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A Fritz John type sufficient optimality theorem in complex space

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A Fritz John type sufficient optimality theorem is proved for nonlinear programming problems in finite dimensional complex space over polyhedral cones, which may include equality as well as inequality constraints.

1. Introduction

Consider the pair of problems:

PROBLEM P1: Minimize $\text{Re} f(z, \bar{z})$
subject to $g(z, \bar{z}) \in S$;

and

PROBLEM P2: Minimize $\text{Re} f(z, \bar{z})$
subject to $g(z, \bar{z}) \in S$, $h(z, \bar{z}) = 0$,

where $f : C^{2n} \to C$, $g : C^{2n} \to C^m$, $h : C^{2n} \to C^l$ are analytic functions and S is a polyhedral cone in C^m .

Abrams and Ben-Israel [2] obtained Kuhn-Tucker type necessary optimality conditions for Problem P1 and Abrams [1] proved that these conditions are sufficient under certain convexity assumptions on f and g . In [5] and [6] Craven and Mond obtained Fritz John type necessary optimality conditions for Problems P1 and P2 respectively. Here, under

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certain convexity assumptions, we prove that conditions in [6] are sufficient for a point z_0 to be global optimal of Problem P2.

2. Preliminaries

Let C^n (R^n) denote n-dimensional complex (real) vector space, R^n_+ the non-negative orthant of R^n and $C^{m\times n}$ the set of $m \times n$ complex matrices. For $A \in C^{m\times n}$, $\bar{A}$, A^T, and A^H respectively denote conjugate, transpose, and conjugate transpose of A. We define

$$C_+ = \{z : z \in C, \text{Re}(z) \geq 0\}$$

and the manifolds

$$L = \left\{(z_1, z_2) \in C^{2l} : z_2 = \bar{z}_1\right\}$$

and

$$M = \left\{(w_1, w_2) \in C^{2m} : w_2 = \bar{w}_1\right\} .$$

For a non-empty set S, $S^* = \{y \in C^m : x \in S \Rightarrow \text{Re}\left(y^H x\right) \geq 0\}$ denotes the polar of S and by $\bar{S}$ we denote the set $\{\bar{x} : x \in S\}$.

Let $g : C^{2n} \to C^m$ be an analytic function and S and T be closed convex cones in C^m and R^m respectively; N be a manifold in C^{2n}.

DEFINITION 1. The function g is said to be strictly convex at z_0 with respect to S on N if for any $z \neq z_0$,

$$g(z, \bar{z}) - g(z_0, \bar{z}_0) - D_z g(z_0, \bar{z}_0)(z-z_0) - D_{\bar{z}} g(z_0, \bar{z}_0)(\bar{z}-\bar{z}_0) \in \text{int} S .$$

DEFINITION 2. The function g is said to have pseudo-convex real part at z_0 with respect to T on N if for any z,

$$\text{Re}\left[D_z g(z_0, \bar{z}_0)(z-z_0)+D_{\bar{z}} g(z_0, \bar{z}_0)(\bar{z}-\bar{z}_0)\right] \in T \Rightarrow \text{Re}\left[g(z, \bar{z})-g(z_0, \bar{z}_0)\right] \in T .$$

For other notations, definitions, and preliminary results we refer to Ben-Israel [3] and Craven and Mond [5].

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3. Sufficient optimality theorem

We shall make use of the following result of Ben-Israel [4].

LEMMA. *Let* $A \in C^{m\times n}$, $x \in C^m$, $y \in C^n$, *and* $S \subset C^m$ *be a polyhedral cone with non-empty interior. Then exactly one of the following two systems has a solution:*

(i) $-Ay \in \operatorname{int} S$,

(ii) $A^H x = 0$, $0 \neq x \in S^*$.

THEOREM. *Let* f, g, h , *and* S *be as in Problem* P2 *and* T *be a polyhedral cone in* C^l . *If*

(i) f *has pseudo-convex real part with respect to* R_+ *on* N ,

(ii) g *is strictly concave with respect to* S *on* N ,

(iii) h *is strictly convex with respect to* T *on* N ,

then a sufficient condition for z_0 *to be an optimal point of Problem* P2 *is the existence of an* $r_0 \in R_+$, $u_0 \in S^*$, *and* $v_0 \in T^*$, *not all zero, such that*

$$(1)\quad r_0\nabla_z f(z_0, \bar{z}_0) + r_0\overline{\nabla_{\bar{z}} f(z_0, \bar{z}_0)} + v_0^H D_z h(z_0, \bar{z}_0) + v_0^T \overline{D_{\bar{z}} h(z_0, \bar{z}_0)} = u_0^H D_z g(z_0, \bar{z}_0) + u_0^T \overline{D_{\bar{z}} g(z_0, \bar{z}_0)}$$

and

$$(2)\qquad \operatorname{Re}\left(u_0, g(z_0, \bar{z}_0)\right) = 0 .$$

Proof. Equation (2) can be written as

$$u_0^H g(z_0, \bar{z}_0) + u_0^T \overline{g(z_0, \bar{z}_0)} = 0 .$$

Let there exist a non-zero vector (r_0, u_0, v_0) , $r_0 \in R_+$, $u_0 \in S^*$, $v_0 \in T^*$, satisfying (1) and (2). Therefore there exists a solution to the system

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$$\begin{bmatrix} \nabla_z f(z_0, \bar{z}_0)+\overline{\nabla_{\bar{z}} f(z_0, \bar{z}_0)} & 0 \\ D_z h(z_0, \bar{z}_0) & 0 \\ \overline{D_{\bar{z}} h(z_0, \bar{z}_0)} & 0 \\ -D_z g(z_0, \bar{z}_0) & g(z_0, \bar{z}_0) \\ \overline{-D_{\bar{z}} g(z_0, \bar{z}_0)} & \overline{g(z_0, \bar{z}_0)} \end{bmatrix}^H \begin{bmatrix} r \\ v_1 \\ v_2 \\ u_1 \\ u_2 \end{bmatrix} = 0 ,$$

$$0 \neq (r, v_1, v_2, u_1, u_2) \in R_+ \times [(T^*\times\overline{T}^*)\cap L] \times [(S^*\times\overline{S}^*)\cap M] .$$

By the lemma, there exists no $p \in C^n$, $q \in C$, such that

$$-\begin{bmatrix} \nabla_z f(z_0, \bar{z}_0)+\overline{\nabla_{\bar{z}} f(z_0, \bar{z}_0)} & 0 \\ D_z h(z_0, \bar{z}_0) & 0 \\ \overline{D_{\bar{z}} h(z_0, \bar{z}_0)} & 0 \\ -D_z g(z_0, \bar{z}_0) & g(z_0, \bar{z}_0) \\ \overline{-D_{\bar{z}} g(z_0, \bar{z}_0)} & \overline{g(z_0, \bar{z}_0)} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

$$\in \operatorname{int}\{C_+\times \operatorname{Cl}[(T\times\overline{T})+L^*] \times \operatorname{Cl}[(S\times\overline{S})+M^*]\}$$

$$= \operatorname{int}C_+ \times \operatorname{int}[(T\times\overline{T})+L^*] \times \operatorname{int}[(S\times\overline{S})+M^*] .$$

Since any vector in $\operatorname{int}[(T\times\overline{T})+L^*]$ is of the form $\begin{bmatrix} t_1+\mu \\ \bar{t}_2-\bar{\mu} \end{bmatrix}$, where $t_1, t_2 \in \operatorname{int}T$ and $\mu \in C^l$, there exists no solution (p, q) to the system

$$(3) \quad \begin{cases} \operatorname{Re}[\nabla_z f(z_0, \bar{z}_0)p+\overline{\nabla_{\bar{z}} f(z_0, \bar{z}_0)p}] < 0 , \\ -D_z h(z_0, \bar{z}_0)p = t_1 + \mu , \\ \overline{-D_{\bar{z}} h(z_0, \bar{z}_0)p} = \bar{t}_2 - \bar{\mu} , \\ D_z g(z_0, \bar{z}_0)p - g(z_0, \bar{z}_0)q = s_1 + \lambda , \\ \overline{D_{\bar{z}} g(z_0, \bar{z}_0)p} - \overline{g(z_0, \bar{z}_0)q} = \bar{s}_2 - \bar{\lambda} , \end{cases}$$

for any $t_1, t_2 \in \operatorname{int}T$; $s_1, s_2 \in \operatorname{int}S$; $\mu \in C^l$ and $\lambda \in C^m$.

Consequently there exists no solution to the system

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$$(4)\qquad \begin{cases} \operatorname{Re}\left[f(z, \bar{z})-f(z_0, \bar{z}_0)\right] < 0 , \\ g(z, \bar{z}) \in S , \\ h(z, \bar{z}) = 0 . \end{cases}$$

For if it did have a solution z_1 , then by pseudo-convexity of the real part of f ,

$$\operatorname{Re}\left[f(z_1, \bar{z}_1)-f(z_0, \bar{z}_0)\right] < 0$$

implies

$$(5)\qquad \operatorname{Re}\left[\nabla_z f(z_0, \bar{z}_0)(z_1-z_0)+\nabla_{\bar{z}} f(z_0, \bar{z}_0)(\bar{z}_1-\bar{z}_0)\right] < 0 .$$

By strict convexity of h ,

$$h(z_1, \bar{z}_1) - h(z_0, \bar{z}_0) - D_z h(z_0, \bar{z}_0)(z_1-z_0) - D_{\bar{z}} h(z_0, \bar{z}_0)(\bar{z}_1-\bar{z}_0) \in \operatorname{int} T ,$$

which with $h(z_0, \bar{z}_0) = 0 = h(z_1, \bar{z}_1)$ gives

$$-D_z h(z_0, \bar{z}_0)(z_1-z_0) - D_{\bar{z}} h(z_0, \bar{z}_0)(\bar{z}_1-\bar{z}_0) = t \in \operatorname{int} T .$$

Now this gives

$$(6)\qquad -D_z h(z_0, \bar{z}_0)(z_1-z_0) = \frac{t}{2} + \mu$$

and

$$(7)\qquad -D_{\bar{z}} h(z_0, \bar{z}_0)(\bar{z}_1-\bar{z}_0) = \frac{t}{2} - \mu$$

for some $\mu \in C^l$.

Similarly, by strict concavity of g we obtain

$$(8)\qquad D_z g(z_0, \bar{z}_0)(z_1-z_0) + \frac{1}{2} g(z_0, \bar{z}_0) = \frac{s}{2} + \lambda$$

$$(9)\qquad D_{\bar{z}} g(z_0, \bar{z}_0)(\bar{z}_1-\bar{z}_0) + \frac{1}{2} g(z_0, \bar{z}_0) = \frac{s}{2} - \lambda ,$$

for some $s \in \operatorname{int} S$ and $\lambda \in C^m$.

Thus (5) to (9) give a contradiction to the fact that the system (3) has no solution. Hence the system (4) has no solution, which implies that z_0 is an optimal point of Problem P2.

REMARK. If the problem is in the form

$$\begin{aligned} &\text{Minimize} \quad \text{Re} f(z, \bar{z}) \\ &\text{subject to} \quad g(z, \bar{z}) \in S , \quad \text{Re} h(z, \bar{z}) = 0 , \end{aligned}$$

then a theorem similar to the above can be proved, requiring the real part of h to be strictly convex with respect to T on N, T being a polyhedral cone in R^l and taking T^*, the polar of T in real vector space.

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