} \\
&= \nabla_{(\boldsymbol{C}}{}^{\boldsymbol{S}}\Gamma_{\boldsymbol{D})\boldsymbol{SAB}} + \sqrt{2}\chi_{(\boldsymbol{C}|\boldsymbol{Q}|}{}^{\boldsymbol{SQ}}\Gamma_{\boldsymbol{D})\boldsymbol{SAB}} \\
&= \frac{1}{2}\mathcal{P}\Gamma_{(\boldsymbol{CD})\boldsymbol{AB}} + \mathcal{D}_{(\boldsymbol{C}}{}^{\boldsymbol{S}}\Gamma_{\boldsymbol{D})\boldsymbol{SAB}} + \sqrt{2}\chi_{(\boldsymbol{C}|\boldsymbol{Q}|}{}^{\boldsymbol{SQ}}\Gamma_{\boldsymbol{D})\boldsymbol{SAB}}
\end{aligned}$$

and that

$$\begin{aligned}
\tau_{\boldsymbol{C}}{}^{\boldsymbol{C'}}\tau_{\boldsymbol{D}}{}^{\boldsymbol{D'}}\nabla^{\boldsymbol{P}}{}_{(\boldsymbol{C'}}\Gamma_{|\boldsymbol{P}|\boldsymbol{D'})\boldsymbol{AB}} &= \nabla^{\boldsymbol{P}}{}_{(\boldsymbol{C}}\Gamma_{|\boldsymbol{P}|\boldsymbol{D})\boldsymbol{AB}} \\
&= -\frac{1}{2}\mathcal{P}\Gamma_{(\boldsymbol{CD})\boldsymbol{AB}} + \mathcal{D}^{\boldsymbol{P}}{}_{(\boldsymbol{C}}\Gamma_{|\boldsymbol{P}|\boldsymbol{D})\boldsymbol{AB}}.
\end{aligned}$$

From the above expressions it follows that

$$\begin{aligned}
\Xi_{\boldsymbol{ABCD}} = {}& \frac{1}{2}\mathcal{P}\Gamma_{(\boldsymbol{CD})\boldsymbol{AB}} - \frac{1}{2}\mathcal{D}_{\boldsymbol{CD}}\Gamma_{\boldsymbol{AB}} + \frac{1}{2}\left(\mathcal{D}_{\boldsymbol{C}}{}^{\boldsymbol{S}}\Gamma_{(\boldsymbol{DS})\boldsymbol{AB}} + \mathcal{D}_{\boldsymbol{D}}{}^{\boldsymbol{S}}\Gamma_{(\boldsymbol{CS})\boldsymbol{AB}}\right) \\
& + \Gamma_{(\boldsymbol{C}}{}^{\boldsymbol{PQ}}{}_{|\boldsymbol{A}|}\Gamma_{\boldsymbol{D})\boldsymbol{PQB}} - \Sigma_{\boldsymbol{C}}{}^{\boldsymbol{PQ}}{}_{\boldsymbol{D}}\Gamma_{\boldsymbol{PQAB}} - R_{\boldsymbol{ABCD}}, \\
\Xi^{*}_{\boldsymbol{ABCD}} = {}& -\frac{1}{2}\mathcal{P}\Gamma_{(\boldsymbol{CD})\boldsymbol{AB}} + \frac{1}{2}\mathcal{D}_{\boldsymbol{CD}}\Gamma_{\boldsymbol{AB}} + \frac{1}{2}\left(\mathcal{D}^{\boldsymbol{P}}{}_{\boldsymbol{C}}\Gamma_{(\boldsymbol{PD})\boldsymbol{AB}} + \mathcal{D}^{\boldsymbol{P}}{}_{\boldsymbol{D}}\Gamma_{(\boldsymbol{PC})\boldsymbol{AB}}\right) \\
& + \Gamma^{\boldsymbol{P}}{}_{(\boldsymbol{C}}{}^{\boldsymbol{Q}}{}_{|\boldsymbol{A}}\Gamma_{\boldsymbol{P}|\boldsymbol{D})\boldsymbol{QB}} + \Sigma^{+}_{\boldsymbol{C}}{}^{\boldsymbol{PQ}}{}_{\boldsymbol{D}}\Gamma_{\boldsymbol{PQAB}} - R^{*}_{\boldsymbol{ABCD}}.
\end{aligned}$$

Constraint equations are obtained from the combination

$$\Xi_{\boldsymbol{ABCD}} + \Xi^{*}_{\boldsymbol{ABCD}} = 0,$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

where

$$\begin{aligned}\Xi_{ABCD} + \Xi^*_{ABCD} = {} & \mathcal{D}^{P}{}_{C}\Gamma_{(PD)AB} + \mathcal{D}^{P}{}_{D}\Gamma_{(PC)AB} \\ & + \Gamma_{(C}{}^{PQ}{}_{|A|}\Gamma_{D)PQB} + \Gamma^{P}{}_{(C}{}^{Q}{}_{|A}\Gamma_{P|D)QB} \\ & + \Sigma^{+}_{C}{}^{PQ}{}_{D}\Gamma_{PQAB} - \Sigma_{C}{}^{PQ}{}_{D}\Gamma_{PQAB} \\ & - R_{ABCD} - R^*_{ABCD},\end{aligned}$$

while the required evolution equations arise from

$$\Xi_{ABCD} - \Xi^*_{ABCD} = 0,$$

with

$$\begin{aligned}\Xi_{ABCD} - \Xi^*_{ABCD} = {} & \mathcal{P}\Gamma_{(CD)AB} - \mathcal{D}_{CD}\Gamma_{AB} \\ & + \Gamma_{(C}{}^{PQ}{}_{|A|}\Gamma_{D)PQB} - \Gamma^{P}{}_{(C}{}^{Q}{}_{|A}\Gamma_{P|D)QB} \\ & - \Sigma^{+}_{C}{}^{PQ}{}_{D}\Gamma_{PQAB} - \Sigma_{C}{}^{PQ}{}_{D}\Gamma_{PQAB} \\ & - R_{ABCD} + R^*_{ABCD}.\end{aligned} \tag{13.28}$$

It can be verified that the system composed by (13.27) and (13.28) leads to a symmetric hyperbolic system for the independent components of Γ_{AB} and $\Gamma_{(CD)AB}$ – up to a suitable normalisation factor. A simple counting argument shows that the system consists of 12 equations, three coming from Equation (13.27) and nine from Equation (13.28). The characteristic polynomial of the system is given by

$$-64(\tau^{\mu}\xi_{\mu})^{6}(g^{\nu\lambda}\xi_{\nu}\xi_{\lambda})^{3}.$$

13.2.3 The conformal gauge source function

The third type of gauge source function to be considered arises from the analysis of the Cotton equation; see Equations (8.31e) and (8.35f). The starting point of the analysis is the spinorial counterpart, Equation (8.37a), associated with the zero quantity

$$\Delta_{CDBB'} \equiv \nabla_{(C}{}^{Q'}L_{D)Q'BB'} + \nabla^{Q}{}_{B'}\Xi\phi_{CDBQ} + \Xi T_{CDBB'}.$$

To deduce a symmetric hyperbolic system from this equation one needs to complete the symmetrised derivative $\nabla_{(C}{}^{Q'}L_{D)Q'BB'}$ with the divergence $\nabla^{QQ'}L_{QQ'BB'}$. Information about this derivative is provided by the contracted Bianchi identity for the Schouten tensor; compare Equation (8.17). In spinorial notation one has

$$\nabla^{QQ'}L_{QQ'BB'} = \frac{1}{6}\nabla_{BB'}R. \tag{13.29}$$

Thus, using

$$\nabla_{(C}{}^{Q'}L_{D)Q'BB'} = \nabla_{C}{}^{Q'}L_{DQ'BB'} + \frac{1}{2}\epsilon_{CD}\nabla^{QQ'}L_{QQ'BB'},$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

one can rewrite the zero quantity $\Delta_{\boldsymbol{CDBB'}}$ as

$$\begin{aligned}\Delta_{\boldsymbol{CDBB'}} = \nabla_{\boldsymbol{C}}{}^{\boldsymbol{Q'}} L_{\boldsymbol{DQ'BB'}} + \frac{1}{12}\epsilon_{\boldsymbol{CD}}\nabla_{\boldsymbol{BB'}}R \\ + \Sigma^{\boldsymbol{Q}}{}_{\boldsymbol{B'}}\phi_{\boldsymbol{CDBQ}} + \Xi T_{\boldsymbol{CDBB'}},\end{aligned} \tag{13.30}$$

where $\Sigma_{\boldsymbol{AA'}} \equiv \nabla_{\boldsymbol{AA'}}\Xi$.

As discussed in Chapter 8, the conformal field equations impose no differential condition on the unphysical Ricci scalar R. Accordingly, R can be specified freely as a function of the coordinates. Thus, if the reduced rescaled Cotton spinor $T_{\boldsymbol{CDBB'}}$ can be rewritten so that it does not explicitly contain derivatives of the matter fields, one can deduce a symmetric hyperbolic system for the components of $L_{\boldsymbol{AA'BB'}}$ from Equation (13.30).

Geometric interpretation

The particular choice of the Ricci scalar fixes the conformal gauge freedom. Thus, it is natural to call $R(x)$ the ***conformal gauge source function***. Given a particular choice of $R(x)$, the transformation law for the Ricci scalar implies a wave equation for the conformal factor realising the prescribed Ricci scalar; see Equation (8.30). This equation can always be solved locally if initial data on a fiduciary hypersurface $\mathcal{S}_\star$ is provided – namely, the values of the conformal factor and its normal derivative on the hypersurface. Conversely, given an unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$ and a conformal factor Ξ linking it to a physical spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ via the standard relation $\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}$, one can compute the corresponding conformal gauge source function $R(x)$.

Space spinor decomposition of the equation for the components of the Schouten tensor

The space spinor decomposition of the equations for the Schouten tensor is based on the expression

$$L_{\boldsymbol{AA'CC'}} = \Phi_{\boldsymbol{AA'CC'}} + \frac{1}{24}\epsilon_{\boldsymbol{AC}}\epsilon_{\boldsymbol{A'C'}}R(x), \tag{13.31}$$

where $\Phi_{\boldsymbol{AA'CC'}}$ denotes the spinorial counterpart of the trace-free part of the Ricci tensor; see Section 3.2.4. The space spinor counterpart of $L_{\boldsymbol{AA'CC'}}$ is defined as

$$\begin{aligned}L_{\boldsymbol{ABCD}} &\equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\tau_{\boldsymbol{D}}{}^{\boldsymbol{C'}}L_{\boldsymbol{AA'CC'}}, \\ &= \Phi_{\boldsymbol{ABCD}} + \frac{1}{24}\epsilon_{\boldsymbol{AC}}\epsilon_{\boldsymbol{BD}}R(x),\end{aligned}$$

where $\Phi_{\boldsymbol{ABCD}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\tau_{\boldsymbol{D}}{}^{\boldsymbol{C'}}\Phi_{\boldsymbol{AA'CC'}}$ so that

$$\Phi_{\boldsymbol{ABCD}} = \Phi_{\boldsymbol{CBAD}} = \Phi_{\boldsymbol{ADCB}},$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

as a consequence of the symmetries of $\Phi_{AA'CC'}$; see Equation (3.44). A spinor with these symmetries can be decomposed as

$$\Phi_{ABCD} = \Phi_{(ABCD)} + \frac{1}{2}\big(\epsilon_{A(B}\Phi_{D)C} + \epsilon_{C(B}\Phi_{D)A}\big) + \frac{1}{3}\Phi h_{ACBD}, \qquad (13.32)$$

where

$$\Phi_{AB} \equiv \Phi_{(AB)Q}{}^{Q}, \qquad \Phi \equiv \Phi_{ABCD}h^{ACBD}.$$

Now, using that

$$\nabla_{A}{}^{Q'}L_{BQ'CC'} = \nabla_{(A}{}^{Q'}L_{B)Q'CC'} - \frac{1}{2}\epsilon_{AB}\nabla^{QQ'}L_{QQ'CC'},$$

together with the contracted Bianchi identity (13.29) one can rewrite the zero quantity $\Delta_{ABCC'}$ as

$$\Delta_{ABCC'} = \nabla_{A}{}^{Q'}L_{BQ'CC'} + \frac{1}{12}\epsilon_{AB}\nabla_{CC'}R(x) + \Sigma^{Q}{}_{C'}\phi_{ABCQ} + \Xi T_{ABCC'}. \qquad (13.33)$$

Defining

$$\Delta_{ABCD} \equiv \tau_{D}{}^{C'}\Delta_{ABCC'},$$

a calculation using (13.33) together with the definitions of the spinors L_{ABCD} and χ_{ABCD}, yields

$$\begin{aligned}\Delta_{ABCD} = {} & \nabla_{A}{}^{Q}L_{BQCD} + \sqrt{2}\chi_{AP}{}^{QP}L_{BQCD} - \sqrt{2}\chi_{A}{}^{QP}{}_{D}L_{BQCP} \\ & + \frac{1}{2}\epsilon_{AB}\nabla_{CD}R(x) + \Sigma^{Q}{}_{D}\phi_{ABCQ} + \Xi T_{ABCD},\end{aligned}$$

where $\Sigma_{AB} \equiv \tau_{B}{}^{Q'}\Sigma_{AQ'}$. Thus, using the decomposition of the operator ∇_{AB} one obtains

$$\begin{aligned}\Delta_{ABCD} = {} & \frac{1}{2}\mathcal{P}L_{BACD} + \mathcal{D}_{A}{}^{Q}L_{BQCD} + \sqrt{2}\chi_{AP}{}^{QP}L_{BQCD} \\ & - \sqrt{2}\chi_{A}{}^{PQ}{}_{D}L_{BPCQ} + \frac{1}{2}\epsilon_{AB}\nabla_{CD}R(x) + \Sigma^{Q}{}_{D}\phi_{ABCQ} + \Xi T_{ABCD}.\end{aligned}$$

To extract the full information of $\Delta_{ABCC'}$ one also needs to consider

$$\Delta^{+}_{ABCD} \equiv \tau_{A}{}^{P'}\tau_{B}{}^{Q'}\tau_{C}{}^{R'}\tau_{D}{}^{S'}\bar{\Delta}_{P'Q'R'S'}.$$

Proceeding as with Δ_{ABCD} one finds that

$$\begin{aligned}\Delta^{+}_{ABCD} = {} & \frac{1}{2}\mathcal{P}L_{ABCD} - \mathcal{D}^{Q}{}_{A}L_{QBDC} + \sqrt{2}\chi^{Q}{}_{A}{}^{R}{}_{B}L_{QRDC} \\ & + \sqrt{2}\chi^{Q}{}_{A}{}^{P}{}_{C}L_{QBDP} - \frac{1}{2}\epsilon_{AB}\nabla_{CD}R(x) + \Sigma^{+R}{}_{D}\phi^{+}_{ABCR} + \Xi T^{+}_{ABCD}.\end{aligned}$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

Given the above expressions for Δ_{ABCD} and Δ^{+}_{ABCD}, suitable symmetric hyperbolic evolution equations for the independent components of the fields $\Phi_{(ABCD)}$, Φ_{AB} and Φ can be found from the combinations

$$\Delta_{(ABCD)} + \Delta^{+}_{(ABCD)} = 0, \tag{13.34a}$$
$$\Delta^{+}_{Q}{}^{Q}{}_{(CD)} - \Delta_{Q}{}^{Q}{}_{(CD)} = 0, \tag{13.34b}$$
$$\Delta_{Q}{}^{Q}{}_{P}{}^{P} + \Delta^{+}_{Q}{}^{Q}{}_{P}{}^{P} = 0, \tag{13.34c}$$

while constraint equations arise from

$$\Delta_{ABCD} - \Delta^{+}_{ABCD} = 0,$$
$$\Delta^{+}_{Q}{}^{Q}{}_{(CD)} + \Delta_{Q}{}^{Q}{}_{(CD)} = 0,$$
$$\Delta_{Q}{}^{Q}{}_{P}{}^{P} - \Delta^{+}_{Q}{}^{Q}{}_{P}{}^{P} = 0.$$

The principal parts of Equations (13.34a)–(13.34c) are given, respectively, by

$$\mathcal{P}\Phi_{(ABCD)} - \mathcal{D}_{(AB}\Phi_{CD)},$$
$$\mathcal{P}\Phi_{AB} + 2\mathcal{D}^{PQ}\Phi_{PQAB} - \frac{1}{3}\mathcal{D}_{AB}\Phi,$$
$$\mathcal{P}\Phi + \mathcal{D}^{PQ}\Phi_{PQ}.$$

The above expressions imply a symmetric hyperbolic system for the independent components of the fields $\Phi_{(ABCD)}$, Φ_{AB} and Φ. The explicit form of this system will not be required in the subsequent discussion but can be readily computed.

13.2.4 The hyperbolic reduction of the Bianchi equation

This section discusses the hyperbolic reduction of the spinorial Bianchi identity. This procedure leads to evolution equations for the components of the rescaled Weyl spinor and is completely analogous to that for the Maxwell equations; see Section 13.1.1. In particular, no gauge source functions are required for this subsystem.

The spinorial Bianchi equation is encoded in the zero quantity

$$\Lambda_{A'BCD} \equiv \nabla^{Q}{}_{A'}\phi_{BCDQ} + T_{CDBA'}.$$

In the following it will be convenient to work with a space spinor version of this zero quantity, namely,

$$\Lambda_{ABCD} \equiv \nabla^{Q}{}_{A}\phi_{BCDQ} + T_{CDBA}, \qquad T_{CDBA} \equiv \tau_{A}{}^{A'}T_{CDBA'}.$$

Using the decomposition (13.5) one can compute that

$$\Lambda_{ABCD} = -\frac{1}{2}\mathcal{P}\phi_{ABCD} + \mathcal{D}^{Q}{}_{A}\phi_{BCDQ} + T_{CDBA}. \tag{13.35}$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

Suitable evolution equations are obtained from the above expression by considering

$$-2\Lambda_{(\boldsymbol{ABCD})} = \mathcal{P}\phi_{\boldsymbol{ABCD}} - 2\mathcal{D}^{\boldsymbol{Q}}{}_{(\boldsymbol{A}}\phi_{\boldsymbol{BCD})\boldsymbol{Q}} + T_{(\boldsymbol{ABCD})} = 0. \tag{13.36}$$

In what follows, this system of evolution equations will be known as the ***standard system***. It gives rise to five independent equations for the five independent components of $\phi_{\boldsymbol{ABCD}}$. Contracting the indices $\boldsymbol{A}$ and $\boldsymbol{B}$ in Equation (13.35) one obtains

$$\Lambda_{\boldsymbol{CD}} \equiv \Lambda^{\boldsymbol{Q}}{}_{\boldsymbol{QCD}} = \mathcal{D}^{\boldsymbol{PQ}}\phi_{\boldsymbol{PQCD}} + T_{\boldsymbol{CDQ}}{}^{\boldsymbol{Q}} = 0,$$

the so-called ***Bianchi constraints***. As in the case of the other constraint equations discussed in the previous sections, the Bianchi constraints may contain derivatives in the time direction.

The hyperbolicity of the standard system

The overall structure of Equation (13.36) suggests that it should imply a symmetric hyperbolic system. In analogy to the Maxwell equations, one considers a slightly modified version of Equation (13.36) given by

$$-2\binom{4}{\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C}+\boldsymbol{D}}\Lambda_{(\boldsymbol{ABCD})} = 0.$$

The principal part of this equation can be written in matricial form as

$$\mathbf{A}^{\mu}\partial_{\mu}\boldsymbol{\phi} \equiv \begin{pmatrix} \tau^{\mu} + 2e_{\boldsymbol{01}}{}^{\mu} & -2e_{\boldsymbol{00}}{}^{\mu} & 0 & 0 & 0 \\ 2e_{\boldsymbol{11}}{}^{\mu} & 4\tau^{\mu} + 4e_{\boldsymbol{01}}{}^{\mu} & -6e_{\boldsymbol{00}}{}^{\mu} & 0 & 0 \\ 0 & 6e_{\boldsymbol{11}}{}^{\mu} & 6\tau^{\mu} & -6e_{\boldsymbol{00}}{}^{\mu} & 0 \\ 0 & 0 & 6e_{\boldsymbol{11}}{}^{\mu} & 4\tau^{\mu} - 4e_{\boldsymbol{01}}{}^{\mu} & -2e_{\boldsymbol{00}}{}^{\mu} \\ 0 & 0 & 0 & 2e_{\boldsymbol{11}}{}^{\mu} & \tau^{\mu} - 2e_{\boldsymbol{01}}{}^{\mu} \end{pmatrix}$$

$$\times\, \partial_{\mu}\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix},$$

with

$$\phi_0 \equiv \phi_{\mathbf{0000}}, \quad \phi_1 \equiv \phi_{\mathbf{0001}}, \quad \phi_2 \equiv \phi_{\mathbf{0011}}, \quad \phi_3 \equiv \phi_{\mathbf{0111}}, \quad \phi_4 \equiv \phi_{\mathbf{1111}}.$$

Using the reality conditions satisfied by the vectors $\boldsymbol{e_{AB}}$, it follows that the matrices of the system are Hermitian. Moreover, one has that $\mathbf{A}^{\mu}\tau_{\mu}$ is positive definite. Thus, the standard evolution system implies a symmetric hyperbolic system for the independent components of $\phi_{\boldsymbol{ABCD}}$. The characteristic matrix of the system is given by

$$\det(\mathbf{A}^\mu \xi_\mu) = 36\big(\tau^\mu \xi_\mu\big)\big(g^{\nu\lambda}\xi_\nu \xi_\lambda\big)\left(\tau^\rho \tau^\sigma + \frac{2}{3}g^{\rho\sigma}\right)\xi_\rho \xi_\sigma.$$

Thus, $\boldsymbol{g}$-null hypersurfaces are characteristics of the standard system.

13.2.5 The hyperbolic reduction of the equations for the conformal factor and its concomitants

Finally, one requires evolution equations for the conformal factor Ξ and its concomitants $\Sigma_{\boldsymbol{AA}'}$ and s. The relevant zero quantities are given by

$$Q_{\boldsymbol{AA}'} \equiv \Sigma_{\boldsymbol{AA}'} - \nabla_{\boldsymbol{AA}'}\Xi, \tag{13.37a}$$

$$Z_{\boldsymbol{AA}'\boldsymbol{BB}'} \equiv \nabla_{\boldsymbol{AA}'}\Sigma_{\boldsymbol{BB}'} + \Xi L_{\boldsymbol{AA}'\boldsymbol{BB}'} - s\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A}'\boldsymbol{B}'} - \frac{1}{2}\Xi^3 T_{\boldsymbol{AA}'\boldsymbol{BB}'}, \tag{13.37b}$$

$$Z_{\boldsymbol{AA}'} \equiv \nabla_{\boldsymbol{AA}'}s + L_{\boldsymbol{AA}'\boldsymbol{CC}'}\nabla^{\boldsymbol{CC}'}\Xi - \frac{1}{2}\Xi^2\nabla^{\boldsymbol{CC}'}\Xi T_{\boldsymbol{AA}'\boldsymbol{CC}'}. \tag{13.37c}$$

Their space spinor counterparts are defined by

$$Q_{\boldsymbol{AB}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A}'}Q_{\boldsymbol{AA}'}, \qquad Z_{\boldsymbol{ABCD}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A}'}\tau_{\boldsymbol{D}}{}^{\boldsymbol{C}'}Z_{\boldsymbol{AA}'\boldsymbol{CC}'}, \qquad Z_{\boldsymbol{AB}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A}'}Z_{\boldsymbol{AA}'}.$$

It is also convenient to make use of the split

$$\Sigma_{\boldsymbol{AB}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A}'}\Sigma_{\boldsymbol{AA}'} = \frac{1}{2}\epsilon_{\boldsymbol{AB}}\Sigma + \Sigma_{(\boldsymbol{AB})}, \qquad \Sigma \equiv \Sigma_{\boldsymbol{Q}}{}^{\boldsymbol{Q}}.$$

From the condition $Q_{\boldsymbol{AB}} = 0$ one obtains the equations

$$\mathcal{P}\Xi = \Sigma, \qquad \mathcal{D}_{\boldsymbol{AB}}\Sigma = \Sigma_{(\boldsymbol{AB})},$$

which are, respectively, an evolution equation for Ξ and a constraint equation. Next, using the identity

$$\tau_{\boldsymbol{B}}{}^{\boldsymbol{A}'}\tau_{\boldsymbol{D}}{}^{\boldsymbol{C}'}\nabla_{\boldsymbol{AA}'}\Sigma_{\boldsymbol{CC}'} = \nabla_{\boldsymbol{AB}}\big(\tau_{\boldsymbol{D}}{}^{\boldsymbol{C}'}\Sigma_{\boldsymbol{CC}'}\big) - \sqrt{2}\Sigma_{\boldsymbol{CP}}\chi_{\boldsymbol{AB}}{}^{\boldsymbol{P}}{}_{\boldsymbol{D}}$$

and the split of $\nabla_{\boldsymbol{AB}}$ it follows that

$$\begin{aligned} Z_{\boldsymbol{ABCD}} = {} & \frac{1}{4}\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{CD}}\mathcal{P}\Sigma + \frac{1}{2}\epsilon_{\boldsymbol{AB}}\mathcal{P}\Sigma_{(\boldsymbol{CD})} + \frac{1}{2}\epsilon_{\boldsymbol{CD}}\mathcal{D}_{\boldsymbol{AB}}\Sigma + \mathcal{D}_{\boldsymbol{AB}}\Sigma_{(\boldsymbol{CD})} \\ & + \frac{1}{\sqrt{2}}\chi_{\boldsymbol{ABCD}} - \sqrt{2}\Sigma_{\boldsymbol{CP}}\chi_{\boldsymbol{AB}}{}^{\boldsymbol{P}}{}_{\boldsymbol{D}} \\ & + \Xi L_{\boldsymbol{ABCD}} - s\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{CD}} - \frac{1}{2}T_{\boldsymbol{ABCD}}. \end{aligned}$$

Evolution equations for Σ and $\Sigma_{(\boldsymbol{AB})}$ are obtained from

$$2Z_{\boldsymbol{AB}}{}^{\boldsymbol{AB}} = 0, \qquad Z_{\boldsymbol{A}}{}^{\boldsymbol{A}}{}_{(\boldsymbol{CD})} = 0,$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

where

$$2Z_{\boldsymbol{AB}}{}^{\boldsymbol{AB}} = \mathcal{P}\Sigma + \sqrt{2}\chi_{\boldsymbol{AB}}{}^{\boldsymbol{AB}}\Sigma - 2\sqrt{2}\chi^{\boldsymbol{ABP}}{}_{\boldsymbol{B}}\Sigma_{(\boldsymbol{AP})} + \Xi L_{\boldsymbol{AB}}{}^{\boldsymbol{AB}} - 4s,$$

$$Z_{\boldsymbol{A}}{}^{\boldsymbol{A}}{}_{(\boldsymbol{CD})} = \mathcal{P}\Sigma_{(\boldsymbol{CD})} + \frac{1}{\sqrt{2}}\chi_{\boldsymbol{A}}{}^{\boldsymbol{A}}{}_{(\boldsymbol{CD})} - \sqrt{2}\Sigma_{(\boldsymbol{C}|\boldsymbol{P}}\chi_{\boldsymbol{A}}{}^{\boldsymbol{AP}}{}_{|\boldsymbol{D})}$$
$$+ \Xi L_{\boldsymbol{A}}{}^{\boldsymbol{A}}{}_{(\boldsymbol{CD})} - \frac{1}{2}\Xi^3 T_{\boldsymbol{A}}{}^{\boldsymbol{A}}{}_{(\boldsymbol{CD})}.$$

The corresponding constraints arise from

$$Z_{(\boldsymbol{ABCD})} = 0, \qquad Z_{(\boldsymbol{AB})\boldsymbol{C}}{}^{\boldsymbol{C}} = 0,$$

with

$$Z_{(\boldsymbol{ABCD})} = \mathcal{D}_{(\boldsymbol{AB}}\Sigma_{\boldsymbol{CD})} + \frac{1}{\sqrt{2}}\chi_{(\boldsymbol{ABCD})}\Sigma - \sqrt{2}\Sigma_{(\boldsymbol{C}|\boldsymbol{P}|}\chi_{|\boldsymbol{AB}}{}^{\boldsymbol{P}}{}_{\boldsymbol{D})}$$
$$+ \Xi L_{(\boldsymbol{ABCD})} - \frac{1}{2}\Xi^3 T_{(\boldsymbol{ABCD})},$$

$$Z_{(\boldsymbol{AB})\boldsymbol{C}}{}^{\boldsymbol{C}} = \mathcal{D}_{\boldsymbol{AB}}\Sigma + \frac{1}{\sqrt{2}}\chi_{(\boldsymbol{AB})\boldsymbol{C}}{}^{\boldsymbol{C}}\Sigma - \sqrt{2}\Sigma_{\boldsymbol{PQ}}\chi_{(\boldsymbol{AB})}{}^{\boldsymbol{PQ}}$$
$$+ \Xi L_{(\boldsymbol{AB})\boldsymbol{C}}{}^{\boldsymbol{C}} - \frac{1}{2}\Xi^3 T_{(\boldsymbol{AB})\boldsymbol{C}}{}^{\boldsymbol{C}}.$$

Finally, similar calculations lead to the expression

$$Z_{\boldsymbol{AB}} = \frac{1}{2}\epsilon_{\boldsymbol{AB}}\mathcal{P}s + \mathcal{D}_{\boldsymbol{AB}}s - \frac{1}{2}L_{\boldsymbol{ABC}}{}^{\boldsymbol{C}}\Sigma + L_{\boldsymbol{ABCD}}\Sigma^{\boldsymbol{CD}}$$
$$+ \frac{1}{4}\Xi^2 T_{\boldsymbol{ABC}}{}^{\boldsymbol{C}} - \frac{1}{2}\Xi^2\Sigma^{\boldsymbol{CD}}T_{\boldsymbol{ABCD}}.$$

The evolution and constraint equations for s are then given, respectively, by

$$Z_{\boldsymbol{A}}{}^{\boldsymbol{A}} = 0, \qquad Z_{(\boldsymbol{AB})} = 0,$$

with

$$Z_{\boldsymbol{A}}{}^{\boldsymbol{A}} = \mathcal{P}s - \frac{1}{2}L_{\boldsymbol{A}}{}^{\boldsymbol{A}}{}_{\boldsymbol{C}}{}^{\boldsymbol{C}} + L_{\boldsymbol{A}}{}^{\boldsymbol{A}}{}_{\boldsymbol{CD}}\Sigma^{\boldsymbol{CD}} + \frac{1}{4}\Xi^2\Sigma T_{\boldsymbol{A}}{}^{\boldsymbol{A}}{}_{\boldsymbol{C}}{}^{\boldsymbol{C}}$$
$$- \frac{1}{2}\Xi^2\Sigma^{\boldsymbol{CD}}T_{\boldsymbol{A}}{}^{\boldsymbol{A}}{}_{\boldsymbol{CD}},$$

$$Z_{(\boldsymbol{AB})} = \mathcal{D}_{\boldsymbol{AB}}s - \frac{1}{2}L_{(\boldsymbol{AB})\boldsymbol{C}}{}^{\boldsymbol{C}}\Sigma + L_{(\boldsymbol{AB})\boldsymbol{CD}}\Sigma^{\boldsymbol{CD}} + \frac{1}{4}\Xi^2\Sigma T_{(\boldsymbol{AB})\boldsymbol{C}}{}^{\boldsymbol{C}}$$
$$- \frac{1}{2}\Xi^2\Sigma^{\boldsymbol{CD}}T_{(\boldsymbol{AB})\boldsymbol{CD}}.$$

Remark. It should be observed that all the evolution equations obtained in this section are transport equations – that is, they involve only the directional derivative $\mathcal{P}$. Accordingly the characteristic polynomial of each of them is just $\tau^\mu\xi_\mu$.

13.3 The subsidiary equations for the standard conformal field equations

After having discussed a set of evolution equations implied by the conformal field Equations (8.38a) and (8.38b), one is now in the position of analysing the construction of the associated subsidiary system. The subsidiary equations constitute a system of evolution equations for the zero quantities encoding the conformal field equations. To prove the ***propagation of the constraints*** it is necessary that these subsidiary evolution equations are homogeneous in the various zero quantities. If this is the case, then Corollary 12.1 implies a unique vanishing solution to the subsidiary equations if the zero quantities are zero initially. The construction of the subsidiary system involves lengthy computations, parts of which are best carried out with spinorial expressions, while others are more conveniently described in tensorial terms. The basic strategy behind the analysis can be understood by first discussing some model equations.

General setup

The general setup for the construction of the subsidiary equations for the conformal field equations is similar to the one for the construction of the evolution equations: one works in an open subset $\mathcal{U} \subset \mathcal{M}$ of the unphysical spacetime manifold; vector and spinor bases are introduced in a similar manner. The key difference lies in the fact that the covariant derivative $\boldsymbol{\nabla}$ is, a priori, not assumed to be the Levi-Civita connection of the metric $\boldsymbol{g}$. Thus, when considering the commutator of covariant derivatives, one has to make use of the general expression involving a non-vanishing torsion tensor. This is because the torsion tensor is, in itself, a zero quantity of the conformal field equations. On similar grounds, one cannot regard the algebraic and geometric curvatures as being equal to each other.

13.3.1 Hyperbolic reduction of model equations

The construction of a system of subsidiary equations for the conformal Einstein equations leads to spinorial equations whose tensorial counterparts are of one of the following forms

$$\nabla_{[\boldsymbol{a}} M_{\boldsymbol{b}]\mathcal{K}} = N_{\boldsymbol{ab}\mathcal{K}}, \tag{13.38a}$$

$$\nabla_{[\boldsymbol{a}} P_{\boldsymbol{bc}]\mathcal{L}} = Q_{\boldsymbol{abc}\mathcal{L}}, \tag{13.38b}$$

where $M_{\boldsymbol{a}\mathcal{K}}$ and $P_{\boldsymbol{ab}\mathcal{L}}$ are some zero quantities with

$$N_{\boldsymbol{ab}\mathcal{K}} = N_{[\boldsymbol{ab}]\mathcal{K}}, \qquad P_{\boldsymbol{ab}\mathcal{L}} = P_{[\boldsymbol{ab}]\mathcal{L}}, \qquad Q_{\boldsymbol{abc}\mathcal{L}} = Q_{[\boldsymbol{abc}]\mathcal{L}},$$

and $_{\mathcal{K}}$ and $_{\mathcal{L}}$ denote an arbitrary string of indices.

Equations (13.38a) and (13.38b) arise from the following observations concerning differential forms; see the Appendix to this chapter for a brief discussion

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

on this and related notions. The fields $M_{a\mathcal{K}}$ and $P_{ab\mathcal{K}}$ can be regarded as the components, respectively, of the *1-form* and *2-form*

$$\boldsymbol{M}_{\mathcal{K}} \equiv M_{\boldsymbol{a}\mathcal{K}}\,\boldsymbol{\omega}^{\boldsymbol{a}}, \qquad \boldsymbol{P}_{\mathcal{L}} \equiv P_{\boldsymbol{ab}\mathcal{L}}\,\boldsymbol{\omega}^{\boldsymbol{a}} \wedge \boldsymbol{\omega}^{\boldsymbol{b}}.$$

Accordingly, Equations (13.38a) and (13.38b) can be written as

$$\mathbf{d}\boldsymbol{M}_{\mathcal{K}} = N_{\boldsymbol{ab}\mathcal{K}}\,\boldsymbol{\omega}^{\boldsymbol{a}} \wedge \boldsymbol{\omega}^{\boldsymbol{b}}, \qquad \mathbf{d}\boldsymbol{P}_{\mathcal{L}} = Q_{\boldsymbol{abc}\mathcal{L}}\,\boldsymbol{\omega}^{\boldsymbol{a}} \wedge \boldsymbol{\omega}^{\boldsymbol{b}} \wedge \boldsymbol{\omega}^{\boldsymbol{c}}.$$

If $\boldsymbol{\tau}$ denotes a timelike vector field, then the Lie derivatives of $\boldsymbol{M}_{\mathcal{K}}$ and $\boldsymbol{P}_{\mathcal{L}}$ along the direction of $\boldsymbol{\tau}$ are given by the so-called ***Cartan's formula***

$$\pounds_{\boldsymbol{\tau}}\boldsymbol{M}_{\mathcal{K}} = i_{\boldsymbol{\tau}}\mathbf{d}\boldsymbol{M}_{\mathcal{K}} + \mathbf{d}(i_{\boldsymbol{\tau}}\boldsymbol{M}_{\mathcal{K}}), \qquad \pounds_{\boldsymbol{\tau}}\boldsymbol{P}_{\mathcal{L}} = i_{\boldsymbol{\tau}}\mathbf{d}\boldsymbol{P}_{\mathcal{L}} + \mathbf{d}(i_{\boldsymbol{\tau}}\boldsymbol{P}_{\mathcal{L}}),$$

where $i_{\boldsymbol{\tau}}$ denotes the operation of contraction between the vector $\boldsymbol{\tau}$ and a differential form; see Frankel (2003). In terms of this notation the evolution equations are given, respectively, by

$$i_{\boldsymbol{\tau}}\boldsymbol{M}_{\mathcal{K}} = 0, \qquad i_{\boldsymbol{\tau}}\boldsymbol{P}_{\mathcal{L}} = 0,$$

so that

$$\pounds_{\boldsymbol{\tau}}\boldsymbol{M}_{\mathcal{K}} = i_{\boldsymbol{\tau}}\mathbf{d}\boldsymbol{M}_{\mathcal{K}}, \qquad \pounds_{\boldsymbol{\tau}}\boldsymbol{P}_{\mathcal{L}} = i_{\boldsymbol{\tau}}\mathbf{d}\boldsymbol{P}_{\mathcal{L}}.$$

The latter can be read as suitable evolution equations for the zero quantities $M_{\boldsymbol{a}\mathcal{K}}$ and $P_{\boldsymbol{ab}\mathcal{L}}$. Their frame component version is given by

$$\nabla_{[\boldsymbol{0}}M_{\boldsymbol{b}]\mathcal{K}} = N_{\boldsymbol{0b}\mathcal{K}}, \qquad \nabla_{[\boldsymbol{0}}P_{\boldsymbol{bc}]\mathcal{L}} = Q_{\boldsymbol{0bc}\mathcal{L}}.$$

Detailed analysis of the first model equation

The spinorial analogue of Equation (13.38a) is given by

$$\nabla_{\boldsymbol{AA}'}M_{\boldsymbol{BB}'\mathcal{K}} - \nabla_{\boldsymbol{BB}'}M_{\boldsymbol{AA}'\mathcal{K}} = 2N_{\boldsymbol{AA}'\boldsymbol{BB}'\mathcal{K}}.$$

Exploiting the antisymmetry one obtains the equivalent expression

$$\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q}'}M_{\boldsymbol{B})\boldsymbol{Q}'\mathcal{K}} = N_{\boldsymbol{A}}{}^{\boldsymbol{Q}'}{}_{\boldsymbol{BQ}'\mathcal{K}}, \qquad N_{\boldsymbol{A}}{}^{\boldsymbol{Q}'}{}_{\boldsymbol{BQ}'\mathcal{K}} = N_{\boldsymbol{B}}{}^{\boldsymbol{Q}'}{}_{\boldsymbol{AQ}'\mathcal{K}}.$$

Defining the space spinor counterpart $M_{\boldsymbol{AB}\mathcal{K}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A}'}M_{\boldsymbol{AA}'\mathcal{K}}$ and using the definition of the spinor $\chi_{\boldsymbol{ABCD}}$ together with the decomposition (13.5) of $\nabla_{\boldsymbol{AB}}$ one obtains the expression

$$\mathcal{P}M_{(\boldsymbol{AB})\mathcal{K}} + 2\mathcal{D}_{(\boldsymbol{A}}{}^{\boldsymbol{P}}M_{\boldsymbol{B})\boldsymbol{P}\mathcal{K}} + 2\sqrt{2}\chi_{(\boldsymbol{A}|\boldsymbol{Q}|}{}^{\boldsymbol{PQ}}M_{\boldsymbol{B})\boldsymbol{P}\mathcal{K}} = N_{\boldsymbol{A}}{}^{\boldsymbol{Q}'}{}_{\boldsymbol{BQ}'\mathcal{K}}.$$

Finally, assuming that the evolution equations implied by the zero quantity $M_{\boldsymbol{AA}'\mathcal{K}}$ are given by $M_{\boldsymbol{Q}}{}^{\boldsymbol{Q}}{}_{\mathcal{K}} = 0$, it follows that $M_{\boldsymbol{BP}\mathcal{K}} = M_{(\boldsymbol{BP})\mathcal{K}}$ and, moreover, that

$$\mathcal{P}M_{(\boldsymbol{AB})\mathcal{K}} + \mathcal{D}_{\boldsymbol{A}}{}^{\boldsymbol{P}}M_{(\boldsymbol{BP})\mathcal{K}} + \mathcal{D}_{\boldsymbol{B}}{}^{\boldsymbol{P}}M_{(\boldsymbol{AP})\mathcal{K}} + 2\sqrt{2}\chi_{(\boldsymbol{A}|\boldsymbol{Q}|}{}^{\boldsymbol{PQ}}M_{\boldsymbol{B})\boldsymbol{P}\mathcal{K}} = N_{\boldsymbol{A}}{}^{\boldsymbol{Q}'}{}_{\boldsymbol{BQ}'\mathcal{K}}.$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

This last expression is a suitable evolution equation for $M_{(\boldsymbol{AB})\mathcal{K}}$ if $N_{\boldsymbol{A}}{}^{\boldsymbol{Q}'}{}_{\boldsymbol{BQ}'\mathcal{K}}$ can be expressed as a linear combination of zero quantities. This computation depends on the particular structure of the conformal equation under consideration.

Detailed analysis of the second model equation

In what follows, let $P_{\boldsymbol{AA}'\boldsymbol{BB}'\mathcal{L}}$ and $Q_{\boldsymbol{AA}'\boldsymbol{BB}'\boldsymbol{CC}'\mathcal{L}}$ denote, respectively, the spinorial counterparts of the fields $P_{\boldsymbol{ab}\mathcal{L}}$ and $Q_{\boldsymbol{abc}\mathcal{L}}$. The spinorial counterpart of Equation (13.38b) can be conveniently written using the spinorial counterpart of the volume form as

$$\epsilon^{\boldsymbol{AA}'\boldsymbol{BB}'\boldsymbol{CC}'}{}_{\boldsymbol{DD}'}\nabla_{\boldsymbol{AA}'}P_{\boldsymbol{BB}'\boldsymbol{CC}'\mathcal{L}} = \epsilon^{\boldsymbol{AA}'\boldsymbol{BB}'\boldsymbol{CC}'}{}_{\boldsymbol{DD}'}Q_{\boldsymbol{AA}'\boldsymbol{BB}'\boldsymbol{CC}'\mathcal{L}}. \quad (13.39)$$

A convenient way of obtaining the space spinor version of this last equation is to consider, alternatively, the expression

$$\epsilon^{\boldsymbol{EFCDGH}}{}_{\boldsymbol{AB}}\nabla_{\boldsymbol{EF}}P_{\boldsymbol{CDGH}\mathcal{L}},$$

where, following standard conventions, one defines

$$P_{\boldsymbol{CDGH}\mathcal{L}} \equiv \tau_{\boldsymbol{D}}{}^{\boldsymbol{C}'}\tau_{\boldsymbol{H}}{}^{\boldsymbol{G}'}P_{\boldsymbol{CC}'\boldsymbol{GG}'\mathcal{L}},$$
$$\epsilon_{\boldsymbol{EFCDGHAB}} \equiv \tau_{\boldsymbol{F}}{}^{\boldsymbol{F}'}\tau_{\boldsymbol{D}}{}^{\boldsymbol{D}'}\tau_{\boldsymbol{H}}{}^{\boldsymbol{H}'}\tau_{\boldsymbol{B}}{}^{\boldsymbol{B}'}\epsilon_{\boldsymbol{EF}'\boldsymbol{CD}'\boldsymbol{GH}'\boldsymbol{AB}'}.$$

A short computation using the expression of the volume form in terms of ϵ-spinors yields

$$\epsilon_{\boldsymbol{EFCDGHAB}} = \mathrm{i}(\epsilon_{\boldsymbol{EG}}\epsilon_{\boldsymbol{CA}}\epsilon_{\boldsymbol{FB}}\epsilon_{\boldsymbol{DH}} - \epsilon_{\boldsymbol{EA}}\epsilon_{\boldsymbol{CG}}\epsilon_{\boldsymbol{FH}}\epsilon_{\boldsymbol{DB}}).$$

Now, exploiting the symmetries of $P_{\boldsymbol{CDGH}\mathcal{L}}$ one can write

$$P_{\boldsymbol{CDGH}\mathcal{L}} = P_{\boldsymbol{CG}\mathcal{L}}\epsilon_{\boldsymbol{DH}} + P^*_{\boldsymbol{DH}\mathcal{L}}\epsilon_{\boldsymbol{CG}},$$

where

$$P_{\boldsymbol{CG}\mathcal{L}} \equiv \frac{1}{2}P_{\boldsymbol{CQG}}{}^{\boldsymbol{Q}}{}_{\mathcal{L}}, \qquad P^*_{\boldsymbol{DH}\mathcal{L}} \equiv \frac{1}{2}P_{\boldsymbol{QD}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{H}\mathcal{L}}.$$

A calculation shows that the above expressions lead to

$$\begin{aligned}\epsilon^{\boldsymbol{EFCDGH}}{}_{\boldsymbol{AB}}\nabla_{\boldsymbol{EF}}P_{\boldsymbol{CDGH}\mathcal{L}} &\\ &= 2\mathrm{i}\left(\nabla_{\boldsymbol{A}}{}^{\boldsymbol{Q}}P^*_{\boldsymbol{BQ}\mathcal{L}} - \nabla^{\boldsymbol{Q}}{}_{\boldsymbol{B}}P_{\boldsymbol{AQ}\mathcal{L}}\right)\\ &= \mathrm{i}\mathcal{P}\left(P^*_{\boldsymbol{AB}\mathcal{L}} + P_{\boldsymbol{AB}\mathcal{L}}\right) + 2\mathrm{i}\mathcal{D}^{\boldsymbol{Q}}{}_{\boldsymbol{A}}P^*_{\boldsymbol{BQ}\mathcal{L}} - 2\mathrm{i}\mathcal{D}^{\boldsymbol{Q}}{}_{\boldsymbol{B}}P_{\boldsymbol{AQ}\mathcal{L}}.\end{aligned}$$

If the evolution equations associated with the zero quantity $P_{\boldsymbol{ABCD}\mathcal{L}}$ are given by the condition

$$P_{\boldsymbol{AB}\mathcal{L}} - P^*_{\boldsymbol{AB}\mathcal{L}} = 0,$$

it follows that

$$\mathcal{P}P_{\boldsymbol{AB}\mathcal{L}} = -\frac{\mathrm{i}}{2}\epsilon^{\boldsymbol{EFCDGH}}{}_{(\boldsymbol{AB})}\nabla_{\boldsymbol{EF}}P_{\boldsymbol{CDGH}\mathcal{L}}.$$

It can be verified that the expression one obtains by working directly with the left-hand side of Equation (13.39) differs from the above expression by homogeneous terms involving $P_{\boldsymbol{CDGH}\mathcal{L}}$ and $\chi_{\boldsymbol{ABCD}}$. To complete the construction of a suitable subsidiary equation for $P_{\boldsymbol{ABCD}\mathcal{L}}$ it is necessary to show that the right-hand side of Equation (13.39) can be expressed as a linear combination of zero quantities – this computation is specific to each zero quantity.

13.3.2 The subsidiary equations for the equations governing the conformal factor and its concomitants

The zero quantities $Q_{\boldsymbol{AA}'}$, $Z_{\boldsymbol{AA}'\boldsymbol{BB}'}$ and $Z_{\boldsymbol{AA}'}$ – see Equations (13.37a)–(13.37c) – lead to subsidiary equations which fall into the class described by the model Equation (13.38a). Accordingly, one will have suitable subsidiary evolution equations for the zero quantities $Q_{\boldsymbol{AA}'}$, $Z_{\boldsymbol{AA}'\boldsymbol{BB}'}$ and $Z_{\boldsymbol{AA}'}$ if the derivatives

$$\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q}'}Q_{\boldsymbol{B})\boldsymbol{Q}'}, \qquad \nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q}'}Z_{\boldsymbol{B})\boldsymbol{Q}'\boldsymbol{CC}'}, \qquad \nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q}'}Z_{\boldsymbol{B})\boldsymbol{Q}'},$$

can be expressed as linear combinations of other zero quantities.

The subsidiary equation for $Q_{\boldsymbol{AA}'}$

A direct computation using the definition of $Q_{\boldsymbol{AA}'}$ shows that

$$\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q}'}Q_{\boldsymbol{B})\boldsymbol{Q}'} = \nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q}'}\Sigma_{\boldsymbol{B})\boldsymbol{Q}'} - \Sigma_{\boldsymbol{A}}{}^{\boldsymbol{QQ}'}{}_{\boldsymbol{B}}\Sigma_{\boldsymbol{QQ}'},$$

where the definition of the torsion spinor – see Equation (8.35a) – has been used to write

$$\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q}'}\nabla_{\boldsymbol{B})\boldsymbol{Q}'}\Xi = \Sigma_{\boldsymbol{A}}{}^{\boldsymbol{QQ}'}{}_{\boldsymbol{B}}\Sigma_{\boldsymbol{QQ}'}, \qquad \Sigma_{\boldsymbol{A}}{}^{\boldsymbol{QQ}'}{}_{\boldsymbol{B}} \equiv \frac{1}{2}\Sigma_{\boldsymbol{A}}{}^{\boldsymbol{P}'\boldsymbol{QQ}'}{}_{\boldsymbol{BP}'}.$$

Finally, using the definition of the zero quantity $Z_{\boldsymbol{AA}'\boldsymbol{BB}'}$ one can eliminate the term $\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q}'}\Sigma_{\boldsymbol{B})\boldsymbol{Q}'}$. Observing that $L_{(\boldsymbol{A}}{}^{\boldsymbol{Q}'}{}_{\boldsymbol{B})\boldsymbol{Q}'} = T_{(\boldsymbol{A}}{}^{\boldsymbol{Q}'}{}_{\boldsymbol{B})\boldsymbol{Q}'} = 0$ – as these are the spinorial counterparts of symmetric rank-2 tensors – one finds

$$\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q}'}\Sigma_{\boldsymbol{B})\boldsymbol{Q}'} = Z_{(\boldsymbol{A}}{}^{\boldsymbol{Q}'}{}_{\boldsymbol{B})\boldsymbol{Q}'},$$

so that one concludes that

$$\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q}'}Q_{\boldsymbol{B})\boldsymbol{Q}'} = Z_{(\boldsymbol{A}}{}^{\boldsymbol{Q}'}{}_{\boldsymbol{B})\boldsymbol{Q}'} - \Sigma_{\boldsymbol{A}}{}^{\boldsymbol{QQ}'}{}_{\boldsymbol{B}}\Sigma_{\boldsymbol{QQ}'},$$

which is a linear combination of zero quantities as required.

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

The subsidiary equation for $Z_{\boldsymbol{AA'BB'}}$

A direct computation starting from the definition of $Z_{\boldsymbol{AA'BB'}}$ yields the expression

$$\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q'}} Z_{\boldsymbol{B})\boldsymbol{Q'CC'}} = \nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q'}}\nabla_{\boldsymbol{B})\boldsymbol{Q'}}\Sigma_{\boldsymbol{CC'}} + \Sigma_{(\boldsymbol{A}}{}^{\boldsymbol{Q'}} L_{\boldsymbol{B})\boldsymbol{Q'CC'}} + \Xi\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q'}} L_{\boldsymbol{B})\boldsymbol{Q'CC'}} + \epsilon_{\boldsymbol{C}(\boldsymbol{A}}\nabla_{\boldsymbol{B})\boldsymbol{C'}}s - \frac{3}{2}\Xi^2\Sigma_{(\boldsymbol{A}}{}^{\boldsymbol{Q'}} T_{\boldsymbol{B})\boldsymbol{Q'CC'}} - \Xi^3\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q'}} T_{\boldsymbol{B})\boldsymbol{Q'CC'}}.$$

Using the commutator

$$\begin{aligned}\nabla_{\boldsymbol{AA'}}\nabla_{\boldsymbol{BB'}}\Sigma_{\boldsymbol{CC'}} - \nabla_{\boldsymbol{BB'}}\nabla_{\boldsymbol{AA'}}\Sigma_{\boldsymbol{CC'}} &= -R^{\boldsymbol{P}}{}_{\boldsymbol{CAA'BB'}}\Sigma_{\boldsymbol{PC'}} - \bar{R}^{\boldsymbol{P'}}{}_{\boldsymbol{C'A'ABB'}}\Sigma_{\boldsymbol{CP'}} \\ &\quad - \Sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{BB'}}\nabla_{\boldsymbol{QQ'}}\Sigma_{\boldsymbol{CC'}},\end{aligned}$$

one finds that

$$\begin{aligned}\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q'}}\nabla_{\boldsymbol{B})\boldsymbol{Q'}}\Sigma_{\boldsymbol{CC'}} &= -R^{\boldsymbol{P}}{}_{\boldsymbol{C}(\boldsymbol{AB})}\Sigma_{\boldsymbol{PC'}} - \bar{R}^{\boldsymbol{P'}}{}_{\boldsymbol{C'}(\boldsymbol{AB})}\Sigma_{\boldsymbol{CP'}} \\ &\quad - \Sigma_{\boldsymbol{A}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{B}}\nabla_{\boldsymbol{QQ'}}\Sigma_{\boldsymbol{CC'}},\end{aligned}$$

where

$$R_{\boldsymbol{ABCD}} \equiv \frac{1}{2}R_{\boldsymbol{ABCQ'D}}{}^{\boldsymbol{Q'}}, \qquad \bar{R}_{\boldsymbol{A'B'CD}} \equiv \frac{1}{2}\bar{R}_{\boldsymbol{A'B'Q'CD}}{}^{\boldsymbol{Q'}}.$$

Using the definitions of the zero quantities $\Delta_{\boldsymbol{CDBB'}}$ and $Z_{\boldsymbol{AA'}}$ to eliminate, respectively, $\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q'}} L_{\boldsymbol{B})\boldsymbol{Q'CC'}}$ and $\nabla_{\boldsymbol{BC'}}s$, one obtains

$$\begin{aligned}\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q'}} Z_{\boldsymbol{B})\boldsymbol{Q'CC'}} &= -R^{\boldsymbol{P}}{}_{\boldsymbol{C}(\boldsymbol{AB})}\Sigma_{\boldsymbol{PC'}} - \bar{R}^{\boldsymbol{P'}}{}_{\boldsymbol{C'}(\boldsymbol{AB})}\Sigma_{\boldsymbol{CP'}} - \Sigma_{\boldsymbol{A}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{B}}\nabla_{\boldsymbol{QQ'}}\Sigma_{\boldsymbol{CC'}} \\ &\quad + \Sigma_{(\boldsymbol{A}}{}^{\boldsymbol{Q'}} L_{\boldsymbol{B})\boldsymbol{Q'CC'}} + \Xi\Delta_{\boldsymbol{ABCC'}} - \Xi\Sigma^{\boldsymbol{Q}}{}_{\boldsymbol{C'}}\phi_{\boldsymbol{ABCQ}} \\ &\quad - \Xi^2 T_{\boldsymbol{ABCC'}} + \epsilon_{\boldsymbol{C}(\boldsymbol{A}}Z_{\boldsymbol{B})\boldsymbol{C'}} - \epsilon_{\boldsymbol{C}(\boldsymbol{A}}L_{\boldsymbol{B})\boldsymbol{C'QQ'}}\Sigma^{\boldsymbol{QQ'}} \\ &\quad - \frac{1}{2}\Xi^2\Sigma^{\boldsymbol{QQ'}}\epsilon_{\boldsymbol{C}(\boldsymbol{A}}T_{\boldsymbol{B})\boldsymbol{C'QQ'}} - \frac{3}{2}\Xi^2\Sigma_{(\boldsymbol{A}}{}^{\boldsymbol{Q'}} T_{\boldsymbol{B})\boldsymbol{Q'CC'}} \\ &\quad - \frac{1}{2}\Xi^3\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q'}} T_{\boldsymbol{B})\boldsymbol{Q'CC'}}.\end{aligned}$$

Next, one uses the zero quantity $\Xi_{\boldsymbol{ABCC'DD'}}$ to eliminate the geometric curvature terms $R^{\boldsymbol{P}}{}_{\boldsymbol{C}(\boldsymbol{AB})}$ and $\bar{R}^{\boldsymbol{P'}}{}_{\boldsymbol{C'}(\boldsymbol{AB})}$. Taking into account the expression of the algebraic curvature in terms of the Schouten tensor and the rescaled Weyl tensor one obtains

$$\begin{aligned}\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q'}} Z_{\boldsymbol{B})\boldsymbol{Q'CC'}} &= -\Xi^{\boldsymbol{P}}{}_{\boldsymbol{C}(\boldsymbol{AB})}\Sigma_{\boldsymbol{PC'}} - \bar{\Xi}^{\boldsymbol{P'}}{}_{\boldsymbol{C'}(\boldsymbol{AB})}\Sigma_{\boldsymbol{CP'}} - \Sigma_{\boldsymbol{A}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{B}}\nabla_{\boldsymbol{QQ'}}\Sigma_{\boldsymbol{CC'}} \\ &\quad + \Xi\Delta_{\boldsymbol{ABCC'}} - \Xi^2 T_{\boldsymbol{ABCC'}} + \epsilon_{\boldsymbol{C}(\boldsymbol{A}}Z_{\boldsymbol{B})\boldsymbol{C'}} \\ &\quad - \frac{1}{2}\Xi^2\Sigma^{\boldsymbol{QQ'}}\epsilon_{\boldsymbol{C}(\boldsymbol{A}}T_{\boldsymbol{B})\boldsymbol{C'QQ'}} - \frac{3}{2}\Xi^2\Sigma_{(\boldsymbol{A}}{}^{\boldsymbol{Q'}} T_{\boldsymbol{B})\boldsymbol{Q'CC'}} \\ &\quad - \frac{1}{2}\Xi^3\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q'}} T_{\boldsymbol{B})\boldsymbol{Q'CC'}},\end{aligned}$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

where

$$\Xi_{ABCD} \equiv \frac{1}{2}\Xi_{ABCQ'D}{}^{Q'}, \qquad \bar{\Xi}_{A'B'CD} \equiv \frac{1}{2}\bar{\Xi}_{A'B'Q'CD}{}^{Q'}.$$

Finally, observing that the definition of the rescaled Cotton tensor implies that $T_{ABCC'} = -\frac{1}{2}\Xi\nabla_{(A}{}^{Q'}T_{B)Q'CC'}$ and exploiting the trace-freeness of the energy-momentum tensor one ends up with the expression

$$\nabla_{(A}{}^{Q'}Z_{B)Q'CC'} = -\Xi^{P}{}_{C(AB)}\Sigma_{PC'} - \bar{\Xi}^{P'}{}_{C'(AB)}\Sigma_{CP'} - \Sigma_{A}{}^{QQ'}{}_{B}\nabla_{QQ'}\Sigma_{CC'} + \Xi\Delta_{ABCC'} + \epsilon_{C(A}Z_{B)C'},$$

which, as required, is a linear combination of zero quantities.

The subsidiary equation for $Z_{AA'}$

In this case one needs to evaluate $\nabla_{(A}{}^{Q'}Z_{B)Q'}$. Making use of the definition of $Z_{AA'}$ one finds that

$$\begin{aligned}\nabla_{(A}{}^{Q'}Z_{B)Q'} = {} & \nabla_{(A}{}^{Q'}\nabla_{B)Q'}s + \nabla_{(A}{}^{Q'}L_{B)Q'PP'}\Sigma^{PP'} + \nabla_{(A}{}^{Q'}\Sigma^{PP'}L_{B)Q'PP'} \\ & - \Xi\Sigma_{(A}{}^{Q'}\Sigma^{PP'}T_{B)Q'PP'} - \frac{1}{2}\Xi^2\nabla_{(A}{}^{Q'}\Sigma^{PP'}T_{B)Q'PP'} \\ & - \frac{1}{2}\Xi^2\Sigma^{PP'}\nabla_{(A}{}^{Q'}T_{B)Q'PP'}.\end{aligned}$$

Using the definition of the torsion tensor in the form

$$\nabla_{(A}{}^{Q'}\nabla_{B)Q'}s = \Sigma_{A}{}^{QQ'}{}_{B}\nabla_{QQ'}s,$$

and the definitions of $\Delta_{ABCC'}$ and $Z_{AA'BB'}$ to eliminate, respectively, $\nabla_{(A}{}^{Q'}L_{B)Q'PP'}$ and $\nabla_{A}{}^{Q'}\Sigma^{PP'}$ one obtains – after some simplifications involving the symmetries of $L_{AA'BB'}$ and $T_{AA'BB'}$ –

$$\begin{aligned}\nabla_{(A}{}^{Q'}Z_{B)Q'} = {} & \Sigma_{(A}{}^{QQ'}{}_{B)}\nabla_{QQ'}s + \Delta_{ABPP'}\Sigma^{PP'} + Z_{(A}{}^{Q'PP'}L_{B)Q'PP'} \\ & - \frac{1}{2}\Xi^2 Z_{(A}{}^{Q'PP'}T_{B)Q'PP'}.\end{aligned}$$

This expresion is a linear combination of zero quantities.

13.3.3 Subsidiary equation for the no-torsion condition

Following the general discussion of Section 13.3.1, one defines

$$\Sigma_{ABCD}{}^{a} \equiv \tau_{B}{}^{A'}\tau_{D}{}^{C'}\Sigma_{AA'}{}^{QQ'}{}_{CC'}e_{QQ'}{}^{a}.$$

One can write

$$\Sigma_{ABCD}{}^{a} = -\Sigma_{AC}{}^{a}\epsilon_{BD} - \Sigma^{+}{}_{BD}{}^{a}\epsilon_{AC},$$

so that, if the evolution equation $\Sigma_{AB}{}^{a} - \Sigma^{+}{}_{AB}{}^{a} = 0$ holds, then

$$\mathcal{P}\Sigma_{AB}{}^{a} = -\frac{\mathrm{i}}{2}\nabla_{EF}\Sigma_{CDGH}{}^{a}\epsilon^{EFCDGH}{}_{(AB)}. \tag{13.40}$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

To conclude the argument one needs to express the right-hand side of the above equation as a linear combination of zero quantities. To this end, one makes use of the first Bianchi identity (2.10) to write

$$\nabla_{[\boldsymbol{a}}\Sigma_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}]} = -\Xi^{\boldsymbol{d}}{}_{[\boldsymbol{cab}]} - \rho^{\boldsymbol{d}}{}_{[\boldsymbol{cab}]} - \Sigma_{[\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\Sigma_{\boldsymbol{c}]}{}^{\boldsymbol{d}}{}_{\boldsymbol{e}},$$

where the zero quantity $\Xi^{\boldsymbol{d}}{}_{\boldsymbol{cab}} \equiv R^{\boldsymbol{d}}{}_{\boldsymbol{cab}} - \rho^{\boldsymbol{d}}{}_{\boldsymbol{cab}}$. By construction, the algebraic curvature has the same algebraic symmetries as the Riemann tensor of a Levi-Civita connection so that, in particular, $\rho^{\boldsymbol{d}}{}_{[\boldsymbol{cab}]} = 0$ and one has that

$$\nabla_{[\boldsymbol{a}}\Sigma_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}]} = -\Xi^{\boldsymbol{d}}{}_{[\boldsymbol{cab}]} - \Sigma_{[\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\Sigma_{\boldsymbol{c}]}{}^{\boldsymbol{d}}{}_{\boldsymbol{e}}.$$

This expression shows that the right-hand side of Equation (13.40) can be written as a linear combination of zero quantities.

13.3.4 Subsidiary equation for the Ricci identity

It follows from the general discussion of Section 13.3.1 that, if the evolution equations $\Xi_{\boldsymbol{ABCD}} - \Xi^*_{\boldsymbol{ABCD}} = 0$ are satisfied, then

$$\mathcal{P}\Xi_{\boldsymbol{ABCD}} = -\frac{\mathrm{i}}{2}\nabla_{\boldsymbol{EF}}\Xi_{\boldsymbol{CDGH}}\epsilon^{\boldsymbol{EFCDGH}}{}_{(\boldsymbol{AB})}. \tag{13.41}$$

To express the right-hand side of this last equation as a linear combination of zero quantities one makes use of the second Bianchi identity (2.11) to obtain

$$\begin{aligned}\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\nabla_{[\boldsymbol{a}}\Xi^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} &= \epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\nabla_{[\boldsymbol{a}}R^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} - \epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\nabla_{[\boldsymbol{a}}\rho^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]}\\ &= -\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\Sigma_{[\boldsymbol{a}}{}^{\boldsymbol{g}}{}_{\boldsymbol{b}}R^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{c}]\boldsymbol{g}} - \epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\nabla_{[\boldsymbol{a}}\rho^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]}.\end{aligned} \tag{13.42}$$

The first term in the right-hand side of the last equation already has the desired form. The second term needs to be examined in more detail. One considers

$$\begin{aligned}\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\nabla_{[\boldsymbol{a}}\rho^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} &= \epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\nabla_{\boldsymbol{a}}\rho^{\boldsymbol{d}}{}_{\boldsymbol{ebc}}\\ &= \Xi\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\nabla_{\boldsymbol{a}}d^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} + \epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\nabla_{\boldsymbol{a}}\Xi d^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} + 2\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}S_{\boldsymbol{eb}}{}^{\boldsymbol{dh}}\nabla_{\boldsymbol{a}}L_{\boldsymbol{ch}},\end{aligned}$$

where in the last line the expression of the algebraic curvature in terms of the Weyl tensor and the Schouten tensor has been used. Now, a computation using the properties of the Hodge dual and the definition of the zero quantity $\Lambda_{\boldsymbol{abc}}$ shows that

$$\begin{aligned}\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\nabla_{\boldsymbol{a}}d^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} &= -\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\nabla_{\boldsymbol{a}}{}^*d^{*\boldsymbol{d}}{}_{\boldsymbol{ebc}} = -2\nabla_{\boldsymbol{a}}{}^*d^{**\boldsymbol{d}}{}_{\boldsymbol{ef}}{}^{\boldsymbol{a}}\\ &= 2\nabla_{\boldsymbol{a}}{}^*d^{\boldsymbol{d}}{}_{\boldsymbol{ef}}{}^{\boldsymbol{a}} = 2\nabla_{\boldsymbol{a}}d^*{}_{\boldsymbol{f}}{}^{\boldsymbol{ad}}{}_{\boldsymbol{e}}\\ &= \epsilon_{\boldsymbol{e}}{}^{\boldsymbol{dgh}}\nabla_{\boldsymbol{a}}d^{\boldsymbol{a}}{}_{\boldsymbol{fgh}} = \epsilon_{\boldsymbol{e}}{}^{\boldsymbol{dgh}}(\Lambda_{\boldsymbol{fgh}} + T_{\boldsymbol{fgh}}).\end{aligned} \tag{13.43}$$

Using the above expression together with the definition of the zero quantities $\Delta_{\boldsymbol{abc}}$ and $Q_{\boldsymbol{a}}$ to eliminate $\nabla_{[\boldsymbol{a}}L_{\boldsymbol{c}]\boldsymbol{f}}$ and $\nabla_{\boldsymbol{a}}\Xi$, respectively, one finds that

$$\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\nabla_{[\boldsymbol{a}}\rho^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} = \epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}Q_{\boldsymbol{a}}d^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} + \Xi\epsilon_{\boldsymbol{e}}{}^{\boldsymbol{dgh}}\Lambda_{\boldsymbol{fgh}} + \epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}S_{\boldsymbol{eb}}{}^{\boldsymbol{dh}}\Delta_{\boldsymbol{ach}}.$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

Substituting this last expression into Equation (13.42) one obtains the required expression for $\nabla_{[a}\Xi^{d}{}_{|e|bc]}$ as a linear combination of zero quantities. The spinorial counterpart of this expression differs from the right-hand side of Equation (13.41) by terms homogeneous in zero quantities involving the spinor χ_{ABCD}.

13.3.5 The subsidiary equations for the Cotton equation

Applying the general discussion of Section 13.3.1 to the zero quantity $\Delta_{AA'BB'CC'}$ associated with the unphysical Bianchi identity leads to the expression

$$\frac{1}{2}\mathcal{P}\big(\Delta^{+}_{ABKL}+\Delta_{ABKL}\big)+\mathcal{D}^{Q}{}_{A}\Delta^{+}_{BQKL}-\mathcal{D}^{Q}{}_{B}\Delta_{AQKL} = -\frac{\mathrm{i}}{2}\epsilon^{EFCDGH}{}_{AB}\nabla_{EF}\Delta_{CDGHKL}. \tag{13.44}$$

To make use of the evolution equations in the above expression it is observed that

$$\Delta_{ABCD}=\Delta_{(ABCD)}+\frac{1}{3}h_{ABCD}\Delta_{PQ}{}^{PQ}+\frac{1}{2}\epsilon_{CD}\Delta_{ABQ}{}^{Q}.$$

Thus, using the evolution equations for the various components of the Schouten tensor one obtains

$$\begin{aligned}
\mathcal{P}\Delta_{(ABKL)} &= -\frac{\mathrm{i}}{2}\nabla_{EF}\Delta_{CDGH(AB}\epsilon^{EFCDGH}{}_{KL)},\\
\mathcal{P}\Delta_{PQ}{}^{PQ} &= -\frac{\mathrm{i}}{2}\epsilon^{EFCDGHKL}\nabla_{EF}\Delta_{CDGHKL},\\
\mathcal{P}\Delta_{ABQ}{}^{Q} &= -\frac{\mathrm{i}}{2}\epsilon^{EFCDGH}{}_{AB}\nabla_{EF}\Delta_{CDGHQ}{}^{Q}.
\end{aligned}$$

To analyse the right-hand sides of the above equations it is more convenient to analyse $\epsilon_{AA'}{}^{BB'CC'DD'}\nabla_{BB'}\Delta_{CC'DD'EE'}$. This expression differs from the right-hand side of Equation (13.44) by terms involving the spinor χ_{ABCD}. For conciseness, the analysis is carried out using tensorial notation. One has that

$$\begin{aligned}
\epsilon_{f}{}^{ecd}\nabla_{e}\Delta_{cdb} = \epsilon_{f}{}^{ecd}\big(&2\nabla_{e}\nabla_{[c}L_{d]b}-\nabla_{e}\Sigma_{a}d^{a}{}_{bcd}-\Sigma_{a}\nabla_{e}d^{a}{}_{bcd}\\
&-\nabla_{e}\Xi T_{cdb}-\Xi\nabla_{e}T_{cdb}\big).
\end{aligned} \tag{13.45}$$

The first term on the right-hand side of the above equation is manipulated using the commutator of covariant derivatives by observing that

$$\begin{aligned}
2\epsilon_{f}{}^{ecd}\nabla_{e}\nabla_{[c}L_{d]b} &= 2\epsilon_{f}{}^{ecd}\nabla_{[e}\nabla_{c]}L_{db}\\
&= -2\epsilon_{f}{}^{ecd}\big(2R^{s}{}_{(d|ec|}L_{b)s}-\Sigma_{e}{}^{s}{}_{c}\nabla_{s}L_{db}\big)\\
&= -2\epsilon_{f}{}^{ecd}\big(2\Xi^{s}{}_{(d|ec|}L_{b)s}+\epsilon_{f}{}^{ecd}\rho^{s}{}_{bec}L_{ds}+\Sigma_{e}{}^{s}{}_{c}\nabla_{s}L_{db}\big),
\end{aligned}$$

where in the third line the identity $\rho^{a}{}_{[bcd]}=0$ has been used. The second term in the last equation does not contain zero quantities. The third term in

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

Equation (13.45) is now cast in a suitable form using the properties of Hodge dual; compare the analogous argument leading to (13.43). One finds that

$$\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{ecd}}\nabla_{\boldsymbol{e}}d_{\boldsymbol{abcd}} = -\epsilon_{\boldsymbol{ab}}{}^{\boldsymbol{gh}}(\Lambda_{\boldsymbol{fgh}} + T_{\boldsymbol{fgh}}).$$

Substituting the above two identities into Equation (13.45) and using the zero quantity $Z_{\boldsymbol{ab}}$ to eliminate $\nabla_{\boldsymbol{a}}\Sigma_{\boldsymbol{b}}$ one obtains

$$\begin{aligned}
&2\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{ecd}}\nabla_{\boldsymbol{e}}\nabla_{[\boldsymbol{c}}L_{\boldsymbol{d}]\boldsymbol{b}} \\
&\qquad = -4\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{ecd}}\Xi^{\boldsymbol{s}}{}_{(\boldsymbol{d}|\boldsymbol{ec}|}L_{\boldsymbol{b})\boldsymbol{s}} - 2\Sigma_{\boldsymbol{e}}{}^{\boldsymbol{s}}{}_{\boldsymbol{c}}\nabla_{\boldsymbol{s}}L_{\boldsymbol{fb}} \\
&\qquad\quad - \Sigma^{\boldsymbol{a}}\epsilon_{\boldsymbol{ab}}{}^{\boldsymbol{gh}}\Lambda_{\boldsymbol{fgh}} - \epsilon_{\boldsymbol{f}}{}^{\boldsymbol{ecd}}Z_{\boldsymbol{ea}}d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} \\
&\qquad\quad - \left\{\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{ecd}}\left(\frac{1}{2}\Xi^3T_{\boldsymbol{ea}}d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} + \nabla_{\boldsymbol{e}}\Xi T_{\boldsymbol{cdb}} - \Xi\nabla_{\boldsymbol{e}}T_{\boldsymbol{cdb}}\right) - \epsilon_{\boldsymbol{ab}}{}^{\boldsymbol{gh}}\Sigma^{\boldsymbol{a}}T_{\boldsymbol{fgh}}\right\},
\end{aligned} \tag{13.46}$$

where the explicit expression of the algebraic curvature has been used to show that

$$\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{ecd}}\big(\Xi d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}L_{\boldsymbol{ea}} - 2\rho^{\boldsymbol{s}}{}_{\boldsymbol{bec}}L_{\boldsymbol{ds}}\big) = 0.$$

Expression (13.46) is, up to the *matter terms* in curly brackets, a linear combination of zero quantities. Whether the terms in curly brackets can be expressed as a linear combination of (matter) zero quantities depends on the particular features of the matter model under consideration.

13.3.6 The subsidiary equations for the Bianchi identity

The construction of the subsidiary equation for the Bianchi identity is similar to that of the subsidiary equation for the Maxwell equations. In this case the relevant zero quantity is given by

$$\Lambda_{\boldsymbol{A}'\boldsymbol{BCD}} \equiv \nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}\phi_{\boldsymbol{BCDQ}} + T_{\boldsymbol{CDBA}'},$$

for which one computes $\nabla^{\boldsymbol{BB}'}\Lambda_{\boldsymbol{B}'\boldsymbol{BCD}}$ in two different manners.

First, making use of the space spinor zero quantity $\Lambda_{\boldsymbol{ABCD}} \equiv \tau_{\boldsymbol{A}}{}^{\boldsymbol{A}'}\Lambda_{\boldsymbol{A}'\boldsymbol{BCD}}$ one has that

$$\begin{aligned}
\nabla^{\boldsymbol{BB}'}\Lambda_{\boldsymbol{B}'\boldsymbol{BCD}} &= -\nabla^{\boldsymbol{BB}'}\big(\tau^{\boldsymbol{P}}{}_{\boldsymbol{B}'}\Lambda_{\boldsymbol{PBCD}}\big) \\
&= \nabla^{\boldsymbol{AB}}\Lambda_{\boldsymbol{ABCD}} - \big(\nabla^{\boldsymbol{BA}'}\tau^{\boldsymbol{A}}{}_{\boldsymbol{A}'}\big)\Lambda_{\boldsymbol{ABCD}}.
\end{aligned}$$

Using Equation (13.8) for the derivative of the spinor $\tau_{\boldsymbol{AA}'}$ and the split of $\nabla_{\boldsymbol{AB}}$ one obtains

$$\mathcal{P}\Lambda_{\boldsymbol{CD}} - 2\mathcal{D}^{\boldsymbol{AB}}\Lambda_{\boldsymbol{ABCD}} - 2\sqrt{2}\chi^{\boldsymbol{B}}{}_{\boldsymbol{P}}{}^{\boldsymbol{PA}}\Lambda_{\boldsymbol{ABCD}} = -2\nabla^{\boldsymbol{BB}'}\Lambda_{\boldsymbol{B}'\boldsymbol{BCD}},$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

where it is recalled that $\Lambda_{CD} \equiv \Lambda^{Q}{}_{QCD}$. Now, a calculation shows that the symmetry $\Lambda_{ABCD} = \Lambda_{A(BCD)}$ implies the decomposition

$$\Lambda_{ABCD} = \Lambda_{(ABCD)} - \frac{3}{4}\epsilon_{A(B}\Lambda_{CD)},$$

so that

$$\begin{aligned}\mathcal{P}\Lambda_{CD} - 2\mathcal{D}^{P}{}_{(C}\Lambda_{D)P} - 2\mathcal{D}^{AB}\Lambda_{(ABCD)} - 2\sqrt{2}\chi^{B}{}_{P}{}^{PA}\Lambda_{(ABCD)}& \\ + \frac{3}{\sqrt{2}}\chi^{B}{}_{P}{}^{PA}\epsilon_{A(B}\Lambda_{CD)} = -2\nabla^{BB'}\Lambda_{B'BCD}.& \qquad (13.47)\end{aligned}$$

As a second way of evaluating $\nabla^{BB'}\Lambda_{B'BCD}$ one makes use of the definition of the zero quantity so that

$$\nabla^{BB'}\Lambda_{B'BCD} = \nabla^{BB'}\nabla^{Q}{}_{B'}\phi_{BCDQ} + \nabla^{BB'}T_{CDBB'}.$$

The first term of the right-hand side is manipulated using the commutator

$$\begin{aligned}&\nabla_{AA'}\nabla_{BB'}\phi_{CDEF} - \nabla_{BB'}\nabla_{AA'}\phi_{CDEF} \\ &\qquad = -R^{S}{}_{CAA'BB'}\phi_{SDEF} - R^{S}{}_{DAA'BB'}\phi_{SCEF} - R^{S}{}_{EAA'BB'}\phi_{SCDF} \\ &\qquad\quad - R^{S}{}_{FAA'BB'}\phi_{SCDE} + \Sigma_{AA'}{}^{PP'}{}_{BB'}\nabla_{PP'}\phi_{CDEF}.\end{aligned}$$

Observe that the torsion $\Sigma_{AA'}{}^{PP'}{}_{BB'}$, being one of the unknowns in the subsidiary system, needs to be included in the commutator. Also, the curvature terms in the above expression are understood to be those of the *geometric curvature*. Contracting the expression of the commutator leads to

$$\begin{aligned}2\nabla^{BB'}\nabla^{Q}{}_{B'}\phi_{BCDQ} = &-R^{S}{}_{C}{}^{BA'Q}{}_{A'}\phi_{SDBQ} - R^{S}{}_{D}{}^{BA'Q}{}_{A'}\phi_{SCBQ} \\ &- R^{S}{}_{B}{}^{BA'Q}{}_{A'}\phi_{SCDQ} - R^{S}{}_{Q}{}^{BA'Q}{}_{A'}\phi_{SCDB} \\ &+ \Sigma^{AA'SS'Q}{}_{A'}\nabla_{SS'}\phi_{CDAQ}.\end{aligned}$$

Using the zero quantity

$$\Xi^{C}{}_{DAA'BB'} = R^{C}{}_{DAA'BB'} - \rho^{C}{}_{DAA'BB'},$$

to eliminate the geometric curvature and the decomposition (13.9) one obtains an expression which is homogenous in zero quantities:

$$\begin{aligned}2\nabla^{BB'}\nabla^{Q}{}_{B'}\phi_{BCDQ} = &-\Xi^{S}{}_{C}{}^{BA'Q}{}_{A'}\phi_{SDBQ} - \Xi^{S}{}_{D}{}^{BA'Q}{}_{A'}\phi_{SCBQ} \\ &- \Xi^{S}{}_{B}{}^{BA'Q}{}_{A'}\phi_{SCDQ} - \Xi^{S}{}_{Q}{}^{BA'Q}{}_{A'}\phi_{SCDB} \\ &+ \Sigma^{AA'SS'Q}{}_{A'}\nabla_{SS'}\phi_{CDAQ}. \qquad (13.48)\end{aligned}$$

In particular, all the terms coming from the algebraic curvature cancel out.

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

Combining Equations (13.47) and (13.48) one obtains the required subsidiary equation. Namely, one has that if $\Lambda_{(\boldsymbol{ABCD})} = 0$, then

$$\begin{aligned}
\mathcal{P}\Lambda^{\boldsymbol{Q}}{}_{\boldsymbol{QCD}} - 2\mathcal{D}^{\boldsymbol{P}}{}_{(\boldsymbol{C}}\Lambda^{\boldsymbol{Q}}{}_{\boldsymbol{D})\boldsymbol{PQ}} + \frac{3}{\sqrt{2}}\chi^{\boldsymbol{B}}{}_{\boldsymbol{P}}{}^{\boldsymbol{PA}}\epsilon_{\boldsymbol{A}(\boldsymbol{B}}\Lambda^{\boldsymbol{Q}}{}_{\boldsymbol{CD})\boldsymbol{Q}}
&= \Xi^{\boldsymbol{S}}{}_{\boldsymbol{C}}{}^{\boldsymbol{BA'Q}}{}_{\boldsymbol{A'}}\phi_{\boldsymbol{SDBQ}} + \Xi^{\boldsymbol{S}}{}_{\boldsymbol{D}}{}^{\boldsymbol{BA'Q}}{}_{\boldsymbol{A'}}\phi_{\boldsymbol{SCBQ}} \\
&\quad + \Xi^{\boldsymbol{S}}{}_{\boldsymbol{B}}{}^{\boldsymbol{BA'Q}}{}_{\boldsymbol{A'}}\phi_{\boldsymbol{SCDQ}} + \Xi^{\boldsymbol{S}}{}_{\boldsymbol{Q}}{}^{\boldsymbol{BA'Q}}{}_{\boldsymbol{A'}}\phi_{\boldsymbol{SCDB}} \\
&\quad - 2\Sigma^{\boldsymbol{AA'SS'Q}}{}_{\boldsymbol{A'}}\nabla_{\boldsymbol{SS'}}\phi_{\boldsymbol{CDAQ}} - 2\nabla^{\boldsymbol{BB'}}T_{\boldsymbol{CDBB'}},
\end{aligned}$$

which is homogeneous in zero quantities if the matter term $\nabla^{\boldsymbol{BB'}}T_{\boldsymbol{CDBB'}}$ can be expressed, in turn, as a homogeneous expression of matter zero quantities.

Alternatively, one can perform the computation with tensorial objects. In this case one looks at

$$\nabla^{\boldsymbol{b}}\Lambda_{\boldsymbol{bcd}} = \nabla^{\boldsymbol{b}}\nabla_{\boldsymbol{a}}d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} - \nabla^{\boldsymbol{b}}T_{\boldsymbol{cdb}}.$$

Again, using the properties of the Hodge dual one can write

$$\begin{aligned}
\nabla^{\boldsymbol{b}}\nabla_{\boldsymbol{a}}d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} &= -\nabla^{\boldsymbol{a}}\nabla^{\boldsymbol{b}}d_{\boldsymbol{abcd}} \\
&= \nabla^{\boldsymbol{a}}\nabla^{\boldsymbol{b}*}d^{*}_{\boldsymbol{abcd}} = \frac{1}{4}\epsilon^{\boldsymbol{abef}}\epsilon^{\boldsymbol{gh}}{}_{\boldsymbol{cd}}\nabla_{\boldsymbol{b}}\nabla_{\boldsymbol{a}}d_{\boldsymbol{efgh}} \\
&= \frac{1}{4}\epsilon^{\boldsymbol{abef}}\epsilon^{\boldsymbol{gh}}{}_{\boldsymbol{cd}}\big(R^{\boldsymbol{s}}{}_{\boldsymbol{eab}}d_{\boldsymbol{fsgh}} + R^{\boldsymbol{s}}{}_{\boldsymbol{gab}}d_{\boldsymbol{hsef}} - \frac{1}{2}\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{s}}{}_{\boldsymbol{b}}\nabla_{\boldsymbol{s}}d_{\boldsymbol{efgh}}\big) \\
&= \frac{1}{4}\epsilon^{\boldsymbol{abef}}\epsilon^{\boldsymbol{gh}}{}_{\boldsymbol{cd}}\big(\Xi^{\boldsymbol{s}}{}_{\boldsymbol{eab}}d_{\boldsymbol{fsgh}} + \Xi^{\boldsymbol{s}}{}_{\boldsymbol{gab}}d_{\boldsymbol{hsef}} - \frac{1}{2}\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{s}}{}_{\boldsymbol{b}}\nabla_{\boldsymbol{s}}d_{\boldsymbol{efgh}}\big).
\end{aligned}$$

Hence, one concludes that $\nabla^{\boldsymbol{b}}\Lambda_{\boldsymbol{bcd}}$, except for the *matter term* $\nabla^{\boldsymbol{b}}T_{\boldsymbol{cdb}}$ can be written as a linear combination of zero quantities.

13.3.7 Summary

In most applications, the detailed form of the evolution and subsidiary equations is not required; general structural properties suffice. These properties are summarised in the following propositions.

It is convenient to group the independent components of the unknowns appearing in the spinorial formulation of the conformal field equations in the following manner:

- $\boldsymbol{\sigma}$ independent components of Ξ, $\Sigma_{\boldsymbol{AA'}}$, s;
- $\boldsymbol{\upsilon}$ independent components of $e^{\mu}_{\boldsymbol{AA'}}$, $\Gamma_{\boldsymbol{AA'BC}}$, $\Phi_{\boldsymbol{AA'BB'}}$;
- $\boldsymbol{\phi}$ independent components of $\phi_{\boldsymbol{ABCD}}$;
- $\boldsymbol{\varphi}$ independent components of matter fields.

Moreover, let $\boldsymbol{e}$ and $\boldsymbol{\Gamma}$ denote, respectively, the independent components of the frame components and the connection coefficients. In terms of these objects one has the following:

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

Proposition 13.1 (***properties of the conformal evolution equations***) *Given arbitrary smooth gauge source functions*

$$F^{\boldsymbol{a}}(x), \qquad F_{\boldsymbol{AB}}(x), \qquad R(x),$$

such that

$$\nabla^{\boldsymbol{QQ}'} e_{\boldsymbol{QQ}'}{}^{\boldsymbol{a}} = F^{\boldsymbol{a}}(x), \qquad \nabla^{\boldsymbol{QQ}'} \Gamma_{\boldsymbol{QQ}'\boldsymbol{AB}} = F_{\boldsymbol{AB}}(x),$$
$$\nabla^{\boldsymbol{QQ}'} L_{\boldsymbol{QQ}'\boldsymbol{BB}'} = \frac{1}{6}\nabla_{\boldsymbol{BB}'}R(x),$$

and assuming that the components of the matter tensors $T_{\boldsymbol{ab}}$ and $T_{\boldsymbol{abc}}$ can be written in such a way that they do not contain derivatives of the matter fields, then the conformal Einstein field equations

$$Q_{\boldsymbol{a}} = 0, \qquad Z_{\boldsymbol{ab}} = 0, \qquad Z_{\boldsymbol{a}} = 0, \qquad \Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0, \qquad \Xi^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = 0,$$
$$\Delta_{\boldsymbol{abc}} = 0, \qquad \Lambda_{\boldsymbol{abc}} = 0,$$

imply a symmetric hyperbolic system of equations for the independent components of the geometric fields $(\boldsymbol{\sigma}, \boldsymbol{\upsilon}, \boldsymbol{\phi})$ of the form

$$\big(\mathbf{I} + \mathbf{A}^0(\boldsymbol{e})\big)\partial_\tau \boldsymbol{\phi} + \mathbf{A}^\alpha(\boldsymbol{e})\partial_\alpha \boldsymbol{\phi} = \mathbf{B}(\boldsymbol{\Gamma})\boldsymbol{\phi} + \mathbf{C}(\boldsymbol{\sigma}, \boldsymbol{\upsilon}, \boldsymbol{\varphi}),$$
$$\big(\mathbf{I} + \mathbf{D}^0(\boldsymbol{e})\big)\partial_\tau \boldsymbol{\upsilon} + \mathbf{D}^\mu(\boldsymbol{e})\partial_\mu \boldsymbol{\upsilon} = \mathbf{E}(\boldsymbol{\Gamma})\boldsymbol{\upsilon} + \mathbf{F}(\boldsymbol{\sigma}, \boldsymbol{\upsilon}, \boldsymbol{\phi}, \boldsymbol{\varphi}),$$
$$\partial_\tau \boldsymbol{\sigma} = \mathbf{G}(\boldsymbol{\Gamma})\boldsymbol{\sigma} + \mathbf{H}(\boldsymbol{\sigma}, \boldsymbol{\upsilon}, \boldsymbol{\varphi}),$$

where $\mathbf{I}$ denotes the identity matrix of the required dimensions,

$$\mathbf{A}^\mu(\boldsymbol{e}), \qquad \mathbf{D}^\mu(\boldsymbol{e})$$

are smooth matrix-valued functions of the components of the frame components,

$$\mathbf{B}(\boldsymbol{\Gamma}), \qquad \mathbf{E}(\boldsymbol{\Gamma}), \qquad \mathbf{G}(\boldsymbol{\Gamma})$$

are smooth matrix-valued functions of the connection coefficients and

$$\mathbf{C}(\boldsymbol{\sigma}, \boldsymbol{\upsilon}, \boldsymbol{\varphi}), \qquad \mathbf{F}(\boldsymbol{\sigma}, \boldsymbol{\upsilon}, \boldsymbol{\phi}, \boldsymbol{\varphi}), \qquad \mathbf{H}(\boldsymbol{\sigma}, \boldsymbol{\upsilon}, \boldsymbol{\varphi})$$

are smooth vector-valued functions with polynomial dependence on their arguments. The characteristics of this system satisfy a characteristic polynomial involving factors of the form

$$\tau^\mu \xi_\mu, \qquad g^{\nu\lambda}\xi_\nu\xi_\lambda, \qquad \left(\tau^\rho\tau^\sigma + \frac{2}{3}g^{\rho\sigma}\right)\xi_\rho\xi_\sigma.$$

Remarks

(i) In the presence of matter, the symmetric hyperbolic system given in the above proposition needs to the supplemented by a symmetric hyperbolic system for the matter fields. As the rescaled Cotton tensor $T_{\boldsymbol{abc}}$ (and hence also the spinor $T_{\boldsymbol{ABCC}'}$) is made up of derivatives of the energy-momentum tensor, the *matter evolution equations* will need to include equations for the matter field derivatives appearing in the *geometric evolution equations*.

(ii) The choice of the gauge source functions is dictated by the particular analysis under consideration.

With regards to the subsidiary system one has the following:

Proposition 13.2 (***properties of the subsidiary evolution system***) *Assume that the evolution equations implied by the conformal Einstein field equations are satisfied and that the energy-momentum tensor $T_{\boldsymbol{ab}}$ is such that the quantities*

$$M_{\boldsymbol{cd}} \equiv \nabla^{\boldsymbol{b}} T_{\boldsymbol{cdb}},$$

$$N_{\boldsymbol{bf}} \equiv \epsilon_{\boldsymbol{f}}{}^{\boldsymbol{ecd}} \left(\frac{1}{2} \Xi^3 T_{\boldsymbol{ea}} d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} + \nabla_{\boldsymbol{e}} \Xi T_{\boldsymbol{cdb}} - \Xi \nabla_{\boldsymbol{e}} T_{\boldsymbol{cdb}} \right) - \epsilon_{\boldsymbol{ab}}{}^{\boldsymbol{gh}} \Sigma^{\boldsymbol{a}} T_{\boldsymbol{fgh}},$$

can be written as homogeneous expressions of the geometric and matter zero quantities. Then the zero quantities encoding the constraint equations implied by the conformal Einstein equations under the hyperbolic reduction procedure leading to Proposition 13.1 satisfy a symmetric hyperbolic system which is a homogeneous expression of zero quantities.

13.4 Hyperbolic reductions using conformal Gaussian systems

This section discusses a hyperbolic reduction procedure based on the properties of congruences of conformal geodesics. The approach discussed in this section makes use of the formulation of the conformal field equations in terms of Weyl connections – the so-called extended conformal field equations. As will be seen, this procedure leads to simpler evolution equations than the ones obtained by the reduction procedure discussed in Section 13.2.

For conciseness of the presentation, the discussion in the rest of this section is restricted to the vacuum case.

13.4.1 Basic set up

In what follows, it is assumed one has a region $\mathcal{U}$ of a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ which is covered by a congruence of conformal geodesics $(\dot{\boldsymbol{x}}(\tau), \boldsymbol{\beta}(\tau))$. For convenience, the vector field tangent to the congruence will be denoted by $\boldsymbol{\tau}$. As discussed in Section 5.5, a canonical representative $\boldsymbol{g}$ of the conformal class $[\tilde{\boldsymbol{g}}]$ is singled out by the requirement

$$\boldsymbol{g}(\boldsymbol{\tau}, \boldsymbol{\tau}) = 1,$$

so that $\boldsymbol{g} = \Theta^2 \tilde{\boldsymbol{g}}$ where the conformal factor Θ satisfies a third-order ordinary differential equation along the congruence of conformal geodesics; see Equation (5.53b). In the case of a vacuum spacetime this equation can be explicitly solved yielding a formula for Θ as a quadratic polynomial in the parameter τ. The conformal factor is completely determined by the three coefficients $\Theta_\star$, $\dot{\Theta}_\star$, $\ddot{\Theta}_\star$ specified, say, on an initial hypersurface $\tilde{\mathcal{S}}_\star$.

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

In the following, $\{\boldsymbol{e_a}\}$ will denote a $\boldsymbol{g}$-orthonormal frame which is Weyl-propagated along the conformal geodesics and such that $\boldsymbol{e_0} = \boldsymbol{\tau}$. As discussed in Section 5.5, to every congruence of conformal geodesics one can associate a Weyl connection $\hat{\boldsymbol{\nabla}}$. This Weyl connection satisfies the relations

$$\hat{\nabla}_{\boldsymbol{\tau}}\boldsymbol{e_a} = 0, \qquad \hat{\boldsymbol{L}}(\boldsymbol{\tau}, \cdot) = 0,$$

where $\hat{\boldsymbol{L}}$ denotes the Schouten tensor of $\hat{\boldsymbol{\nabla}}$; see Equation (5.41). In terms of frame components, the above conditions can be rewritten as

$$\hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{a}}{}_{\boldsymbol{b}} = 0, \qquad \hat{L}_{\boldsymbol{0a}} = 0. \tag{13.49}$$

In particular, it follows that the covector $\boldsymbol{f}$ which defines the Weyl connection $\hat{\boldsymbol{\nabla}}$ satisfies

$$f_{\boldsymbol{0}} = 0.$$

The gauge choice can be refined further by choosing the parameter of the conformal geodesics τ as the time coordinate. Thus, one has the additional gauge condition

$$\boldsymbol{e_0} = \boldsymbol{\partial}_\tau, \qquad \text{so that} \qquad \boldsymbol{e_0}{}^\mu = \delta_0{}^\mu. \tag{13.50}$$

In most applications, initial data for the congruence of conformal geodesics will be prescribed on the initial hypersurface $\mathcal{S}_\star$. On $\mathcal{S}_\star$ choose some local coordinates (x^α). Assuming that each curve of the congruence of conformal geodesics intersects $\mathcal{S}_\star$ only once, one can extend coordinates on $\mathcal{S}_\star$ off the hypersurface by requiring them to be constant along the conformal geodesic which intersects $\mathcal{S}_\star$ at the point with coordinates (x^α); see Figure 13.1. The spacetime coordinates (τ, x^α) one obtains by this procedure are known as ***conformal Gaussian coordinates***. More generally, the collection of the conformal factor Θ, Weyl-propagated frame vectors $\{\boldsymbol{e_a}\}$ and coordinates (τ, x^α) extended off some initial hypersurface $\mathcal{S}_\star$ using a congruence of conformal geodesics will be known as a ***conformal Gaussian gauge system***.

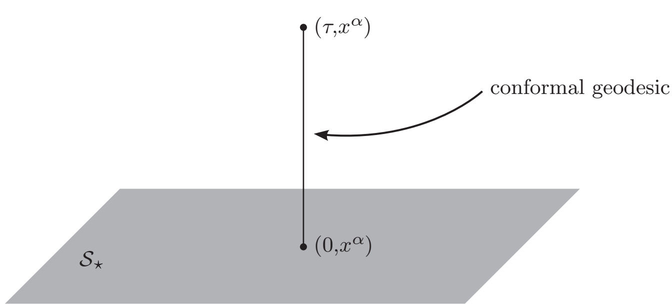

Figure 13.1 Schematic depiction of the construction of conformal Gaussian coordinates. The coordinates (x^α) of a point $p \in \mathcal{S}_\star$ are propagated off the hypersurface along the unique conformal geodesic passing through p; see the main text for further details.

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

Remarks

(i) The specific choice of the data for the conformal Gaussian gauge system on an initial hypersurface $\mathcal{S}$ is dictated by the particular geometric setting under consideration.
(ii) The discussion in this section can be adapted, with minor changes, to the case of a congruence of so-called *conformal curves* in non-vacuum spacetimes; see, for example, Lübbe and Valiente Kroon (2012) for further details.

13.4.2 Hyperbolic reduction of a model equation

The general ideas behind the procedure of hyperbolic reduction using conformal Gaussian systems are best illustrated with a model equation. All the extended conformal equations, except for the unphysical Bianchi identity, are of the form

$$\hat{\nabla}_{\boldsymbol{a}} M_{\boldsymbol{b}\mathcal{K}} - \hat{\nabla}_{\boldsymbol{b}} M_{\boldsymbol{a}\mathcal{K}} = N_{\boldsymbol{ab}\mathcal{K}}, \tag{13.51}$$

where $M_{\boldsymbol{a}\mathcal{K}}$ and $N_{\boldsymbol{ab}\mathcal{K}} = N_{[\boldsymbol{ab}]\mathcal{K}}$ denote the components of some tensorial quantities with respect to the frame $\{\boldsymbol{e_a}\}$ and $_\mathcal{K}$ denotes an arbitrary set of tensor indices. To derive an evolution equation along the direction given by the congruence of curves, one sets $\boldsymbol{a} = \boldsymbol{0}$ so that

$$\hat{\nabla}_{\boldsymbol{0}} M_{\boldsymbol{b}\mathcal{K}} - \hat{\nabla}_{\boldsymbol{b}} M_{\boldsymbol{0}\mathcal{K}} = N_{\boldsymbol{0b}\mathcal{K}},$$

or, more explicitly,

$$\boldsymbol{e_0}(M_{\boldsymbol{b}\mathcal{K}}) - \boldsymbol{e_b}(M_{\boldsymbol{0}\mathcal{K}}) = N_{\boldsymbol{0b}\mathcal{K}} + \hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} M_{\boldsymbol{c}\mathcal{K}} + \hat{\Gamma}_{\boldsymbol{0}}{}^{\mathcal{L}}{}_{\mathcal{K}} M_{\boldsymbol{b}\mathcal{L}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{0}} M_{\boldsymbol{c}\mathcal{K}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\mathcal{L}}{}_{\mathcal{K}} M_{\boldsymbol{0}\mathcal{L}}.$$

If the gauge conditions (13.49) are taken into account and coordinates are chosen such that $\boldsymbol{e_0} = \boldsymbol{\partial}_\tau$, then the above equation reduces to

$$\partial_\tau M_{\boldsymbol{b}\mathcal{K}} - \boldsymbol{e_b}(M_{\boldsymbol{0}\mathcal{K}}) = N_{\boldsymbol{0b}\mathcal{K}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{0}} M_{\boldsymbol{c}\mathcal{K}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\mathcal{L}}{}_{\mathcal{K}} M_{\boldsymbol{0}\mathcal{L}}. \tag{13.52}$$

This last equation is not a completely satisfactory evolution equation for the components $M_{\boldsymbol{a}\mathcal{K}}$ as it does not yield information about $\partial_\tau M_{\boldsymbol{0}\mathcal{K}}$ – notice that by setting $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{0}$ in (13.51) both sides of the equation vanish as a result of the skew symmetry of the equation. To read Equation (13.52) as a suitable evolution equation one needs to know the value of the time component $M_{\boldsymbol{0}\mathcal{K}}$ either as a result of symmetries of the tensor $M_{a\mathcal{K}}$ or through some gauge condition. In any of these cases, Equation (13.52) is just a ***transport equation*** along the congruence of conformal curves, and, accordingly, it trivially gives rise to a symmetric hyperbolic subsystem of equations.

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

Analysis in terms of spinors

In view of subsequent applications, the properties of the spinorial counterpart of Equation (13.51) are now analysed. In this case one has

$$\hat{\nabla}_{\boldsymbol{AA'}}M_{\boldsymbol{BB'}\mathcal{K}} - \hat{\nabla}_{\boldsymbol{BB'}}M_{\boldsymbol{AA'}\mathcal{K}} = N_{\boldsymbol{AA'BB'}\mathcal{K}}, \tag{13.53}$$

where $\mathcal{K}$ denotes an arbitrary string of spinorial indices. In view of its antisymmetry, Equation (13.53) is completely equivalent to the pair of contracted equations

$$\hat{\nabla}_{(\boldsymbol{A}|\boldsymbol{P'}|}M_{\boldsymbol{B})}{}^{\boldsymbol{P'}}{}_{\mathcal{K}} = \frac{1}{2}N_{(\boldsymbol{A}|\boldsymbol{P'}|\boldsymbol{B})}{}^{\boldsymbol{P'}}{}_{\mathcal{K}}, \tag{13.54a}$$

$$\hat{\nabla}_{\boldsymbol{P}(\boldsymbol{A'}}M_{\boldsymbol{B'})}{}^{\boldsymbol{P}}{}_{\mathcal{K}} = \frac{1}{2}N_{\boldsymbol{P}(\boldsymbol{A'}}{}^{\boldsymbol{P}}{}_{\boldsymbol{B'})\mathcal{K}}. \tag{13.54b}$$

Thus, not unsurprisingly, one has arrived at a situation similar to the one analysed in Section 13.2. Namely, one has equations containing a symmetrised spinorial curl. A symmetric hyperbolic system can then be obtained if suitable information about the divergence $\hat{\nabla}^{\boldsymbol{PP}}M_{\boldsymbol{PP'}\mathcal{K}}$ is available.

The next step in the procedure consists of introducing the space spinor version of $M_{\boldsymbol{AA'}\mathcal{K}}$, namely, $M_{\boldsymbol{BB'}\mathcal{K}} = -\tau^{\boldsymbol{P}}{}_{\boldsymbol{B'}}M_{\boldsymbol{BP}\mathcal{K}}$ so that

$$\begin{aligned}\tau_{\boldsymbol{P}}{}^{\boldsymbol{A'}}\tau_{\boldsymbol{Q}}{}^{\boldsymbol{B'}}\hat{\nabla}_{\boldsymbol{AA'}}M_{\boldsymbol{BB'}\mathcal{K}} &= -\tau_{\boldsymbol{P}}{}^{\boldsymbol{A'}}\tau_{\boldsymbol{Q}}{}^{\boldsymbol{B'}}\left(\tau^{\boldsymbol{R}}{}_{\boldsymbol{B'}}\hat{\nabla}_{\boldsymbol{AA'}}M_{\boldsymbol{BR}\mathcal{K}} + M_{\boldsymbol{BR}\mathcal{K}}\hat{\nabla}_{\boldsymbol{AA'}}\tau^{\boldsymbol{R}}{}_{\boldsymbol{B'}}\right)\\ &= \hat{\nabla}_{\boldsymbol{AP}}M_{\boldsymbol{BQ}\mathcal{K}} - \sqrt{2}M_{\boldsymbol{BR}\mathcal{K}}\,\hat{\chi}_{\boldsymbol{AP}}{}^{\boldsymbol{R}}{}_{\boldsymbol{Q}},\end{aligned}$$

where it has been used that $\sqrt{2}\hat{\chi}_{\boldsymbol{ABCD}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\tau_{\boldsymbol{D}}{}^{\boldsymbol{C'}}\hat{\nabla}_{\boldsymbol{AA'}}\tau_{\boldsymbol{CC'}}$ consistent with formula (4.17). From the above identity together with Equations (13.54a) and (13.54b) one obtains

$$\hat{\nabla}_{(\boldsymbol{A}|\boldsymbol{P}|}M_{\boldsymbol{B})}{}^{\boldsymbol{P}}{}_{\mathcal{K}} = \frac{1}{2}N_{(\boldsymbol{A}|\boldsymbol{P}|\boldsymbol{B})}{}^{\boldsymbol{P}}{}_{\mathcal{K}} - \sqrt{2}M_{(\boldsymbol{A}}{}^{\boldsymbol{R}}{}_{|\mathcal{K}|}\,\hat{\chi}_{\boldsymbol{B})\boldsymbol{PR}}{}^{\boldsymbol{P}},$$

$$\hat{\nabla}_{\boldsymbol{A}(\boldsymbol{P}}M^{\boldsymbol{A}}{}_{\boldsymbol{Q})\mathcal{K}} = \frac{1}{2}N_{\boldsymbol{A}(\boldsymbol{P}}{}^{\boldsymbol{A}}{}_{\boldsymbol{Q})\mathcal{K}} + \sqrt{2}M^{\boldsymbol{A}}{}_{\boldsymbol{R}\mathcal{K}}\,\hat{\chi}_{\boldsymbol{A}(\boldsymbol{P}}{}^{\boldsymbol{R}}{}_{\boldsymbol{Q})}.$$

Using the decomposition

$$\hat{\nabla}_{\boldsymbol{AB}} = \frac{1}{2}\epsilon_{\boldsymbol{AB}}\hat{\mathcal{P}} + \hat{\mathcal{D}}_{\boldsymbol{AB}},$$

with $\hat{\mathcal{P}} \equiv \tau^{\boldsymbol{AA'}}\hat{\nabla}_{\boldsymbol{AA'}}$ and $\hat{\mathcal{D}}_{\boldsymbol{AB}} \equiv \tau_{(\boldsymbol{A}}{}^{\boldsymbol{A'}}\hat{\nabla}_{\boldsymbol{B})\boldsymbol{A'}}$ – compare Equation (4.16) – and writing $M_{\boldsymbol{AB}\mathcal{K}}$ as

$$M_{\boldsymbol{AB}\mathcal{K}} = \frac{1}{2}\epsilon_{\boldsymbol{AB}}m_{\mathcal{K}} + m_{\boldsymbol{AB}\mathcal{K}}$$

where $m_{\mathcal{K}} \equiv M_{\boldsymbol{Q}}{}^{\boldsymbol{Q}}{}_{\mathcal{K}}$ and $m_{\boldsymbol{AB}\mathcal{K}} \equiv M_{(\boldsymbol{AB})\mathcal{K}}$, one obtains

$$\begin{aligned}&\frac{1}{2}\hat{\mathcal{P}}m_{\boldsymbol{AB}\mathcal{K}} - \frac{1}{2}\hat{\mathcal{D}}_{\boldsymbol{AB}}m_{\mathcal{K}} - \hat{\mathcal{D}}_{\boldsymbol{P}(\boldsymbol{A}}m_{\boldsymbol{B})}{}^{\boldsymbol{P}}{}_{\mathcal{K}}\\ &\qquad\qquad = -\frac{1}{2}N_{(\boldsymbol{A}|\boldsymbol{P}|\boldsymbol{B})}{}^{\boldsymbol{P}}{}_{\mathcal{K}} + \sqrt{2}M_{(\boldsymbol{A}}{}^{\boldsymbol{R}}{}_{|\mathcal{K}|}\,\hat{\chi}_{\boldsymbol{B})\boldsymbol{PR}}{}^{\boldsymbol{P}}\end{aligned}$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

$$\frac{1}{2}\hat{\mathcal{P}}m_{\boldsymbol{PQ}\mathcal{K}} - \frac{1}{2}\hat{\mathcal{D}}_{\boldsymbol{PQ}}m_{\mathcal{K}} + \hat{\mathcal{D}}_{\boldsymbol{A}(\boldsymbol{P}}m^{\boldsymbol{A}}{}_{\boldsymbol{Q})\mathcal{K}}$$
$$= \frac{1}{2}N_{\boldsymbol{A}(\boldsymbol{P}}{}^{\boldsymbol{A}}{}_{\boldsymbol{Q})\mathcal{K}} + \sqrt{2}M^{\boldsymbol{A}}{}_{\boldsymbol{R}\mathcal{K}}\,\hat{\chi}_{\boldsymbol{A}(\boldsymbol{P}}{}^{\boldsymbol{R}}{}_{\boldsymbol{Q})}.$$

Taking linear combinations of the latter equations one finally arrives at

$$\hat{\mathcal{P}}m_{\boldsymbol{AB}\mathcal{K}} - \hat{\mathcal{D}}_{\boldsymbol{AB}}m_{\mathcal{K}} = E_{\boldsymbol{AB}\mathcal{K}}, \tag{13.55a}$$

$$\hat{\mathcal{D}}_{\boldsymbol{P}(\boldsymbol{A}}m^{\boldsymbol{P}}{}_{\boldsymbol{B})\mathcal{K}} = C_{\boldsymbol{AB}\mathcal{K}}, \tag{13.55b}$$

where $E_{\boldsymbol{AB}\mathcal{K}}$ and $C_{\boldsymbol{AB}\mathcal{K}}$ are some expressions not involving derivatives of $M_{\boldsymbol{AB}\mathcal{K}}$ whose precise form is not relevant for the subsequent discussion. Equation (13.55a) can be read as an evolution equation for the *spatial components* $m_{\boldsymbol{AB}\mathcal{K}}$ if the *time component* $m_{\mathcal{K}}$ is known. Observe that the reduction procedure does not produce an equivalent equation for $m_{\mathcal{K}}$ consistent with the discussion of Equation (13.51).

13.4.3 The evolution equations in the frame formalism

To obtain some intuition into the structural properties of the evolution equations, it is convenient to look first at the form of the equations in a tensor frame formalism. Accordingly, one considers the vacuum extended conformal field equations as given in Section 8.4.1; see Equations (8.46).

The required evolution equations for the frame components, connection coefficients and components of the Schouten tensor are obtained from the conditions

$$\hat{\Sigma}_{\boldsymbol{0b}} = 0, \qquad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{d0b}} = 0, \qquad \hat{\Delta}_{\boldsymbol{0bc}} = 0.$$

In particular, the evolution equation for the covector $\boldsymbol{f}$ defining the Weyl connection is given by

$$\hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{c0b}} = 0.$$

Using the definitions of the zero quantities given in Equations (8.44a)–(8.44c), recalling that in the vacuum case $T_{\boldsymbol{cdb}} = 0$, and making use of the gauge conditions (13.49) and (13.50), one obtains the evolution equations

$$\partial_\tau e_{\boldsymbol{b}}{}^{\mu} = -\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}e_{\boldsymbol{f}}{}^{\mu},$$
$$\partial_\tau \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} = -\hat{\Gamma}_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}} + \delta_{\boldsymbol{0}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{bd}} + \delta_{\boldsymbol{d}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{b0}} - \eta_{\boldsymbol{0d}}\eta^{\boldsymbol{fc}}\hat{L}_{\boldsymbol{bf}} + \Theta d^{\boldsymbol{c}}{}_{\boldsymbol{d0b}},$$
$$\partial_\tau \hat{L}_{\boldsymbol{bc}} = -\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}\hat{L}_{\boldsymbol{fc}} + d_{\boldsymbol{f}}d^{\boldsymbol{f}}{}_{\boldsymbol{c0b}}.$$

These equations contain derivatives only in the τ direction – that is, they are transport equations along the conformal geodesics.

The evolution equations for the components of the rescaled Weyl tensor are obtained by resorting to an electric-magnetic decomposition; see Section 11.1.2. Using Equations (11.9) and (11.10) for the decomposition of a Weyl candidate in the equations

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

$$\nabla^{\boldsymbol{a}} d_{\boldsymbol{abcd}} = 0, \qquad \nabla^{\boldsymbol{a}} d^*_{\boldsymbol{abcd}} = 0,$$

one obtains the expressions

$$\begin{aligned}\Lambda^*_{(\boldsymbol{b}|\boldsymbol{0}|\boldsymbol{d})} &= \boldsymbol{e_0}(E_{\boldsymbol{bd}}) + D_{\boldsymbol{a}} B_{\boldsymbol{c}(\boldsymbol{b}} \epsilon_{\boldsymbol{d})}{}^{\boldsymbol{ac}} + 2a_{\boldsymbol{a}} \epsilon^{\boldsymbol{ac}}{}_{(\boldsymbol{b}} B_{\boldsymbol{d})\boldsymbol{c}} - 3\chi_{(\boldsymbol{b}}{}^{\boldsymbol{c}} E_{\boldsymbol{d})\boldsymbol{c}} \\ &\quad - \epsilon_{\boldsymbol{b}}{}^{\boldsymbol{ac}} \epsilon_{\boldsymbol{d}}{}^{\boldsymbol{ef}} E_{\boldsymbol{ac}} \chi_{\boldsymbol{cf}} + \chi E_{\boldsymbol{bd}} = 0, \\ \Lambda_{(\boldsymbol{b}|\boldsymbol{0}|\boldsymbol{d})} &= \boldsymbol{e_0}(B_{\boldsymbol{bd}}) - D_{\boldsymbol{a}} E_{\boldsymbol{c}(\boldsymbol{b}} \epsilon_{\boldsymbol{d})}{}^{\boldsymbol{ac}} - 2a_{\boldsymbol{a}} \epsilon^{\boldsymbol{ac}}{}_{(\boldsymbol{b}} E_{\boldsymbol{d})\boldsymbol{c}} - 3\chi^{\boldsymbol{a}}{}_{(\boldsymbol{b}} B_{\boldsymbol{d})\boldsymbol{a}} \\ &\quad - \epsilon_{\boldsymbol{b}}{}^{\boldsymbol{ac}} \epsilon_{\boldsymbol{d}}{}^{\boldsymbol{ef}} B_{\boldsymbol{ac}} \chi_{\boldsymbol{cf}} + \chi B_{\boldsymbol{bd}} = 0,\end{aligned}$$

where

$$h_{\boldsymbol{ab}} \equiv g_{\boldsymbol{ab}} - \tau_{\boldsymbol{a}}\tau_{\boldsymbol{b}}, \qquad \chi_{\boldsymbol{ab}} = h_{\boldsymbol{a}}{}^{\boldsymbol{c}} \nabla_{\boldsymbol{c}} \tau_{\boldsymbol{b}}, \qquad \chi = h^{\boldsymbol{ab}} \chi_{\boldsymbol{ab}}, \qquad a_{\boldsymbol{a}} \equiv \tau^{\boldsymbol{b}} \nabla_{\boldsymbol{b}} \tau_{\boldsymbol{a}}.$$

The above form of the equations is completely general. In the particular case of a conformal Gaussian gauge system one has $\boldsymbol{e_0} = \boldsymbol{\partial}_\tau$.

13.4.4 The evolution equations in the spinorial formalism

To discuss the spinorial version of the evolution equations one makes use of the extended conformal field equations

$$\hat{\Sigma}_{\boldsymbol{AA'BB'}} = 0, \qquad \hat{\Xi}^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} = 0, \qquad \hat{\Delta}_{\boldsymbol{CC'DD'BB'}} = 0, \qquad \hat{\Lambda}_{\boldsymbol{BB'CD}} = 0,$$

with the zero quantities as given in (8.53a)–(8.53e). These equations are regarded as differential conditions on the fields

$$\boldsymbol{e_{AA'}}{}^{\mu}, \qquad \hat{\Gamma}_{\boldsymbol{AA'BC}}, \qquad \hat{L}_{\boldsymbol{AA'BB'}}, \qquad \phi_{\boldsymbol{ABCD}}.$$

Moreover, one considers the spinor $\tau^{AA'}$ – the counterpart of the vector $\boldsymbol{\tau}$, with normalisation $\tau_{AA'}\tau^{AA'} = 2$. In terms of a spinor dyad $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$ adapted to $\tau^{AA'}$ one has

$$\tau^{AA'} = \epsilon_{\boldsymbol{0}}{}^{A} \epsilon_{\boldsymbol{0'}}{}^{A'} + \epsilon_{\boldsymbol{1}}{}^{A} \epsilon_{\boldsymbol{1'}}{}^{A'}.$$

In what follows, all spinorial objects will be expressed with respect to this basis. In particular, the components of $\tau^{AA'}$ with respect to $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$ will be denoted by $\tau^{\boldsymbol{AA'}}$.

The gauge conditions (13.49) and (13.50) in the spinorial formalism take the form

$$\tau^{\boldsymbol{AA'}} \boldsymbol{e_{AA'}} = \sqrt{2}\boldsymbol{\partial}_\tau, \qquad \tau^{\boldsymbol{AA'}} \hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} = 0, \qquad \tau^{\boldsymbol{AA'}} \hat{L}_{\boldsymbol{AA'BB'}} = 0. \quad (13.56)$$

For future use, it is recalled that the reduced spin Weyl connection coefficients $\hat{\Gamma}_{\boldsymbol{CC'AB}}$ can be written in terms of the unphysical Levi-Civita connection coefficients $\Gamma_{\boldsymbol{CC'AB}}$ and the covector $f_{\boldsymbol{AA'}}$ as

$$\hat{\Gamma}_{\boldsymbol{CC'AB}} = \Gamma_{\boldsymbol{CC'AB}} - \epsilon_{\boldsymbol{AC}} f_{\boldsymbol{BC'}}.$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

Combining the above with the gauge conditions one obtains

$$\tau^{\boldsymbol{CC'}}\Gamma_{\boldsymbol{CC'AB}} = -\tau_{\boldsymbol{A}}{}^{\boldsymbol{C'}} f_{\boldsymbol{BC'}} \tag{13.57}$$

and, furthermore, that

$$\hat{\Gamma}_{\boldsymbol{CC'AB}} = \Gamma_{\boldsymbol{CC'AB}} - \epsilon_{\boldsymbol{AC}}\tau^{\boldsymbol{QQ'}}\Gamma_{\boldsymbol{QQ'PB}}\tau^{\boldsymbol{P}}{}_{\boldsymbol{C'}}.$$

In the gauge given by conditions (13.56) *the connection coefficients* $\hat{\Gamma}_{\boldsymbol{CC'AB}}$ *can be fully expressed in terms of the coefficients* $\Gamma_{\boldsymbol{CC'AB}}$*, and vice versa.* Comparing Equation (13.57) with the definition in Equation (4.17), one sees that the spinor $f_{AA'}$ encodes the acceleration of the congruence of conformal geodesics. In particular, if $f_{AA'} = 0$, then the congruence consists of standard geodesics and one obtains a Gaussian gauge system.

The reduced symmetric hyperbolic system of evolution equations can be deduced from the following contractions of the conformal field equations

$$\tau^{\boldsymbol{AA'}}\hat{\Sigma}_{\boldsymbol{AA'}}{}^{\boldsymbol{PP'}}{}_{\boldsymbol{BB'}}e_{\boldsymbol{PP'}}{}^{\mu} = 0, \qquad \tau^{\boldsymbol{CC'}}\hat{\Xi}_{\boldsymbol{ABCC'DD'}} = 0,$$
$$\tau^{\boldsymbol{AA'}}\hat{\Delta}_{\boldsymbol{AA'BB'CC'}} = 0, \qquad \tau_{(\boldsymbol{A}}{}^{\boldsymbol{A'}}\hat{\Lambda}_{|\boldsymbol{A'}|\boldsymbol{BCD})} = 0.$$

Explicitly, for the first three equations one has

$$\begin{aligned}
\sqrt{2}\partial_\tau e_{\boldsymbol{AA'}}{}^{\mu} &= -\left(\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B}}\tau^{\boldsymbol{BQ'}} + \bar{\hat{\Gamma}}_{\boldsymbol{A'A}}{}^{\boldsymbol{Q'}}{}_{\boldsymbol{B'}}\tau^{\boldsymbol{QB'}}\right)e_{\boldsymbol{QQ'}}{}^{\mu},\\
\sqrt{2}\partial_\tau\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} &= -\left(\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{P}}{}_{\boldsymbol{Q}}\hat{\Gamma}_{\boldsymbol{PQ'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} + \bar{\hat{\Gamma}}_{\boldsymbol{AA'}}{}^{\boldsymbol{P'}}{}_{\boldsymbol{Q'}}\hat{\Gamma}_{\boldsymbol{QP'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}\right)\tau^{\boldsymbol{QQ'}}\\
&\quad + \hat{L}_{\boldsymbol{AA'CQ'}}\tau^{\boldsymbol{BQ'}} + \Theta\phi^{\boldsymbol{B}}{}_{\boldsymbol{CQA}}\tau^{\boldsymbol{Q}}{}_{\boldsymbol{A'}},\\
\sqrt{2}\partial_\tau\hat{L}_{\boldsymbol{AA'BB'}} &= -\left(\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{P}}{}_{\boldsymbol{Q}}\hat{L}_{\boldsymbol{PQ'BB'}} + \bar{\hat{\Gamma}}_{\boldsymbol{A'A}}{}^{\boldsymbol{P'}}{}_{\boldsymbol{Q'}}\hat{L}_{\boldsymbol{QP'BB'}}\right)\tau^{\boldsymbol{QQ'}}\\
&\quad - d^{\boldsymbol{PP'}}\left(\phi_{\boldsymbol{PAQB}}\epsilon_{\boldsymbol{P'B'}}\tau^{\boldsymbol{Q}}{}_{\boldsymbol{A'}} + \bar{\phi}_{\boldsymbol{P'A'Q'B'}}\epsilon_{\boldsymbol{PB}}\tau_{\boldsymbol{A}}{}^{\boldsymbol{Q'}}\right).
\end{aligned}$$

Following the same procedure discussed in Section 13.2.4 one finds, for the Bianchi identity, that

$$\mathcal{P}\phi_{\boldsymbol{ABCD}} - 2\mathcal{D}_{(\boldsymbol{A}}{}^{\boldsymbol{Q}}\phi_{\boldsymbol{BCD})\boldsymbol{Q}} = 0. \tag{13.58}$$

Observe that this last expression is, for convenience, expressed in terms of the Levi-Civita connection $\boldsymbol{\nabla}$.

The space spinor split of the evolution equations

A more detailed version of the evolution equations is obtained by resorting to the space spinor formalism, and, in particular, to the split of the connection coefficients as given in Section 4.3.1.

Following the general strategy behind the space spinor formalism, it is convenient to define

$$\hat{\Gamma}_{\boldsymbol{ABCD}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\hat{\Gamma}_{\boldsymbol{AA'CD}}, \qquad \Gamma_{\boldsymbol{ABCD}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\Gamma_{\boldsymbol{AA'CD}}, \qquad f_{\boldsymbol{AB}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}f_{\boldsymbol{AA'}},$$
$$\Theta_{\boldsymbol{ABCD}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\tau_{\boldsymbol{D}}{}^{\boldsymbol{C'}}\hat{L}_{\boldsymbol{AA'CC'}}$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

In particular, one has

$$\hat{\Gamma}_{\boldsymbol{ABCD}} = \Gamma_{\boldsymbol{ABCD}} - \epsilon_{\boldsymbol{CA}} f_{\boldsymbol{DB}}.$$

As a consequence of the gauge conditions (13.56) it follows that

$$f_{\boldsymbol{AB}} = f_{(\boldsymbol{AB})}, \qquad \Gamma_{\boldsymbol{Q}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{AB}} = -f_{\boldsymbol{AB}}, \qquad \hat{L}_{\boldsymbol{Q}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{AB}} = 0.$$

Defining, as in Section 4.3.1, the spinors $\chi_{\boldsymbol{ABCD}}$ and $\xi_{\boldsymbol{ABCD}}$ via

$$\chi_{\boldsymbol{ABCD}} \equiv -\frac{1}{\sqrt{2}}(\Gamma_{\boldsymbol{ABCD}} + \Gamma^{+}_{\boldsymbol{ABCD}}), \qquad \xi_{\boldsymbol{ABCD}} \equiv \frac{1}{\sqrt{2}}(\Gamma_{\boldsymbol{ABCD}} - \Gamma^{+}_{\boldsymbol{ABCD}}),$$

one obtains from the metricity of the connection $\boldsymbol{\nabla}$ that

$$\begin{aligned} \Gamma_{\boldsymbol{ABCD}} &= \frac{1}{\sqrt{2}}(\xi_{\boldsymbol{ABCD}} - \chi_{\boldsymbol{ABCD}}) \\ &= \frac{1}{\sqrt{2}}(\xi_{\boldsymbol{ABCD}} - \chi_{(\boldsymbol{AB})\boldsymbol{CD}}) - \frac{1}{2}\epsilon_{\boldsymbol{AB}} f_{\boldsymbol{CD}}. \end{aligned}$$

Exploiting the gauge conditions, the spinor $\Theta_{\boldsymbol{ABCD}}$ can be decomposed into

$$\Theta_{\boldsymbol{ABCD}} = \Theta_{\boldsymbol{AB}(\boldsymbol{CD})} + \frac{1}{2}\epsilon_{\boldsymbol{CD}}\Theta_{\boldsymbol{ABQ}}{}^{\boldsymbol{Q}}.$$

In addition, it is convenient to introduce the ***electric and magnetic parts of the rescaled Weyl spinor*** $\phi_{\boldsymbol{ABCD}}$ via

$$\eta_{\boldsymbol{ABCD}} \equiv \frac{1}{2}(\phi_{\boldsymbol{ABCD}} + \phi^{+}_{\boldsymbol{ABCD}}), \qquad \mu_{\boldsymbol{ABCD}} \equiv -\frac{\mathrm{i}}{2}(\phi_{\boldsymbol{ABCD}} - \phi^{+}_{\boldsymbol{ABCD}}).$$

A calculation using the above definitions yields the detailed system:

$$\partial_\tau e_{\boldsymbol{AB}}{}^{0} = -\chi_{(\boldsymbol{AB})}{}^{\boldsymbol{PQ}} e_{\boldsymbol{PQ}}{}^{0} - f_{\boldsymbol{AB}}, \tag{13.59a}$$

$$\partial_\tau e_{\boldsymbol{AB}}{}^{\alpha} = -\chi_{(\boldsymbol{AB})}{}^{\boldsymbol{PQ}} e_{\boldsymbol{PQ}}{}^{\alpha}, \tag{13.59b}$$

$$\begin{aligned} \partial_\tau \xi_{\boldsymbol{ABCD}} = {} & -\chi_{(\boldsymbol{AB})}{}^{\boldsymbol{PQ}}\xi_{\boldsymbol{PQCD}} + \frac{1}{\sqrt{2}}(\epsilon_{\boldsymbol{AC}}\chi_{(\boldsymbol{BD})\boldsymbol{PQ}} + \epsilon_{\boldsymbol{BD}}\chi_{(\boldsymbol{AC})\boldsymbol{PQ}}) f^{\boldsymbol{PQ}} \\ & - \sqrt{2}\chi_{(\boldsymbol{AB})(\boldsymbol{C}}{}^{\boldsymbol{E}} f_{\boldsymbol{D})\boldsymbol{E}} - \frac{1}{2}(\epsilon_{\boldsymbol{AC}}\Theta_{\boldsymbol{BDQ}}{}^{\boldsymbol{Q}} + \epsilon_{\boldsymbol{BD}}\Theta_{\boldsymbol{ACQ}}{}^{\boldsymbol{Q}}) \\ & - \mathrm{i}\Theta\mu_{\boldsymbol{ABCD}}, \end{aligned} \tag{13.59c}$$

$$\partial_\tau f_{\boldsymbol{AB}} = -\chi_{(\boldsymbol{AB})}{}^{\boldsymbol{PQ}} f_{\boldsymbol{PQ}} + \frac{1}{\sqrt{2}}\Theta_{\boldsymbol{ABQ}}{}^{\boldsymbol{Q}}, \tag{13.59d}$$

$$\partial_\tau \chi_{(\boldsymbol{AB})\boldsymbol{CD}} = -\chi_{(\boldsymbol{AB})}{}^{\boldsymbol{PQ}}\chi_{\boldsymbol{PQCD}} - \Theta_{\boldsymbol{AB}(\boldsymbol{CD})} + \Theta\eta_{\boldsymbol{ABCD}}, \tag{13.59e}$$

$$\begin{aligned} \partial_\tau \Theta_{\boldsymbol{CD}(\boldsymbol{AB})} = {} & -\chi_{(\boldsymbol{CD})}{}^{\boldsymbol{PQ}}\Theta_{\boldsymbol{PQ}(\boldsymbol{AB})} - \partial_\tau\Theta\eta_{\boldsymbol{ABCD}} \\ & + \mathrm{i}\sqrt{2} d^{\boldsymbol{P}}{}_{(\boldsymbol{A}}\mu_{\boldsymbol{B})\boldsymbol{CDP}}, \end{aligned} \tag{13.59f}$$

$$\partial_\tau \Theta_{\boldsymbol{ABQ}}{}^{\boldsymbol{Q}} = -\chi_{(\boldsymbol{AB})}{}^{\boldsymbol{EF}}\Theta_{\boldsymbol{EFQ}}{}^{\boldsymbol{Q}} + \sqrt{2} d^{\boldsymbol{PQ}}\eta_{\boldsymbol{ABPQ}}. \tag{13.59g}$$

Remark. The term $\partial_\tau\Theta$ in the second term of the left-hand side of Equation (13.59f) arises from the fact that, in a conformal Gaussian system, the time component of the covector $\boldsymbol{d}$ is given by $\dot{\Theta}$; see Proposition 5.1.

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

Setting

$$\phi_0 \equiv \phi_{\mathbf{0000}}, \quad \phi_1 \equiv \phi_{\mathbf{0001}}, \quad \phi_2 \equiv \phi_{\mathbf{0011}}, \quad \phi_3 \equiv \phi_{\mathbf{0111}}, \quad \phi_4 \equiv \phi_{\mathbf{1111}},$$

the standard Bianchi system, Equation (13.58), explicitly reads

$$\begin{aligned}
&(\sqrt{2} + 2e_{\mathbf{01}}{}^0)\partial_\tau\phi_0 - 2e_{\mathbf{11}}{}^0\partial_\tau\phi_1 + 2e_{\mathbf{01}}{}^\alpha\partial_\alpha\phi_0 - 2e_{\mathbf{11}}{}^\alpha\partial_\alpha\phi_1\\
&\qquad = -6\Gamma_{\mathbf{1111}}\phi_2 + (4\Gamma_{\mathbf{1110}} + 8\Gamma_{\mathbf{0111}})\phi_1 + (2\Gamma_{\mathbf{1100}} - 8\Gamma_{\mathbf{0101}})\phi_0,\\
&(\sqrt{2} + 2e_{\mathbf{01}}{}^0)\partial_\tau\phi_1 - 2e_{\mathbf{11}}{}^0\partial_\tau\phi_2 - 2e_{\mathbf{11}}{}^\alpha\partial_\alpha\phi_2 + 2e_{\mathbf{01}}{}^\alpha\partial_\alpha\phi_1\\
&\qquad = -4\Gamma_{\mathbf{1111}}\phi_3 + (6\Gamma_{(\mathbf{01})\mathbf{11}} - 3f_{\mathbf{11}})\phi_2\\
&\qquad\quad + (4\Gamma_{\mathbf{1100}} - 4\Gamma_{(\mathbf{01})\mathbf{01}} + 2f_{\mathbf{01}})\phi_1 - (2\Gamma_{(\mathbf{01})\mathbf{00}} + f_{\mathbf{00}})\phi_0,\\
&\sqrt{2}\partial_\tau\phi_2 - e_{\mathbf{11}}{}^0\partial_\tau\phi_3 + e_{\mathbf{00}}{}^0\partial_\tau\phi_1 - e_{\mathbf{11}}{}^\alpha\partial_\alpha\phi_3 + e_{\mathbf{00}}{}^\alpha\partial_\alpha\phi_1\\
&\qquad = -\Gamma_{\mathbf{1111}}\phi_4 - 2(\Gamma_{\mathbf{1101}} + f_{\mathbf{11}})\phi_3 + 3(\Gamma_{\mathbf{0011}} + \Gamma_{\mathbf{1100}})\phi_2\\
&\qquad\quad - 2(\Gamma_{\mathbf{0001}} - f_{\mathbf{00}})\phi_1 - \Gamma_{\mathbf{0000}}\phi_0,\\
&(\sqrt{2} - 2e_{\mathbf{01}}{}^0)\partial_\tau\phi_3 + 2e_{\mathbf{00}}{}^0\partial_\tau\phi_2 - 2e_{\mathbf{01}}{}^\alpha\partial_\alpha\phi_3 + 2e_{\mathbf{00}}{}^\alpha\partial_\alpha\phi_2\\
&\qquad = -(2\Gamma_{(\mathbf{01})\mathbf{11}} + f_{\mathbf{11}})\phi_4 + (2\Gamma_{\mathbf{0011}} - 4\Gamma_{(\mathbf{01})\mathbf{01}} - 2f_{\mathbf{01}})\phi_3\\
&\qquad\quad + (6\Gamma_{(\mathbf{01})\mathbf{00}} + 3f_{\mathbf{00}})\phi_2 - 4\Gamma_{\mathbf{0000}}\phi_1,\\
&(\sqrt{2} - 2e_{\mathbf{01}}{}^0)\partial_\tau\phi_4 + 2e_{\mathbf{00}}{}^0\partial_\tau\phi_3 - 2e_{\mathbf{01}}{}^\alpha\partial_\alpha\phi_4 + 2e_{\mathbf{00}}{}^\alpha\partial_\alpha\phi_3\\
&\qquad = (2\Gamma_{\mathbf{0011}} - 8\Gamma_{\mathbf{1010}})\phi_4 + (4\Gamma_{\mathbf{0001}} + 8\Gamma_{\mathbf{1000}})\phi_3 - 6\Gamma_{\mathbf{0000}}\phi_2.
\end{aligned}$$

For completeness, the constraints

$$\Lambda_{\boldsymbol{AB}} \equiv \mathcal{D}^{\boldsymbol{PQ}}\phi_{\boldsymbol{PQAB}} = 0$$

are also given in explicit form:

$$\begin{aligned}
&e_{\mathbf{11}}{}^0\partial_\tau\phi_4 - 2e_{\mathbf{01}}{}^0\partial_\tau\phi_3 + e_{\mathbf{00}}{}^0\partial_\tau\phi_2 + e_{\mathbf{11}}{}^\alpha\partial_\alpha\phi_4 - 2e_{\mathbf{01}}{}^\alpha\partial_\alpha\phi_3 + e_{\mathbf{00}}{}^\alpha\partial_\alpha\phi_2\\
&\qquad = -(2\Gamma_{(\mathbf{01})\mathbf{11}} - 4\Gamma_{\mathbf{1110}})\phi_4 + (2\Gamma_{\mathbf{0011}} - 4\Gamma_{(\mathbf{01})\mathbf{01}} - 4\Gamma_{\mathbf{1100}})\phi_3\\
&\qquad\quad + 6\Gamma_{(\mathbf{01})\mathbf{00}}\phi_2 - 2\Gamma_{\mathbf{0000}}\phi_1,\\
&e_{\mathbf{11}}{}^0\partial_\tau\phi_3 - 2e_{\mathbf{01}}{}^0\partial_\tau\phi_2 + e_{\mathbf{00}}{}^0\partial_\tau\phi_1 + e_{\mathbf{11}}{}^\alpha\partial_\alpha\phi_3 - 2e_{\mathbf{01}}{}^\alpha\partial_\alpha\phi_2 + e_{\mathbf{00}}{}^\alpha\partial_\alpha\phi_1\\
&\qquad = \Gamma_{\mathbf{1111}}\phi_4 - (4\Gamma_{(\mathbf{01})\mathbf{11}} - 2\Gamma_{\mathbf{1101}})\phi_3 + 3(\Gamma_{\mathbf{0011}} - \Gamma_{\mathbf{1100}})\phi_2\\
&\qquad\quad - (2\Gamma_{\mathbf{0001}} - 4\Gamma_{(\mathbf{01})\mathbf{00}})\phi_1 - \Gamma_{\mathbf{0000}}\phi_0,\\
&e_{\mathbf{11}}{}^0\partial_\tau\phi_2 - 2e_{\mathbf{01}}{}^0\partial_\tau\phi_1 + e_{\mathbf{00}}{}^0\partial_\tau\phi_0 + e_{\mathbf{11}}{}^\alpha\partial_\alpha\phi_2 - 2e_{\mathbf{01}}{}^\alpha\partial_\alpha\phi_1 + e_{\mathbf{00}}{}^\alpha\partial_\alpha\phi_0\\
&\qquad = 2\Gamma_{\mathbf{1111}}\phi_3 - 6\Gamma_{(\mathbf{01})\mathbf{11}}\phi_2 + (4\Gamma_{\mathbf{0011}} + 4\Gamma_{(\mathbf{01})\mathbf{01}} - 2\Gamma_{\mathbf{1100}})\phi_1\\
&\qquad\quad - (4\Gamma_{\mathbf{0001}} - 2\Gamma_{(\mathbf{01})\mathbf{00}})\phi_0.
\end{aligned}$$

These *constraint equations* contain time derivatives of the components of the Weyl spinor. Furthermore, as the congruence of conformal curves is, in general, not hypersurface orthogonal, the constraint equations are not intrinsic to the leaves of a foliation.

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

The boundary adapted system

The standard system (13.36) is not the only symmetric hyperbolic evolution system that can be extracted from the Bianchi equation. In certain applications, such as the ones involving evolution domains with a timelike boundary, another form of the evolution equations is more convenient. In what follows, the system extracted from

$$-2\Lambda_{(\mathbf{0000})} = 0, \quad -2\Lambda_{(\mathbf{0001})} - \frac{1}{2}C_{\mathbf{00}} = 0, \quad -2\Lambda_{(\mathbf{0011})} = 0, \tag{13.60a}$$

$$-2\Lambda_{(\mathbf{0111})} + \frac{1}{2}C_{\mathbf{11}} = 0, \quad -2\Lambda_{\mathbf{1111}} = 0, \tag{13.60b}$$

will be known as the ***boundary adapted system***. In the following, it will be shown that it is, indeed, symmetric hyperbolic. The principal part of the boundary adapted system can be written as

$$\mathbf{A}^\mu \partial_\mu \boldsymbol{\phi} = \begin{pmatrix} \tau^\mu + 2e_{\mathbf{01}}{}^\mu & -2e_{\mathbf{00}}{}^\mu & 0 & 0 & 0 \\ 2e_{\mathbf{11}}{}^\mu & 2\tau^\mu & -2e_{\mathbf{00}}{}^\mu & 0 & 0 \\ 0 & 2e_{\mathbf{11}}{}^\mu & 2\tau^\mu & -2e_{\mathbf{00}}{}^\mu & 0 \\ 0 & 0 & 2e_{\mathbf{11}}{}^\mu & 2\tau^\mu & -2e_{\mathbf{00}}{}^\mu \\ 0 & 0 & 0 & 2e_{\mathbf{11}}{}^\mu & \tau^\mu - 2e_{\mathbf{01}}{}^\mu \end{pmatrix} \partial_\mu \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}, \tag{13.61}$$

so that the matrices $\mathbf{A}^\mu$ are Hermitian, and, in particular, $\mathbf{A}^\mu \tau_\mu$ is positive definite. The characteristic polynomial is given by

$$\det(\mathbf{A}^\mu \xi_\mu) = 4\big(\tau^\mu \xi_\mu\big)\big(g^{\nu\lambda}\xi_\nu\xi_\lambda\big)\big(l^{\rho\sigma}\xi_\rho\xi_\sigma\big),$$

where $l^{\rho\sigma} \equiv \tau^\rho\tau^\sigma + e_{\mathbf{00}}{}^{(\rho}e_{\mathbf{11}}{}^{\sigma)}$. In Chapter 14, it will be seen that when τ^μ is tangent to a timelike hypersurface, then the pull-back of $l_{\mu\nu}$ gives the components of the intrinsic three-dimensional Lorentzian metric implied by $\boldsymbol{g}$ on the hypersurface.

Explicitly, the boundary adapted system takes the form

$$\begin{aligned}
&(\sqrt{2} + 2e_{\mathbf{01}}{}^0)\partial_\tau\phi_0 - 2e_{\mathbf{11}}{}^0\partial_\tau\phi_1 + 2e_{\mathbf{01}}{}^\alpha\partial_\alpha\phi_0 - 2e_{\mathbf{11}}{}^\alpha\partial_\alpha\phi_1 \\
&\qquad = -6\Gamma_{\mathbf{1111}}\phi_2 + (4\Gamma_{\mathbf{1110}} + 8\Gamma_{\mathbf{0111}})\phi_1 + (2\Gamma_{\mathbf{1100}} - 8\Gamma_{\mathbf{0101}})\phi_0, \\
&\sqrt{2}\partial_\tau\phi_1 - e_{\mathbf{11}}{}^0\partial_\tau\phi_2 + e_{\mathbf{00}}{}^0\partial_\tau\phi_0 - e_{\mathbf{11}}{}^\alpha\partial_\alpha\phi_2 + e_{\mathbf{00}}{}^\alpha\partial_\alpha\phi_4 \\
&\qquad = -2\Gamma_{\mathbf{1111}}\phi_3 - 3f_{\mathbf{11}}\phi_2 + (2\Gamma_{\mathbf{1100}} + 4\Gamma_{\mathbf{0011}} + 2f_{\mathbf{01}})\phi_1 - (4\Gamma_{\mathbf{0001}} - f_{\mathbf{00}})\phi_0, \\
&\sqrt{2}\partial_\tau\phi_2 - e_{\mathbf{11}}{}^0\partial_\tau\phi_3 + e_{\mathbf{00}}{}^0\partial_\tau\phi_1 - e_{\mathbf{11}}{}^\alpha\partial_\alpha\phi_3 + e_{\mathbf{00}}{}^\alpha\partial_\alpha\phi_1 \\
&\qquad = -\Gamma_{\mathbf{1111}}\phi_4 - 2(\Gamma_{\mathbf{1101}} + f_{\mathbf{11}})\phi_3 + 3(\Gamma_{\mathbf{0011}} + \Gamma_{\mathbf{1100}})\phi_2 \\
&\qquad\quad - 2(\Gamma_{\mathbf{0001}} - f_{\mathbf{00}})\phi_1 - \Gamma_{\mathbf{0000}}\phi_0, \\
&\sqrt{2}\partial_\tau\phi_3 - e_{\mathbf{11}}{}^0\partial_\tau\phi_4 + e_{\mathbf{00}}{}^0\partial_\tau\phi_2 - e_{\mathbf{11}}{}^\alpha\partial_\alpha\phi_4 + e_{\mathbf{00}}{}^\alpha\partial_\alpha\phi_2 \\
&\qquad = -(4\Gamma_{\mathbf{1110}} + f_{\mathbf{11}})\phi_2 + (2\Gamma_{\mathbf{0011}} + 4\Gamma_{\mathbf{1100}} - 2f_{\mathbf{01}})\phi_3 + 3f_{\mathbf{00}}\phi_2 - 2\Gamma_{\mathbf{0000}}\phi_1, \\
&(\sqrt{2} - 2e_{\mathbf{01}}{}^0)\partial_\tau\phi_4 + 2e_{\mathbf{00}}{}^0\partial_\tau\phi_3 - 2e_{\mathbf{01}}{}^\alpha\partial_\alpha\phi_4 + 2e_{\mathbf{00}}{}^\alpha\partial_\alpha\phi_3 \\
&\qquad = (2\Gamma_{\mathbf{0011}} - 8\Gamma_{\mathbf{1010}})\phi_4 + (4\Gamma_{\mathbf{0001}} + 8\Gamma_{\mathbf{1000}})\phi_3 - 6\Gamma_{\mathbf{0000}}\phi_2.
\end{aligned}$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

13.4.5 The construction of a subsidiary system

This section addresses the construction of a system of subsidiary equations for the evolution equations discussed in the previous section. The particular problem at hand consists of constructing evolution equations for the zero quantities

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}, \qquad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}, \qquad \hat{\Delta}_{\boldsymbol{abc}}, \qquad \Lambda_{\boldsymbol{abc}},$$

encoding the extended conformal field equations. In addition, in the present hyperbolic reduction procedure, one also needs to construct evolution equations for the additional zero quantities

$$\delta_{\boldsymbol{a}}, \qquad \gamma_{\boldsymbol{ab}}, \qquad \varsigma_{\boldsymbol{ab}},$$

which play the role of constraints of the conformal equations; see Equations (8.47a)–(8.47c) for their definitions. The necessity of these extra zero quantities can be traced back to Proposition 8.3.

As in the case of the analysis of the subsidiary equations for the hyperbolic reduction procedure using gauge source functions, the subsidiary equations need to be homogeneous in zero quantities so that the vanishing of the latter on an initial hypersurface readily implies a unique vanishing solution. The basic assumption in the construction of the subsidiary system is that the evolution equations associated to the extended conformal field equations are satisfied. That is, one assumes that

$$\hat{\Sigma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0, \qquad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{d0b}} = 0, \qquad \hat{\Delta}_{\boldsymbol{0bc}} = 0,$$

hold, together with either the standard or the boundary adapted system for the components of the Weyl spinor. The aforementioned evolution equations have been constructed using the gauge conditions

$$f_{\boldsymbol{0}} = 0, \qquad \hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = 0, \qquad \hat{L}_{\boldsymbol{0b}} = 0,$$

which, therefore, can also be used in the construction of the subsidiary system. Note also, that in the present gauge $d_{\boldsymbol{0}} = \Theta\beta_{\boldsymbol{0}} = \nabla_{\boldsymbol{0}}\Theta$ so that one has

$$\delta_{\boldsymbol{0}} = 0.$$

Similarly,

$$\gamma_{\boldsymbol{0b}} = \hat{L}_{\boldsymbol{0b}} - \hat{\nabla}_{\boldsymbol{0}}\beta_{\boldsymbol{b}} - \frac{1}{2}S_{\boldsymbol{0b}}{}^{\boldsymbol{ef}}\beta_{\boldsymbol{e}}\beta_{\boldsymbol{f}} + \lambda\Theta^{-2}\eta_{\boldsymbol{0b}} = 0$$

by virtue of the gauge conditions and the evolution equation

$$\tilde{\nabla}_{\boldsymbol{0}}\beta_{\boldsymbol{a}} + \beta_{\boldsymbol{0}}\beta_{\boldsymbol{a}} - \frac{1}{2}\eta_{\boldsymbol{0a}}(\beta_{\boldsymbol{e}}\beta^{\boldsymbol{e}} - 2\lambda\Theta^{-2}) = 0, \qquad (13.62)$$

for the covector $\beta_{\boldsymbol{a}}$. Finally, one has

$$\varsigma_{\boldsymbol{0b}} = -\hat{L}_{\boldsymbol{b0}} - \hat{\nabla}_{\boldsymbol{0}}f_{\boldsymbol{b}} + \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}f_{\boldsymbol{e}} = 0,$$

as a result of the evolution equation for the covector $\boldsymbol{f}$.

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

The construction of subsidiary equations is similar in spirit to the one discussed in Section 13.3. There are, however, certain differences. The most conspicuous one is the fact that one is now working with a connection which is non-metric.

The subsidiary equation for the no-torsion condition

To construct the subsidiary equation for the no-torsion condition one considers the totally antisymmetric covariant derivative $\hat{\nabla}_{[\boldsymbol{a}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}]}$ and observes that

$$\begin{aligned}3\hat{\nabla}_{[\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}]} &= \hat{\nabla}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}} + \hat{\nabla}_{\boldsymbol{b}}\hat{\Sigma}_{\boldsymbol{c}}{}^{\boldsymbol{d}}{}_{\boldsymbol{0}} + \hat{\nabla}_{\boldsymbol{c}}\hat{\Sigma}_{\boldsymbol{0}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} \\ &= \hat{\nabla}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{c}}{}^{\boldsymbol{d}}{}_{\boldsymbol{e}} - \hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{e}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}}.\end{aligned} \tag{13.63}$$

On the other hand, from the first Bianchi identity, Equation (2.10), and the definition of $\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{cab}}$ one obtains

$$\hat{\nabla}_{[\boldsymbol{a}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}]} = -\hat{\Xi}^{\boldsymbol{d}}{}_{[\boldsymbol{cab}]} - \hat{\Sigma}_{[\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\hat{\Sigma}_{\boldsymbol{c}]}{}^{\boldsymbol{d}}{}_{\boldsymbol{e}}, \tag{13.64}$$

where it has been used that $\hat{\rho}^{\boldsymbol{d}}{}_{[\boldsymbol{cab}]} = 0$ by construction. The desired evolution equation is obtained from combining Equations (13.63) and (13.64). More precisely, one has

$$\hat{\nabla}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}} = -\frac{1}{3}\hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{e}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} - \frac{1}{3}\hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{e}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} - \hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{0bc}}.$$

This evolution equation has the required homogeneous form.

The subsidiary equation for the Ricci identity

In this case, one considers the totally symmetrised covariant derivative $\hat{\nabla}_{[\boldsymbol{a}}\hat{\Xi}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]}$. A direct computation shows that

$$\begin{aligned}3\hat{\nabla}_{[\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} &= \hat{\nabla}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} + \hat{\nabla}_{\boldsymbol{b}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{ec0}} + \hat{\nabla}_{\boldsymbol{c}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{e0b}} \\ &= \hat{\nabla}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{ecf}} - \hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{efb}}.\end{aligned}$$

Using the second Bianchi identity, Equation (2.11), and the definition of $\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{ebc}}$ one arrives at the expression

$$\hat{\nabla}_{[\boldsymbol{a}}\hat{\Xi}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} = -\hat{\Sigma}_{[\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}}\hat{R}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{c}]\boldsymbol{f}} - \hat{\nabla}_{[\boldsymbol{a}}\hat{\rho}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]}.$$

The first term on the right-hand side is already of the required form. The second one needs to be analysed in more detail. It is recalled that

$$\hat{\rho}^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} \equiv C^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} + 2S_{\boldsymbol{e}[\boldsymbol{b}}{}^{\boldsymbol{df}}\hat{L}_{\boldsymbol{c}]\boldsymbol{f}}.$$

Thus,

$$\hat{\nabla}_{[\boldsymbol{a}}\hat{\rho}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} = \hat{\nabla}_{[\boldsymbol{a}}C^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} + 2S_{\boldsymbol{e}[\boldsymbol{b}}{}^{\boldsymbol{df}}\hat{\nabla}_{\boldsymbol{a}}\hat{L}_{\boldsymbol{c}]\boldsymbol{f}}.$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

In order to further expand this expression one considers $\epsilon_{f}{}^{abc}\hat{\nabla}_{a}\hat{\rho}^{d}{}_{ebc}$. A direct calculation shows that

$$\hat{\nabla}_{[a}C^{d}{}_{|e|bc]} = \nabla_{[a}C^{d}{}_{|e|bc]} + \delta_{[a}{}^{d}f_{|f}C^{f}{}_{e|bc]} + \eta_{e[a}f^{f}C^{d}{}_{|f|bc]}. \tag{13.65}$$

Moreover, one has

$$\begin{aligned}\epsilon_{f}{}^{abc}\nabla_{a}C^{d}{}_{ebc} &= -\epsilon_{f}{}^{abc}\nabla_{a}{}^{*}C^{*d}{}_{ebc}\\ &= -2\nabla_{a}{}^{*}C^{d}{}_{ef}{}^{a} = 2\nabla_{a}C^{*a}{}_{f}{}^{d}{}_{e}\\ &= -\epsilon_{e}{}^{dgh}\nabla_{a}C^{a}{}_{fgh}.\end{aligned}$$

Thus, using that $C^{c}{}_{dab} = \Theta d^{c}{}_{dab}$ and the definition of the zero quantity Λ_{abc} one concludes that

$$\epsilon_{f}{}^{abc}\hat{\nabla}_{a}C^{d}{}_{ebc} = \Theta\epsilon_{e}{}^{dgh}\Lambda_{fgh} + 2\nabla^{g}\Theta d^{*d}{}_{efg} + 2\Theta f^{g}d^{*}_{gef}{}^{d} + 2\Theta f^{g}d^{*d}{}_{gfe}.$$

A similar computation using the definition of $\hat{\Delta}_{abc}$ yields

$$2\epsilon_{f}{}^{abc}S_{eb}{}^{dg}\hat{\Delta}_{acg} = 2\Theta\beta_{g}d^{*g}{}_{ef}{}^{d} - 2\Theta\beta_{g}d^{*gd}{}_{fe}.$$

Thus, using the symmetries of $d^{*c}{}_{dab}$ and the definition of δ_{a} one concludes that

$$\epsilon_{f}{}^{abc}\hat{\nabla}_{a}\hat{\rho}^{d}{}_{ebc} = \Theta\epsilon_{e}{}^{dgh}\Lambda_{fgh} - 2\Theta\delta^{g}d^{*d}{}_{efg} + \epsilon_{f}{}^{abc}S_{eb}{}^{dg}\hat{\Delta}_{acg}.$$

Alternatively, using the properties of the generalised Hodge duals † and ‡ defined in Equation (2.24), one can write

$$\hat{\nabla}_{[a}\hat{\rho}^{d}{}_{|e|bc]} = \frac{1}{6}\Theta\epsilon^{f}{}_{abc}\epsilon_{e}{}^{dgh}\Lambda_{fgh} - \frac{1}{3}\Theta\epsilon^{f}{}_{abc}\delta^{g}d^{*d}{}_{efg} - S_{e[b}{}^{dg}\hat{\Delta}_{ac]g}.$$

Combining the expressions, one obtains the required evolution equation. Namely, one has

$$\begin{aligned}\hat{\nabla}_{0}\hat{\Xi}^{d}{}_{ebc} = \hat{\Gamma}_{b}{}^{f}{}_{0}\hat{\Xi}^{d}{}_{ecf} + \hat{\Gamma}_{c}{}^{f}{}_{0}\hat{\Xi}^{d}{}_{efb} - \hat{\Sigma}_{b}{}^{f}{}_{c}\hat{R}^{d}{}_{e0f} - \frac{1}{2}\Theta\epsilon^{f}{}_{0bc}\epsilon_{e}{}^{dgh}\Lambda_{fgh}\\ + \epsilon^{f}{}_{0bc}\delta^{g}d^{*d}{}_{efg} + 3S_{e0}{}^{dg}\hat{\Delta}_{cbg},\end{aligned}$$

which is homogeneous in the zero quantities.

The subsidiary equation for the Cotton equation

In this case one considers the skew derivative $\hat{\nabla}_{[a}\hat{\Delta}_{bc]d}$. A direct computation yields

$$\begin{aligned}3\hat{\nabla}_{[0}\hat{\Delta}_{bc]d} &= \hat{\nabla}_{0}\hat{\Delta}_{bcd} + \hat{\nabla}_{b}\hat{\Delta}_{c0d} + \hat{\nabla}_{c}\hat{\Delta}_{0bd}\\ &= \hat{\nabla}_{0}\hat{\Delta}_{bcd} - \hat{\Gamma}_{b}{}^{e}{}_{0}\hat{\Delta}_{ced} - \hat{\Gamma}_{c}{}^{e}{}_{0}\hat{\Delta}_{ebd}.\end{aligned}$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

On the other hand, using the definition of $\hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{cab}}$ and the symmetries of $\hat{\rho}^{\boldsymbol{e}}{}_{\boldsymbol{cab}}$ one obtains

$$\begin{aligned}\hat{\nabla}_{[\boldsymbol{a}}\hat{\Delta}_{\boldsymbol{bc}]\boldsymbol{d}} &= 2\hat{\nabla}_{[\boldsymbol{a}}\hat{\nabla}_{\boldsymbol{b}}\hat{L}_{\boldsymbol{c}]\boldsymbol{d}} - \hat{\nabla}_{[\boldsymbol{a}}d_{|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} - d_{\boldsymbol{e}}\hat{\nabla}_{[\boldsymbol{a}}d^{\boldsymbol{e}}{}_{|\boldsymbol{d}|\boldsymbol{bc}]}\\ &= -\hat{\Xi}^{\boldsymbol{e}}{}_{[\boldsymbol{cab}]}\hat{L}_{\boldsymbol{ed}} - \hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{d}[\boldsymbol{ab}}\hat{L}_{\boldsymbol{c}]\boldsymbol{e}} - \hat{\rho}^{\boldsymbol{e}}{}_{\boldsymbol{d}[\boldsymbol{ab}}\hat{L}_{\boldsymbol{c}]\boldsymbol{e}} + \hat{\Sigma}_{[\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\hat{\nabla}_{|\boldsymbol{e}|}\hat{L}_{\boldsymbol{c}]\boldsymbol{d}}\\ &\quad - \hat{\nabla}_{[\boldsymbol{a}}d_{|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} - d_{\boldsymbol{e}}\hat{\nabla}_{[\boldsymbol{a}}d^{\boldsymbol{e}}{}_{|\boldsymbol{d}|\boldsymbol{bc}]}.\end{aligned}$$

Using the definition of $\delta_{\boldsymbol{a}}$ and $\gamma_{\boldsymbol{ab}}$ one finds that

$$\hat{\nabla}_{[\boldsymbol{a}}d_{|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} = -\Theta\delta_{[\boldsymbol{a}}\beta_{|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} - \Theta\gamma_{[\boldsymbol{a}|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} - \Theta f_{[\boldsymbol{a}}\beta_{|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} + \Theta\hat{L}_{[\boldsymbol{a}|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]}.$$

Finally, a calculation similar to the one carried out in the previous subsection shows that

$$\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\nabla_{\boldsymbol{a}}d^{\boldsymbol{e}}{}_{\boldsymbol{dbc}} = \epsilon_{\boldsymbol{d}}{}^{\boldsymbol{egh}}\nabla_{\boldsymbol{a}}d^{\boldsymbol{a}}{}_{\boldsymbol{fgh}},$$

so that using Equation (13.65) and the properties of the generalised duals † and ‡ – see Equation (2.24) – one finds that

$$\hat{\nabla}_{[\boldsymbol{a}}d^{\boldsymbol{e}}{}_{|\boldsymbol{d}|\boldsymbol{bc}]} = \frac{1}{6}\epsilon_{\boldsymbol{abc}}{}^{\boldsymbol{f}}\epsilon_{\boldsymbol{d}}{}^{\boldsymbol{egh}}\Lambda_{\boldsymbol{fgh}} + \delta_{[\boldsymbol{a}}{}^{\boldsymbol{e}}f_{|\boldsymbol{f}}d^{\boldsymbol{f}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} + \eta_{\boldsymbol{d}[\boldsymbol{a}}f^{\boldsymbol{f}}d^{\boldsymbol{e}}{}_{|\boldsymbol{f}|\boldsymbol{bc}]}.$$

Combining the above expressions and using the properties of the decomposition of $\hat{\rho}^{\boldsymbol{d}}{}_{\boldsymbol{dab}}$ one obtains the expression

$$\begin{aligned}\hat{\nabla}_{[\boldsymbol{a}}\hat{\Delta}_{\boldsymbol{bc}]\boldsymbol{d}} &= -\hat{\Xi}^{\boldsymbol{e}}{}_{[\boldsymbol{cab}]}\hat{L}_{\boldsymbol{ed}} - \hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{d}[\boldsymbol{ab}}\hat{L}_{\boldsymbol{c}]\boldsymbol{e}} + \hat{\Sigma}_{[\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\hat{\nabla}_{|\boldsymbol{e}|}\hat{L}_{\boldsymbol{c}]\boldsymbol{d}}\\ &\qquad + \Theta\delta_{[\boldsymbol{a}}\beta_{|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} + \Theta\gamma_{[\boldsymbol{a}|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} - \frac{1}{6}\epsilon_{\boldsymbol{abc}}{}^{\boldsymbol{f}}\epsilon_{\boldsymbol{d}}{}^{\boldsymbol{egh}}\Lambda_{\boldsymbol{fgh}}\beta_{\boldsymbol{e}},\end{aligned}$$

and, finally, the evolution equation

$$\begin{aligned}\hat{\nabla}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{bcd}} &= \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{ced}} + \hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{ebd}} - \hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{0bc}}\hat{L}_{\boldsymbol{ed}} + \delta_{\boldsymbol{b}}d_{\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{dc0}} + \delta_{\boldsymbol{c}}d_{\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d0b}}\\ &\qquad + \Theta\gamma_{\boldsymbol{be}}d^{\boldsymbol{e}}{}_{\boldsymbol{dc0}} + \Theta\gamma_{\boldsymbol{ce}}d^{\boldsymbol{e}}{}_{\boldsymbol{d0b}} - \frac{1}{2}\epsilon_{\boldsymbol{0bc}}{}^{\boldsymbol{f}}\epsilon_{\boldsymbol{d}}{}^{\boldsymbol{egh}}\Lambda_{\boldsymbol{fgh}}\beta_{\boldsymbol{e}},\end{aligned}$$

which is homogeneous in zero quantities as required.

The subsidiary equations for the physical Bianchi identity

The argument to show the propagation of the Bianchi identity in the present context is similar to the one discussed in Section 13.3.6. In particular, the zero quantity $\Lambda_{\boldsymbol{ABCD}}$ satisfies Equation (13.47). It remains to compute $\nabla^{\boldsymbol{b}}\Lambda_{\boldsymbol{bcd}}$ and express it in terms of zero quantities associated with the extended conformal field equations. A calculation using the commutator of the covariant derivative $\boldsymbol{\nabla}$ yields

$$\begin{aligned}2\nabla^{\boldsymbol{b}}\Lambda_{\boldsymbol{bcd}} &= 2\nabla^{\boldsymbol{b}}\nabla^{\boldsymbol{a}}d_{\boldsymbol{abcd}} = 2\nabla^{[\boldsymbol{b}}\nabla^{\boldsymbol{a}]}d_{\boldsymbol{abcd}}\\ &= 2R^{\boldsymbol{e}}{}_{[\boldsymbol{c}}{}^{\boldsymbol{ba}}d_{\boldsymbol{d}]\boldsymbol{eab}} - 2R^{\boldsymbol{e}}{}_{\boldsymbol{a}}{}^{\boldsymbol{ba}}d_{\boldsymbol{ebcd}} + \Sigma_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{a}}\nabla_{\boldsymbol{e}}d^{\boldsymbol{ab}}{}_{\boldsymbol{cd}}.\end{aligned}$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

Now, it is recalled that if $\hat{\boldsymbol{\nabla}} - \boldsymbol{\nabla} = \boldsymbol{S}(\boldsymbol{f})$, then $\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = \Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$. Moreover, using the formula relating the curvature tensors of the connections $\hat{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$, Equation (5.25b), the definitions of the zero quantities $\hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ and $\varsigma_{\boldsymbol{ab}}$ and the symmetries of $d_{\boldsymbol{abcd}}$, one concludes that

$$\nabla^{\boldsymbol{b}}\Lambda_{\boldsymbol{bcd}} = \hat{\Xi}^{\boldsymbol{e}}{}_{[\boldsymbol{c}}{}^{\boldsymbol{ba}}d_{\boldsymbol{d}]\boldsymbol{eab}} - \hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{a}}{}^{\boldsymbol{ba}}d_{\boldsymbol{ebcd}} + \frac{1}{2}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{a}}\nabla_{\boldsymbol{e}}d^{\boldsymbol{ab}}{}_{\boldsymbol{cd}} + \varsigma^{\boldsymbol{ab}}d_{\boldsymbol{abcd}}.$$

This expression is homogeneous in zero quantities and, thus, also its spinorial counterpart $\nabla^{\boldsymbol{AA'}}\Lambda_{\boldsymbol{A'ACD}}$. Consequently, if the *standard evolution equations* hold, it follows from Equation (13.47) and the calculations in the previous paragraph that

$$\mathcal{P}\Lambda_{\boldsymbol{AB}} - \mathcal{D}_{(\boldsymbol{A}}{}^{\boldsymbol{P}}\Lambda_{\boldsymbol{B})\boldsymbol{P}} - \frac{3}{\sqrt{2}}\chi^{\boldsymbol{P}}{}_{\boldsymbol{Q}}{}^{\boldsymbol{SQ}}\epsilon_{\boldsymbol{S}(\boldsymbol{A}}\Lambda_{\boldsymbol{BP})} = 2\nabla^{\boldsymbol{QQ'}}\Lambda_{\boldsymbol{QQ'AB}}$$

is homogeneous in zero quantities.

Finally, in the case of the *boundary adapted system*, one obtains a symmetric hyperbolic system of evolution equations of the form

$$\mathcal{P}\Lambda_{\boldsymbol{00}} + \mathcal{D}_{\boldsymbol{00}}\Lambda_{\boldsymbol{01}} = U_{\boldsymbol{00}}, \tag{13.66a}$$

$$\mathcal{P}\Lambda_{\boldsymbol{01}} + \mathcal{D}_{\boldsymbol{00}}\Lambda_{\boldsymbol{11}} - \mathcal{D}_{\boldsymbol{11}}\Lambda_{\boldsymbol{00}} = U_{\boldsymbol{01}}, \tag{13.66b}$$

$$\mathcal{P}\Lambda_{\boldsymbol{11}} - \mathcal{D}_{\boldsymbol{11}}\Lambda_{\boldsymbol{01}} = U_{\boldsymbol{11}}, \tag{13.66c}$$

where $U_{\boldsymbol{00}}$, $U_{\boldsymbol{01}}$ and $U_{\boldsymbol{11}}$ are expressions homogeneous in zero quantities.

The subsidiary equations for the auxiliary zero quantities

Direct computations show that

$$\begin{aligned}
2\hat{\nabla}_{[\boldsymbol{0}}\delta_{\boldsymbol{b}]} &= \hat{\nabla}_{\boldsymbol{0}}\delta_{\boldsymbol{b}} + \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\delta_{\boldsymbol{e}},\\
2\hat{\nabla}_{[\boldsymbol{0}}\gamma_{\boldsymbol{b}]\boldsymbol{c}} &= \hat{\nabla}_{\boldsymbol{0}}\gamma_{\boldsymbol{bc}} + \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\gamma_{\boldsymbol{ec}},\\
3\hat{\nabla}_{[\boldsymbol{0}}\varsigma_{\boldsymbol{bc}]} &= \hat{\nabla}_{\boldsymbol{0}}\varsigma_{\boldsymbol{bc}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\varsigma_{\boldsymbol{ce}} - \hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\varsigma_{\boldsymbol{eb}}.
\end{aligned}$$

For $\delta_{\boldsymbol{a}}$ one finds, using the definitions of the various zero quantities, that

$$\begin{aligned}
\hat{\nabla}_{[\boldsymbol{a}}\delta_{\boldsymbol{b}]} &= \hat{\nabla}_{\boldsymbol{a}}\beta_{\boldsymbol{b}} - \hat{\nabla}_{\boldsymbol{a}}f_{\boldsymbol{b}} - \Theta^{-1}\hat{\nabla}_{[\boldsymbol{a}}\hat{\nabla}_{\boldsymbol{b}]}\Theta\\
&= -\gamma_{[\boldsymbol{ab}]} + \varsigma_{\boldsymbol{ab}} - \frac{1}{2}\Theta^{-1}\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\hat{\nabla}_{\boldsymbol{e}}\Theta.
\end{aligned}$$

A lengthier computation yields

$$\begin{aligned}
2\hat{\nabla}_{[\boldsymbol{a}}\gamma_{\boldsymbol{b}]\boldsymbol{c}} =\ & 2\hat{\nabla}_{[\boldsymbol{a}}\hat{L}_{\boldsymbol{b}]\boldsymbol{c}} - 2\hat{\nabla}_{[\boldsymbol{a}}\hat{\nabla}_{\boldsymbol{b}]}\beta_{\boldsymbol{c}} + 2S_{\boldsymbol{c}[\boldsymbol{a}}{}^{\boldsymbol{ef}}\beta_{|\boldsymbol{e}|}\hat{\nabla}_{\boldsymbol{b}]}\beta_{\boldsymbol{f}}\\
&- 2\lambda\Theta^{-3}\hat{\nabla}_{[\boldsymbol{a}}\Theta\eta_{\boldsymbol{b}]\boldsymbol{c}} - 2\lambda\Theta^{-2}f_{[\boldsymbol{a}}\eta_{\boldsymbol{b}]\boldsymbol{c}}\\
=\ & \hat{\Delta}_{\boldsymbol{abc}} + \beta_{\boldsymbol{e}}\hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{cab}} - \hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\hat{\nabla}_{\boldsymbol{e}}\beta_{\boldsymbol{c}} + 2\beta_{\boldsymbol{c}}\gamma_{[\boldsymbol{ab}]} - 2\beta_{[\boldsymbol{a}}\gamma_{\boldsymbol{b}]\boldsymbol{c}} + \eta_{\boldsymbol{c}[\boldsymbol{a}}\beta^{\boldsymbol{e}}\gamma_{\boldsymbol{b}]\boldsymbol{e}}\\
&+ 2\lambda\Theta^{-2}\delta_{[\boldsymbol{a}}\eta_{\boldsymbol{b}]\boldsymbol{c}} + \beta_{[\boldsymbol{a}}\eta_{\boldsymbol{b}]\boldsymbol{c}}\beta_{\boldsymbol{e}}\beta^{\boldsymbol{e}} - 2\lambda\Theta^{-2}\eta_{\boldsymbol{c}[\boldsymbol{a}}\beta_{\boldsymbol{b}]}.
\end{aligned}$$

Similarly, using $d^{\boldsymbol{e}}{}_{[\boldsymbol{cab}]} = 0$, one obtains

$$\begin{aligned}\hat{\nabla}_{[\boldsymbol{a}}\varsigma_{\boldsymbol{bc}]} &= \hat{\nabla}_{[[\boldsymbol{a}}\hat{L}_{\boldsymbol{b}]\boldsymbol{c}]} - \hat{\nabla}_{[[\boldsymbol{a}}\hat{\nabla}_{\boldsymbol{b}]}f_{\boldsymbol{c}]}\\ &= \frac{1}{2}\hat{\Delta}_{[\boldsymbol{abc}]} - \frac{1}{2}d_{\boldsymbol{e}}d^{\boldsymbol{e}}{}_{[\boldsymbol{cab}]} + \frac{1}{2}\hat{R}^{\boldsymbol{e}}{}_{[\boldsymbol{cab}]}f_{\boldsymbol{e}} - \frac{1}{2}\hat{\Sigma}_{[\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\hat{\nabla}_{|\boldsymbol{e}|}f_{\boldsymbol{c}]}\\ &= \frac{1}{2}\hat{\Delta}_{[\boldsymbol{abc}]} + \frac{1}{2}\hat{\Xi}^{\boldsymbol{e}}{}_{[\boldsymbol{cab}]}f_{\boldsymbol{e}} - \frac{1}{2}\hat{\Sigma}_{[\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\hat{\nabla}_{|\boldsymbol{e}|}f_{\boldsymbol{c}]}.\end{aligned}$$

Hence, one obtains the evolution equations

$$\begin{aligned}&\hat{\nabla}_{\boldsymbol{0}}\delta_{\boldsymbol{i}} = \gamma_{\boldsymbol{i0}} - \hat{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\delta_{\boldsymbol{e}},\\ &\hat{\nabla}_{\boldsymbol{0}}\gamma_{\boldsymbol{ic}} = -\gamma_{\boldsymbol{jc}}\hat{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{0}} - \beta_{\boldsymbol{0}}\gamma_{\boldsymbol{ic}} - \beta_{\boldsymbol{c}}\gamma_{\boldsymbol{i0}} + \eta_{\boldsymbol{0c}}(\beta^{\boldsymbol{e}}\gamma_{\boldsymbol{ie}} - 2\lambda\Theta^{-2}\delta_{\boldsymbol{i}}),\\ &\hat{\nabla}_{\boldsymbol{0}}\varsigma_{\boldsymbol{jk}} = \hat{\Gamma}_{\boldsymbol{j}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\varsigma_{\boldsymbol{ke}} + \hat{\Gamma}_{\boldsymbol{k}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\varsigma_{\boldsymbol{ej}} + \frac{1}{2}\hat{\Delta}_{\boldsymbol{jk0}} + \frac{1}{2}\hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{0jk}}f_{\boldsymbol{e}} + \frac{1}{2}\hat{\Sigma}_{\boldsymbol{j}}{}^{\boldsymbol{e}}{}_{\boldsymbol{k}}\hat{\Gamma}_{\boldsymbol{e}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}f_{\boldsymbol{f}},\end{aligned}$$

where, in the last equation relation, (13.62) has been used to get further cancellation of terms.

13.4.6 Summary of the analysis

It is convenient to group the independent components of the spinorial fields in the extended conformal field equations as:

$\hat{\boldsymbol{\upsilon}}$ independent components of$e_{\boldsymbol{AA'}}{}^{\mu}$, $\hat{\Gamma}_{\boldsymbol{AA'BC}}$, $\hat{L}_{\boldsymbol{AA'BB'}}$,

$\boldsymbol{\phi}$ independent components of$\phi_{\boldsymbol{ABCD}}$.

Also, let $\boldsymbol{e}$ and $\hat{\boldsymbol{\Gamma}}$ denote, respectively, the independent components of the frame and connection coefficients. In terms of the above definitions one has:

Proposition 13.3 (***properties of the conformal evolution equations***) *The extended conformal vacuum Einstein field equations*

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0, \qquad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = 0, \qquad \hat{\Delta}_{\boldsymbol{abc}} = 0, \qquad \Lambda_{\boldsymbol{abc}} = 0,$$

expressed in terms of a conformal Gaussian gauge imply a symmetric hyperbolic system for the components of $(\hat{\boldsymbol{\upsilon}}, \boldsymbol{\phi})$ *of the form*

$$\begin{aligned}&\partial_\tau\hat{\boldsymbol{\upsilon}} = \mathbf{K}\hat{\boldsymbol{\upsilon}} + \mathbf{Q}(\hat{\boldsymbol{\Gamma}})\hat{\boldsymbol{\upsilon}} + \mathbf{L}(x)\boldsymbol{\phi},\\ &\big(\mathbf{I} + \mathbf{A}^0(\boldsymbol{e})\big)\partial_\tau\boldsymbol{\phi} + \mathbf{A}^\alpha(\boldsymbol{e})\partial_\alpha\boldsymbol{\phi} = \mathbf{B}(\hat{\boldsymbol{\Gamma}})\boldsymbol{\phi},\end{aligned}$$

where $\mathbf{I}$ *is the* 5×5 *unit matrix,* $\mathbf{K}$ *is a constant matrix,* $\mathbf{Q}(\hat{\boldsymbol{\Gamma}})$ *is a smooth matrix-valued function,* $\mathbf{L}(x)$ *is a smooth matrix-valued function of the coordinates,* $\mathbf{A}^\mu(\boldsymbol{e})$ *are Hermitian matrices depending smoothly on the frame coefficients* $\boldsymbol{e}$ *and* $\mathbf{B}(\hat{\boldsymbol{\Gamma}})$ *is a smooth matrix-valued function of the connection coefficients. In the case of the standard Bianchi system, the characteristic polynomial consists of the factors*

$$\tau^\mu\xi_\mu, \qquad g^{\mu\nu}\xi_\mu\xi_\nu, \qquad \left(\tau^\mu\tau^\nu + \frac{2}{3}g^{\mu\nu}\right)\xi_\mu\xi_\nu,$$

while for the boundary-adapted Bianchi system one has the factors

$$\tau^\mu \xi_\mu, \qquad g^{\mu\nu}\xi_\mu\xi_\nu, \qquad \big(\tau^\mu\tau^\nu + e_{\mathbf{00}}{}^{(\mu}e_{\mathbf{11}}{}^{\nu)}\big)\xi_\mu\xi_\nu.$$

Remark. It is important to emphasise the relative simplicity of the evolution system provided by Proposition 13.3 compared with the one given in Proposition 13.1. This structure reinforces the intuition that the Weyl tensor encodes the degrees of freedom of the gravitational field.

With regards to the subsidiary system one obtains a result analogous to Proposition 13.2:

Proposition 13.4 (***properties of the subsidiary evolution***) *Assume that the conditions*

$$\hat{\Sigma}_{\mathbf{0}}{}^{\mathbf{c}}{}_{\mathbf{b}} = 0, \qquad \hat{\Xi}^{\mathbf{c}}{}_{\mathbf{d0b}} = 0, \qquad \hat{\Delta}_{\mathbf{0bc}} = 0,$$

hold and that the associated evolution equations are expressed in terms of a conformal Gaussian gauge system. Moreover, let the independent components of the rescaled Weyl spinor satisfy either the standard or the boundary-adapted evolution system. Then, the independent components of the zero quantities

$$\hat{\Sigma}_{\mathbf{a}}{}^{\mathbf{c}}{}_{\mathbf{b}}, \qquad \hat{\Xi}^{\mathbf{c}}{}_{\mathbf{dab}}, \qquad \hat{\Delta}_{\mathbf{abc}}, \qquad \Lambda_{\mathbf{abc}}, \qquad \delta_{\mathbf{a}}, \qquad \gamma_{\mathbf{ab}}, \qquad \varsigma_{\mathbf{ab}},$$

which are not determined by either the evolution equations or gauge conditions satisfy a symmetric hyperbolic system which is homogeneous in zero quantities.

Controlling the conformal Gaussian gauge

The *conformal Gaussian* hyperbolic reduction procedure is based on the assumption of the existence of a non-singular (i.e. non-intersecting) congruence of conformal geodesics. While this assumption may be valid close to an initial hypersurface, it may fail at later times. To analyse the potential breakdown of the gauge, one appends to the evolution system given in Proposition 13.4 an evolution equation for the components of the deviation vector of the congruence; see Section 5.5.7.

In what follows, let $\boldsymbol{z}$ denote a ***separation vector*** for the congruence of conformal geodesics. Accordingly, one has

$$[\dot{\boldsymbol{x}}, \boldsymbol{z}] = 0.$$

Thus, writing $\boldsymbol{z} = z^{\mathbf{a}}\boldsymbol{e}_{\mathbf{a}}$ where $\{\boldsymbol{e}_{\mathbf{a}}\}$ is a Weyl propagated frame such that $\boldsymbol{e}_{\mathbf{0}} = \dot{\boldsymbol{x}}$, it follows that

$$\boldsymbol{e}_{\mathbf{0}}(z^{\mathbf{a}})\boldsymbol{e}_{\mathbf{a}} = z^{\mathbf{a}}[\boldsymbol{e}_{\mathbf{a}}, \boldsymbol{e}_{\mathbf{0}}].$$

Using the conformal field equation $\hat{\Sigma}_{\mathbf{a}}{}^{\mathbf{c}}{}_{\mathbf{b}} = 0$ and using that, in the present gauge, $\boldsymbol{e}_{\mathbf{0}} = \boldsymbol{\partial}_\tau$ and $\hat{\Gamma}_{\mathbf{0}}{}^{\mathbf{c}}{}_{\mathbf{b}} = 0$, the above expression can be rewritten as

$$\partial_\tau z^{\mathbf{a}} = \hat{\Gamma}_{\mathbf{b}}{}^{\mathbf{a}}{}_{\mathbf{0}} z^{\mathbf{b}}.$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

Now, let $z_{\boldsymbol{AA'}}$ denote the spinorial counterpart of $z^{\boldsymbol{a}}$. Defining the space spinor counterpart $z_{\boldsymbol{AB}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}} z_{\boldsymbol{AA'}}$ and using the split

$$z_{\boldsymbol{AB}} = \frac{1}{2} z \epsilon_{\boldsymbol{AB}} + z_{(\boldsymbol{AB})},$$

a computation similar to the one used to derive the evolution equations yields the following evolution equations for the irreducible components of $z_{\boldsymbol{AB}}$:

$$\partial_\tau z = f_{\boldsymbol{AB}} z^{(\boldsymbol{AB})}, \tag{13.67a}$$

$$\partial_\tau z_{(\boldsymbol{AB})} = \chi_{\boldsymbol{CD}(\boldsymbol{AB})} z^{(\boldsymbol{CD})}. \tag{13.67b}$$

The congruence of conformal geodesics will be non-intersecting as long as $z_{(\boldsymbol{AB})} \neq 0$.

13.5 Other hyperbolic reduction strategies

The hyperbolic reduction procedures discussed in Sections 13.2 and 13.4 do not exhaust the possible strategies to extract evolution equations from the conformal Einstein field equations. Indeed, other approaches have been put forward in the literature.

13.5.1 Hyperbolic reductions for the metric conformal field equations

Numerical evaluations of solutions to the vacuum conformal Einstein field equations have been carried out in Hübner (1999a,b, 2001a) using the metric formulation of the equations; see Equations (8.28a)–(8.28e). As the metric conformal field equations contain no equation which can be read as a differential equation for the components of the unphysical metric $\boldsymbol{g}$, one needs to supplement the equations in some manner. Assuming that suitable evolution equations can be found for the components of the conformal fields Ξ, Σ_a, s, L_{ab} and $d^a{}_{bcd}$ in some local coordinates $x = (x^\mu)$, the conformal metric $\boldsymbol{g}$ can be computed from the components of the Schouten tensor, $L_{\mu\nu}$, using ***generalised wave coordinates***; see the Appendix to this chapter for the vacuum Einstein field equations and the remark at the end of Section 8.2.5. More precisely, the components $g_{\mu\nu}$ of $\boldsymbol{g}$ are given as the solutions to the equations

$$\begin{aligned}
&\square g_{\mu\nu} - 2\nabla_{(\mu} F_{\nu)} - 2 g_{\lambda\rho} g^{\sigma\tau} \Gamma_\sigma{}^\lambda{}_\mu \Gamma_\tau{}^\rho{}_\nu - 4\Gamma_\lambda{}^\sigma{}_\rho g^{\lambda\tau} g_{\sigma(\mu} \Gamma_{\nu)}{}^\rho{}_\tau \\
&\quad = -4L_{\mu\nu} - \frac{1}{3} R(x) g_{\mu\nu}, \\
&\square x^\mu = -F^\mu(x), \qquad \text{that is,} \qquad \Gamma^\mu = F^\mu(x),
\end{aligned}$$

where $F^\mu(x)$ are some suitable ***coordinate gauge source functions*** and it has been used that $R_{ab} = 2L_{ab} + \frac{1}{6} R(x) g_{ab}$. Observe that in the right-hand side of the first of the above equations one has the Ricci scalar R, which, following the discussion from previous sections, is to be treated as a further gauge source function.

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

From an analytic point of view, the approach described in the previous paragraphs leads to an evolution system with equations of mixed order. This type of system requires a more general notion of hyperbolicity than the one discussed in Chapter 12: the so-called ***Leray hyperbolicity***; see, for example, Choquet-Bruhat (2008) and Rendall (2008).

13.5.2 Wave equations for the conformal fields

One way of avoiding mixed-order evolution systems is to construct wave equations for the components of the conformal fields Ξ, Σ_a, s, L_{ab} and $d^a{}_{bcd}$. This strategy has been pursued in Paetz (2015) for the metric (vacuum) conformal Einstein field equations. More precisely, it has been shown that by introducing suitable gauge source functions, the conformal field equations can be rewritten as a system of quasilinear wave equations for the conformal fields. An alternative reformulation can be obtained using spinors; see Gasperín and Valiente Kroon (2015). This approach is briefly discussed in the remainder of this section.

Wave equations for the concomitants of the conformal factor

Wave equations for the fields Ξ, $\Sigma_{\boldsymbol{AA'}}$ and s can be obtained from the following derivatives of the relevant zero quantities:

$$\nabla^{\boldsymbol{AA'}} Q_{\boldsymbol{AA'}} = 0, \qquad \nabla^{\boldsymbol{AA'}} Z_{\boldsymbol{AA'BB'}} = 0, \qquad \nabla^{\boldsymbol{AA'}} Z_{\boldsymbol{AA'}} = 0.$$

A direct computation renders the equations

$$\begin{aligned}
&\square\Xi - \nabla^{\boldsymbol{AA'}}\Sigma_{\boldsymbol{AA'}} = 0,\\
&\square\Sigma_{\boldsymbol{BB'}} + \Sigma^{\boldsymbol{AA'}} L_{\boldsymbol{AA'BB'}} + \Xi\nabla^{\boldsymbol{AA'}} L_{\boldsymbol{AA'BB'}} - \nabla_{\boldsymbol{BB'}} s = 0,\\
&\square s + \Sigma^{\boldsymbol{CC'}}\nabla^{\boldsymbol{AA'}} L_{\boldsymbol{AA'CC'}} + \nabla^{\boldsymbol{AA'}}\Sigma^{\boldsymbol{CC'}} L_{\boldsymbol{AA'CC'}} = 0.
\end{aligned}$$

The wave equation satisfied by the rescaled Weyl spinor

Recalling the definition of the zero quantity $\Lambda_{\boldsymbol{B'BCD}}$, one has

$$\begin{aligned}
\nabla_{\boldsymbol{A}}{}^{\boldsymbol{B'}}\Lambda_{\boldsymbol{B'BCD}} &= \nabla_{\boldsymbol{A}}{}^{\boldsymbol{B'}}\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{B'}}\phi_{\boldsymbol{BCDQ}}\\
&= -\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{B'}}\nabla_{\boldsymbol{Q})\boldsymbol{B'}}\phi_{\boldsymbol{BCD}}{}^{\boldsymbol{Q}} + \frac{1}{2}\epsilon_{\boldsymbol{AQ}}\nabla^{\boldsymbol{PP'}}\nabla_{\boldsymbol{PP'}}\phi_{\boldsymbol{BCD}}{}^{\boldsymbol{Q}}\\
&= \square_{\boldsymbol{AQ}}\phi_{\boldsymbol{BCD}}{}^{\boldsymbol{Q}} - \frac{1}{2}\square\phi_{\boldsymbol{ABCD}},
\end{aligned}$$

where $\square_{\boldsymbol{AB}}$ denotes the box operator discussed in Section 3.2.5. A further calculation shows that

$$\square_{\boldsymbol{AQ}}\phi_{\boldsymbol{BCD}}{}^{\boldsymbol{Q}} = 6\Xi\phi^{\boldsymbol{PQ}}{}_{(\boldsymbol{AB}}\phi_{\boldsymbol{CD})\boldsymbol{PQ}} - \frac{1}{4}R(x)\phi_{\boldsymbol{ABCD}}.$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

Thus, the condition $-2\nabla_{\boldsymbol{A}}{}^{\boldsymbol{Q}'}\Lambda_{\boldsymbol{BQ'CD}} = 0$ implies the wave equation

$$\square\phi_{\boldsymbol{ABCD}} - 12\Xi\phi^{\boldsymbol{PQ}}{}_{(\boldsymbol{AB}}\phi_{\boldsymbol{CD})\boldsymbol{PQ}} + \frac{1}{2}R(x)\phi_{\boldsymbol{ABCD}} = 0$$

for the components of the rescaled Weyl spinor as long as the conformal gauge source function $R(x)$ is explicitly provided.

The wave equation satisfied by the components of the Schouten spinor

To construct an equation for the Schouten spinor, one considers the expression

$$-2\nabla^{\boldsymbol{C}}{}_{\boldsymbol{C}'}\Delta_{\boldsymbol{CDBB}'} = 0,$$

as given by Equation (13.30) together with the decomposition (13.31) for the Schouten tensor in terms of the spinor $\Phi_{\boldsymbol{AA'BB'}}$ and the Ricci scalar. Accordingly, one has

$$\begin{aligned}2\nabla^{\boldsymbol{C}}{}_{\boldsymbol{C}'}\Delta_{\boldsymbol{CDBB}'} &= \nabla^{\boldsymbol{C}}{}_{\boldsymbol{C}'}\nabla_{\boldsymbol{C}}{}^{\boldsymbol{Q}'}\Phi_{\boldsymbol{DQ'BB'}} + \frac{1}{2}\epsilon_{\boldsymbol{DB}}\nabla^{\boldsymbol{C}}{}_{\boldsymbol{C}'}\nabla_{\boldsymbol{CB}'}R(x)\\ &\quad + \nabla^{\boldsymbol{C}}{}_{\boldsymbol{C}'}\Sigma^{\boldsymbol{Q}}{}_{\boldsymbol{B}'}\phi_{\boldsymbol{CDBQ}} + \Sigma^{\boldsymbol{Q}}{}_{\boldsymbol{B}'}\nabla^{\boldsymbol{C}}{}_{\boldsymbol{C}'}\phi_{\boldsymbol{CDBQ}},\end{aligned}$$

where

$$\begin{aligned}\nabla^{\boldsymbol{C}}{}_{\boldsymbol{C}'}\nabla_{\boldsymbol{C}}{}^{\boldsymbol{Q}'}\Phi_{\boldsymbol{DQ'BB'}} &= -\nabla^{\boldsymbol{C}}{}_{(\boldsymbol{C}'}\nabla_{|\boldsymbol{C}|\boldsymbol{Q}')}\Phi_{\boldsymbol{D}}{}^{\boldsymbol{Q}'}{}_{\boldsymbol{BB}'}\\ &\quad - \frac{1}{2}\epsilon_{\boldsymbol{C'Q'}}\nabla^{\boldsymbol{C}}{}_{\boldsymbol{P}'}\nabla_{\boldsymbol{C}}{}^{\boldsymbol{P}'}\Phi_{\boldsymbol{D}}{}^{\boldsymbol{Q}'}{}_{\boldsymbol{BB}'}\\ &= -\bar{\square}_{\boldsymbol{C'Q'}}\Phi_{\boldsymbol{D}}{}^{\boldsymbol{Q}'}{}_{\boldsymbol{BB}'} - \frac{1}{2}\square\Phi_{\boldsymbol{DC'BB'}},\\ \nabla^{\boldsymbol{C}}{}_{\boldsymbol{C}'}\nabla_{\boldsymbol{CB}'}R(x) &= \frac{1}{2}\epsilon_{\boldsymbol{C'B'}}\square R(x).\end{aligned}$$

Thus, using that

$$\begin{aligned}\bar{\square}_{\boldsymbol{C'Q'}}\Phi_{\boldsymbol{D}}{}^{\boldsymbol{Q}'}{}_{\boldsymbol{BB}'} &= \Phi^{\boldsymbol{PQ}'}{}_{\boldsymbol{BB}'}\Phi_{\boldsymbol{DC'PQ'}} + \Phi_{\boldsymbol{D}}{}^{\boldsymbol{Q'P}}{}_{\boldsymbol{B}'}\Phi_{\boldsymbol{BC'PQ'}}\\ &\quad + \Xi\bar{\phi}_{\boldsymbol{C'Q'B'S'}}\Phi_{\boldsymbol{D}}{}^{\boldsymbol{Q}'}{}_{\boldsymbol{B}}{}^{\boldsymbol{S}'}\\ &\quad - \frac{1}{8}R(x)\Phi_{\boldsymbol{DC'BB'}} - \frac{1}{24}R(x)\Phi_{\boldsymbol{DB'BC'}},\end{aligned}$$

one obtains the desired wave equation for the components of $\Phi_{\boldsymbol{AA'BB'}}$. Finally, a suitable subsidiary equation to ensure that the conformal gauge source function $R(x)$ is, indeed, the Ricci scalar of the connection $\boldsymbol{\nabla}$ can be obtained from the contracted Bianchi identity (13.29).

13.6 Further reading

The original references for the hyperbolic reduction procedure based on the use of spinors and gauge source functions are Friedrich (1983, 1991) – in particular, the latter reference contains a discussion of the hyperbolic reduction of the Einstein-Yang-Mills equations. The hyperbolic reduction procedure using a conformal Gaussian system was first discussed in Friedrich (1995, 1998c); extensions of these ideas to the non-vacuum case using conformal curves have been given in Lübbe and Valiente Kroon (2012). An alternative discussion of hyperbolic reductions of the conformal field equations using space spinors can be found in Frauendiener (1998a,b). A gauge source function-based hyperbolic reduction of the conformal Einstein-Euler system for a perfect fluid with a radiation equation of state has been described in Lübbe and Valiente Kroon (2013b). A discussion of the hyperbolic reduction of the conformal Einstein-scalar field system using gauge source functions is given in Hübner (1995).

A general discussion of the procedure of hyperbolic reduction of the *standard* Einstein field equations in the vacuum and matter case can be found in Friedrich and Rendall (2000), where, for example, the case of the Einstein-Dirac system is discussed. A related reference is Reula (1998). More specific discussions of hyperbolic reductions for the vacuum Einstein field equations and their associated subsidiary evolution systems can be found in Friedrich (1996, 2005). A *Lagrangian* hyperbolic reduction for the Einstein-Euler system has been discussed in Friedrich (1998b). Extensions of this Lagrangian approach have been obtained for the equations of relativistic magnetohydrodynamics coupled to gravity – the so-called Einstein-Euler-Maxwell system – in Pugliese and Valiente Kroon (2012) and for the Einstein-charged scalar field system in Pugliese and Valiente Kroon (2013).

Readers interested in the hyperbolic reductions of the Einstein field equations used in numerical relativity are referred to the monographs by Alcubierre (2008) and Baumgarte and Shapiro (2010) as an entry point to the extensive literature.

Appendix A.1: the reduced Einstein field equations

This chapter has been primarily focused on hyperbolic reductions for the conformal Einstein field equations in their spinorial formulation. In order to put the discussion into a more general context, it is useful to briefly consider the hyperbolic reduction procedure of the (standard) Einstein field equations using ***generalised wave coordinates***. This procedure is essentially the one used in the seminal work by Fourès-Bruhat (1952) where the well-posedness of the Cauchy problem in general relativity was first established.

For simplicity, in the following, the discussion is restricted to the vacuum case so that the Einstein field equations are equivalent to

$$\tilde{R}_{ab} = 0. \tag{13.68}$$

Given general coordinates $x = (x^\mu)$, the Ricci tensor can be explicitly written in terms of the components of the metric tensor $\tilde{\boldsymbol{g}}$ and its first and second partial derivatives as

$$\begin{aligned}\tilde{R}_{\mu\nu} = &-\frac{1}{2}\tilde{g}^{\lambda\rho}\partial_\lambda\partial_\rho\tilde{g}_{\mu\nu} + \tilde{\nabla}_{(\mu}\tilde{\Gamma}_{\nu)}\\ &+ \tilde{g}_{\lambda\rho}\tilde{g}^{\sigma\tau}\tilde{\Gamma}_\sigma{}^\lambda{}_\mu\tilde{\Gamma}_\tau{}^\rho{}_\nu + 2\tilde{\Gamma}_\lambda{}^\sigma{}_\rho\tilde{g}^{\lambda\tau}\tilde{g}_{\sigma(\mu}\tilde{\Gamma}_{\nu)}{}^\rho{}_\tau,\end{aligned} \tag{13.69}$$

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press

where it is recalled that the Christoffel symbols $\tilde{\Gamma}_\mu{}^\nu{}_\lambda$ can be written in terms of partial derivatives of the metric tensor as

$$\tilde{\Gamma}_\mu{}^\nu{}_\lambda = \frac{1}{2}\tilde{g}^{\nu\rho}(\partial_\mu \tilde{g}_{\rho\lambda} + \partial_\lambda \tilde{g}_{\mu\rho} - \partial_\rho \tilde{g}_{\mu\lambda}),$$

and one has defined

$$\tilde{\Gamma}^\nu \equiv \tilde{g}^{\mu\lambda}\tilde{\Gamma}_\mu{}^\nu{}_\lambda,$$

the so-called ***contracted Christoffel symbols***. The principal part of the vacuum Einstein field equation (13.68) is given by the terms

$$-\frac{1}{2}\tilde{g}^{\lambda\rho}\partial_\lambda\partial_\rho \tilde{g}_{\mu\nu} + \tilde{\nabla}_{(\mu}\tilde{\Gamma}_{\nu)}.$$

The first term in the above expression is hyperbolic as it coincides with the principal part of the D'Alambertian operator $\tilde{\square} \equiv \tilde{\nabla}^\mu\tilde{\nabla}_\mu$ acting on the components $\tilde{g}_{\mu\nu}$. If the second term in the principal part can be removed one would obtain a system of non-linear wave equations for $\tilde{g}_{\mu\nu}$.

Generalised wave coordinates

A systematic approach to the construction of coordinates $x = (x^\mu)$ is to require the coordinates to satisfy the equation

$$\tilde{\square} x^\mu = -F^\mu(x), \tag{13.70}$$

where the ***coordinate gauge source functions*** $F^\mu(x)$ are arbitrary smooth functions of the coordinates x. In the particular case where $F^\mu(x) = 0$ one talks of ***wave coordinates***, called ***harmonic coordinates*** in older accounts. In order to unravel the consequences of Equation (13.70), one treats the coordinates x^μ as scalar fields over $\tilde{\mathcal{M}}$. Accordingly, a direct computation gives

$$\begin{aligned} \tilde{\nabla}_\nu x^\mu &= \partial_\nu x^\mu = \delta_\nu{}^\mu, \\ \tilde{\nabla}_\lambda\tilde{\nabla}_\nu x^\mu &= \partial_\lambda \delta_\nu{}^\mu - \tilde{\Gamma}_\lambda{}^\rho{}_\nu \delta_\rho{}^\mu = -\tilde{\Gamma}_\nu{}^\mu{}_\lambda, \end{aligned}$$

so that

$$\tilde{\square} x^\mu = \tilde{g}^{\nu\lambda}\tilde{\Gamma}_\nu{}^\mu{}_\lambda = -\tilde{\Gamma}^\mu. \tag{13.71}$$

A natural way of prescribing initial conditions for Equation (13.70) on a hypersurface $\tilde{\mathcal{S}}_\star$ with normal ν^a is to set $x^0 = 0$ with $\nu^\mu\partial_\mu x^0 = 1$ while setting the spatial coordinates (x^α) to be equal to some given coordinates on $\tilde{\mathcal{S}}_\star$ and requiring that $\nu^\mu\partial_\mu x^\alpha = 0$. Given this data, the general theory of hyperbolic differential equations ensures the existence of a solution to Equation (13.70), and as a result of Equation (13.71), one concludes that

$$\tilde{\Gamma}^\mu = F^\mu(x). \tag{13.72}$$

Moreover, if the coordinate differentials $\mathbf{d}x^\mu$ are chosen initially to be pointwise independent on the initial hypersurface $\tilde{\mathcal{S}}_\star$, then they will also remain pointwise independent close to $\tilde{\mathcal{S}}_\star$. Thus, by a suitable choice of coordinates, the contracted Christoffel symbols can be made to agree, locally, with any prescribed set of functions $F^\mu(x)$. These coordinate gauge source functions and the data for Equation (13.71) uniquely determine the coordinates. Conversely, given any metric $\tilde{\boldsymbol{g}}$, any coordinate system is characterised by some suitable gauge source function and initial data. The domain on which the coordinates $x = (x^\mu)$ form a good coordinate system depends on the initial data, the coordinate gauge source functions and the metric $\tilde{\boldsymbol{g}}$ itself. Consequently, there is little that can be said, a priori, about the domain of existence of the coordinates.

The reduced Einstein equation and the subsidiary evolution equation

Substituting Equation (13.72) into the Einstein field equations in the form given by (13.69) one finds that

$$-\frac{1}{2}\tilde{g}^{\lambda\rho}\partial_\lambda\partial_\rho\tilde{g}_{\mu\nu} + \tilde{\nabla}_{(\mu}F_{\nu)}(x) + \tilde{g}_{\lambda\rho}\tilde{g}^{\sigma\tau}\tilde{\Gamma}_\sigma{}^\lambda{}_\mu\tilde{\Gamma}_\tau{}^\rho{}_\nu + 2\tilde{\Gamma}_\lambda{}^\sigma{}_\rho\tilde{g}^{\lambda\tau}\tilde{g}_{\sigma(\mu}\tilde{\Gamma}_{\nu)}{}^\rho{}_\tau = 0, \tag{13.73}$$

where $F_\mu(x) \equiv g_{\mu\nu}F^\nu(x)$. This equation is a system of quasilinear wave equations for the components of the metric tensor $\tilde{\boldsymbol{g}}$. For this system, the local Cauchy problem with data on a spacelike hypersurface $\tilde{\mathcal{S}}_\star$ is well posed – one can show the existence and uniqueness of solutions and their continuous dependence on the data; see, for example, Friedrich and Rendall (2000). Equation (13.73) is known as the ***reduced Einstein field equation***.

The introduction of a specific system of coordinates via the gauge source functions $F^\mu(x)$ breaks the tensoriality of the Einstein field equation (13.68). Given a solution to the reduced Einstein field equation (13.73) the latter will imply a solution to the actual Einstein field equations as long as the coordinates $x = (x^\mu)$ satisfy Equation (13.71) for the chosen coordinate source function $F^\mu(x)$ appearing in the reduced equation. To prove that this is the case one needs to construct a suitable *subsidiary evolution equation*.

A suitable subsidiary equation for the hyperbolic reduction procedure under consideration can be obtained by observing that the reduced Einstein field equation, Equation (13.73), can be written as

$$\tilde{R}_{\mu\nu} = \tilde{\nabla}_{(\mu}Q_{\nu)}, \qquad Q_\mu \equiv \tilde{\Gamma}_\mu - F_\mu(x), \tag{13.74}$$

where $\tilde{\Gamma}_\mu = \tilde{g}_{\mu\nu}\tilde{\Gamma}^\nu$. Now, from the contracted Bianchi identity in the form

$$\tilde{\nabla}^\mu\left(\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\tilde{g}_{\mu\nu}\right) = 0,$$

it follows, by substituting Equation (13.74), that

$$\tilde{\square}Q_\nu + \tilde{R}^\mu{}_\nu Q_\mu = 0.$$

From the homogeneity on Q_μ of this wave equation, it follows that if $Q_\nu = 0$ and $\tilde{\nabla}_\mu Q_\nu = 0$ on some initial hypersurface and if $\tilde{g}_{\mu\nu}$ satisfies the reduced Einstein field equations, then $\tilde{\Gamma}^\mu = F^\mu(x)$ at later times.

Appendix A.2: differential forms

Let $\mathcal{M}$ be a four-dimensional manifold. A ***p*-form** $\boldsymbol{\alpha}$ on $\mathcal{M}$ is a totally antisymmetric covariant tensor of rank p. Thus, if $\alpha_{i_1\cdots i_p}$ is the abstract index version of $\boldsymbol{\alpha}$, one has that

$$\alpha_{i_1\cdots i_p} = \alpha_{[i_1\cdots i_p]}.$$

Given $q \in \mathcal{M}$, the space of p-forms at q is denoted by $\Lambda^p|_q(\mathcal{M})$, while the bundle of p-forms over $\mathcal{M}$ is denoted by $\Lambda^p(\mathcal{M})$. In particular, 0-forms are scalar fields so that $\Lambda^0(\mathcal{M}) = \mathcal{X}(\mathcal{M})$ and 1-forms are covectors – accordingly, $\Lambda^1(\mathcal{M}) = T^*(\mathcal{M})$. A counting argument readily shows that $\dim \Lambda^p|_q(\mathcal{M}) = 4!/p!(4-p)!$ – thus, in four dimensions any 4-form is proportional to the volume form. Given a p-form $\boldsymbol{\alpha}$ and a q-form $\boldsymbol{\beta}$, their ***wedge product*** $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ is defined, using abstract index notation, as

$$(\alpha \wedge \beta)_{a_1\cdots a_p b_1\cdots b_q} \equiv \frac{(p+q)!}{p!q!} \alpha_{[a_1\cdots a_p}\beta_{b_1\cdots b_q]}.$$

Given local coordinates $x = (x^\mu)$ in $\mathcal{M}$, a 1-form $\boldsymbol{\alpha}$ can be written as $\boldsymbol{\alpha} = \alpha_\mu \mathbf{d}x^\mu$. More generally, for a p-form one has the expansion

$$\boldsymbol{\alpha} = \alpha_{\mu_1\cdots\mu_p} \mathbf{d}x^{\mu_1} \wedge \cdots \wedge \mathbf{d}x^{\mu_p}.$$

It can be verified that

$$\mathbf{d}x^\mu \wedge \mathbf{d}x^\nu = \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu - \mathbf{d}x^\nu \otimes \mathbf{d}x^\mu.$$

Given a p-form $\boldsymbol{\alpha}$ and a vector $\boldsymbol{v} = v^\mu \boldsymbol{\partial}_\mu$, one defines the ***contraction*** $i_{\boldsymbol{v}}\boldsymbol{\alpha}$ as the $(p-1)$-form

$$i_{\boldsymbol{v}}\boldsymbol{\alpha} \equiv v^\nu \alpha_{\nu\mu_1\cdots\mu_{p-1}} \mathbf{d}x^{\mu_1} \wedge \cdots \wedge \mathbf{d}x^{\mu_{p-1}}.$$

The ***exterior derivative*** $\mathbf{d}\boldsymbol{\alpha}$ is the $(p+1)$-form defined via the relation

$$\mathbf{d}\boldsymbol{\alpha} \equiv \partial_{[\mu_1}\alpha_{\mu_2\cdots\mu_{p+1}]} \mathbf{d}x^{\mu_1} \wedge \cdots \wedge \mathbf{d}x^{\mu_{p+1}}.$$

It follows from the commutativity of partial derivatives that $\mathbf{d}^2\boldsymbol{\alpha} = 0$.

Finally, it observed that the Lie derivative of a p-form can be computed using ***Cartan's formula***:

$$\pounds_{\boldsymbol{v}}\boldsymbol{\alpha} = i_{\boldsymbol{v}}\mathbf{d}\boldsymbol{\alpha} + \mathbf{d}i_{\boldsymbol{v}}\boldsymbol{\alpha}.$$

Further details on the above expressions can be found in, for example, Frankel (2003).

https://doi.org/10.1017/9781009291347.017 Published online by Cambridge University Press