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A PRIME ESSENTIAL RING THAT GENERATES A SPECIAL ATOM

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Abstract

A special atom (respectively, supernilpotent atom) is a minimal element of the lattice $\mathbb{S}$ of all special radicals (respectively, a minimal element of the lattice $\mathbb{K}$ of all supernilpotent radicals). A semiprime ring R is called prime essential if every nonzero prime ideal of R has a nonzero intersection with each nonzero two-sided ideal of R. We construct a prime essential ring R such that the smallest supernilpotent radical containing R is not a supernilpotent atom but where the smallest special radical containing R is a special atom. This answers a question put by Puczylowski and Roszkowska.

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1. Introduction

In this paper, all rings are associative and all classes of rings are closed under isomorphisms and contain the one-element ring zero. The fundamental definitions and properties of radicals can be found in [1, 12]. A class μ of rings is called hereditary if μ is closed under ideals. If μ is a hereditary class of rings, then $\mathcal{U}(\mu)$ denotes the upper radical determined by μ, that is, the class of all rings which have no nonzero homomorphic images in μ. As usual, for a radical ρ, the ρ radical of a ring R is denoted by $\rho(R)$ and the class of all ρ-semisimple rings is denoted by $\mathcal{S}(\rho)$. The class of all prime rings is denoted by π and $\beta = \mathcal{U}(\pi)$ denotes the prime radical. For a radical ρ, let $\pi(\rho) = \mathcal{S}(\rho) \cap \pi$. The notation $I \triangleleft R$ means that I is a two-sided ideal of a ring R. An ideal I of a ring R is called a prime (respectively, semiprime) ideal of R if $R/I \in \pi$ (respectively, $R/I \in \mathcal{S}(\beta)$). A ring R is called simple if, for every $I \triangleleft R$, either $I = 0$ or $I = R$. An ideal I of a ring R is called essential in R if $I \cap J \neq 0$ for any nonzero two-sided ideal J of R. A ring R is called an essential extension of a ring I if I is an essential ideal of R. A class μ of rings is called essentially closed if $\mu = \mu_k$, where $\mu_k = \{R\colon R \text{ is an essential extension of some } I \in \mu\}$. A hereditary and essentially closed class of prime rings is called a special class and the upper radical determined by a special class is called a special radical. A hereditary radical containing the prime radical β is called a supernilpotent radical.

It is well known [1, 15] that the families of special and supernilpotent radicals form complete lattices. We denote these lattices by $\mathbb{S}$ and $\mathbb{K}$, respectively. The smallest special (respectively, the smallest supernilpotent) radical containing a ring A will be denoted by $\widehat{l_A}$ (respectively, $\overline{l_A}$). It is easy to check that if α is a supernilpotent radical, then $\pi(\alpha)$ is a special class. Thus the upper radical $\widehat{\alpha}$ determined by this class is the smallest special radical containing α.

The problem of a description of special atoms (that is, the minimal elements in $\mathbb{S}$) and supernilpotent atoms (that is, the minimal elements in $\mathbb{K}$) was raised in [1]. Then it was studied in [2, 3, 6–8, 13] and [14]. This problem is also related to other long-standing open problems in radical theory such as the problem of Gardner which asks whether β is extra special [10] or the problem of Leavitt which asks whether there exists the smallest special class that determines the prime radical β [9]. Thus there is a motivation for studying special and supernilpotent atoms. One way to describe special or supernilpotent atoms is to study properties of rings that generate them.

All special (respectively, supernilpotent) atoms known to this day are of the form $\widehat{l_R}$ (respectively, $\overline{l_R}$), for some nonzero $*$-ring R, that is, a semiprime ring R such that $R/I \in \beta$ for every nonzero ideal I of R [7]. Therefore, in [14, Question 6], Puczylowski and Roszkowska pose a natural question which asks whether there exists a non $*$-ring R that generates a special atom. A nonsemiprime ring R that meets those prerequisites was constructed in [4]. But, clearly, it is sufficient to restrict our search for such rings to semiprime rings.

A semiprime ring R is called prime essential [11] if, for every nonzero prime ideal P of R, $P \cap I \neq 0$ whenever I is a nonzero two-sided ideal of R. Since the class $\mathcal{E}$ of all prime essential rings is hereditary [11] and every supernilpotent radical $\alpha \supsetneq \beta$ with $\mathcal{E} \subseteq \mathcal{S}(\alpha)$ is not special [5], it follows that every special radical $\alpha \supsetneq \beta$ contains a nonzero prime essential ring. It is therefore interesting to know whether every nonzero prime essential ring generates a special atom or contains a nonzero $*$-ring.

In this paper, we construct a prime essential ring R such that $\overline{l_R}$ is not a supernilpotent atom but where $\widehat{l_R}$ is a special atom. Prime essential rings cannot be $*$-rings since the latter must be prime, while the former cannot be. Therefore our example answers the question of Puczylowski and Roszkowska in the positive. It also gives impetus for further research related to special or supernilpotent atoms as well as the structure of prime essential rings.

2. Main results

We start with the following construction. Let $\mathbb{Q}$ be the field of all rational numbers. Then the set $B := \{b_i : i \in \mathbb{Q}\}$ forms a semigroup with respect to the multiplication given by

$$b_i b_j := b_{\max(i,j)}.$$

Let $R := \mathbb{Z}_2[B]$ be the semigroup ring of the semigroup B over the two element field $\mathbb{Z}_2$.

THEOREM 2.1. *The ring $R := \mathbb{Z}_2[B]$ is a commutative prime essential ring, each subring of which is an idempotent ring, and no nonzero ideal of R is a simple ring.*

PROOF. Clearly, R is a commutative ring. We will show that R is a Boolean ring (that is, $r^2 = r$ for every $r \in R$) which does not have an ideal isomorphic to $\mathbb{Z}_2$. Indeed, let $0 \neq r = b_{i_1} + \cdots + b_{i_m}$, where $i_1 < \cdots < i_m$. Since $b_i b_j = b_j b_i$ for every $b_i, b_j \in B$, $r^2 = (b_{i_1} + \cdots + b_{i_m})^2 = b_{i_1}^2 + \cdots + b_{i_m}^2 = b_{i_1} + \cdots + b_{i_m} = r$. This clearly implies that $S^2 = S$ for every subring S of R.

Let $0 \neq I \lhd R$. Then, there exists a nonzero element $r = b_{i_1} + \cdots + b_{i_k} \in I$ with $i_1 < \cdots < i_k$. If k is even, then, for every $i_{k-1} < j < i_k$, $0 \neq r(b_j + b_{i_k}) = b_j + b_{i_k} \in I$ and then $r \notin K := R(b_j + b_{i_k}) \lhd I$. If k is odd, then, for every $i_k < j$, $0 \neq rb_j = kb_j = b_j \in I$ and then $r \notin K := Rb_j \lhd I$. Thus, in either case, I contains a nonzero ideal $K \neq I$ which shows that I is not a simple ring. Since $\mathbb{Z}_2$ is a simple ring, this shows, in particular, that no nonzero ideal of R is isomorphic to $\mathbb{Z}_2$. Since every Boolean ring without ideals isomorphic to $\mathbb{Z}_2$ is prime essential [11, Example 3], it follows that R is a prime essential ring. □

REMARK 2.2. Recall that a subring A of a ring R is accessible in R if $A = I_0 \lhd I_1 \lhd \cdots \lhd I_n = R$, for some natural number n. Since the lower radical l_μ containing a hereditary class μ is hereditary [12], the smallest supernilpotent radical $\overline{l_\mu}$ containing any given class μ of rings is

$$l_{\{A:A \text{ is an accessible subring of some ring } R\in\mu\}\cup\beta} = \mathcal{U}(\mathcal{S}(l_{\{A:A \text{ is an accessible subring of some ring } R\in\mu\}}) \cap \mathcal{S}(\beta)).$$

Since $l_{\{A:A \text{ is an accessible subring of some ring } R\in\mu\}}$ is a hereditary radical, it follows that

$$\mathcal{S}(l_{\{A:A \text{ is an accessible subring of some ring } R\in\mu\}})$$

is essentially closed, and hence

$$\mathcal{S}(l_{\{A:A \text{ is accessible subring of some ring } R\in\mu\}}) \cap \pi$$

is essentially closed and hereditary for nonzero ideals and thus is a special class. Therefore, the smallest special radical $\widehat{l_\mu}$ containing μ is

$$\mathcal{U}(\mathcal{S}(l_{\{A:A \text{ is accessible subring of some ring } R\in\mu\}}) \cap \pi).$$

THEOREM 2.3. *Let R be the ring constructed above. Then $\overline{l_R}$ is not a supernilpotent atom but $\widehat{l_R}$ is a special atom.*

PROOF. Let r be a nonzero element of R. It follows, from the proof of Theorem 2.1, that R is a Boolean ring, which implies that R is commutative and $r = r^2$ so that $r \in Rr \lhd R$. Moreover, for every $x \in R$, $r(xr) = (xr)r = xr^2 = xr$, which shows that r is the identity element of Rr. Since any ideal of a Boolean ring is Boolean, it therefore follows that Rr is a Boolean ring with identity r. Then Rr can be homomorphically mapped onto a

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simple ring $\overline{Rr}$ with identity that is a prime ring. Since $\mathbb{Z}_2$ is the only Boolean prime ring, it therefore follows that $\overline{Rr} \simeq \mathbb{Z}_2$.

Now, since $\overline{l_R}$ is hereditary and $Rr \lhd R \in \overline{l_R}$, it follows that $Rr \in \overline{l_R}$. Then, since radical classes are homomorphically closed, $\mathbb{Z}_2 \in \overline{l_R} \backslash \beta$. This implies that $\beta \lneqq \overline{l_{\mathbb{Z}_2}} \leqslant \overline{l_R}$. Now, by Remark 2.2, $\overline{l_{\mathbb{Z}_2}} = l_{\{\mathbb{Z}_2\} \cup \beta}$. Thus, if $R \in \overline{l_{\mathbb{Z}_2}}$, then, since $R \in S(\beta)$, R would contain a nonzero ideal isomorphic to $\mathbb{Z}_2$, which contradicts Theorem 2.1. Thus $R \in \overline{l_R} \backslash \overline{l_{\mathbb{Z}_2}}$. Hence $\beta \lneqq \overline{l_{\mathbb{Z}_2}} \lneqq \overline{l_R}$, which shows that $\overline{l_R}$ is not a supernilpotent atom.

Similarly, since special radicals are hereditary, it follows that $\beta \lneqq \widehat{l_{\mathbb{Z}_2}} \lneqq \widehat{l_R}$, as $\mathbb{Z}_2 \in \widehat{l_{\mathbb{Z}_2}} \backslash \beta$. Moreover, since $\mathbb{Z}_2$ is a nonzero $*$-ring, it follows from [13] that $\widehat{l_{\mathbb{Z}_2}} = \mathcal{U}(\pi \backslash_{\{\mathbb{Z}_2\}})$ is a special atom. Suppose that $R \notin \widehat{l_{\mathbb{Z}_2}}$. Then R is semiprime so it has a homomorphic image in $\pi \backslash_{\{\mathbb{Z}_2\}}$. But this image, like R, is Boolean, so we have a contradiction. Thus $R \in \widehat{l_{\mathbb{Z}_2}}$, which implies that $\widehat{l_R} \leqslant \widehat{l_{\mathbb{Z}_2}}$. Consequently, $\widehat{l_R} = \widehat{l_{\mathbb{Z}_2}}$. But, since $\widehat{l_{\mathbb{Z}_2}}$ is a special atom, this shows that so is $\widehat{l_R}$, which ends the proof. □

In view of Theorem 2.3, two natural questions spring to mind.

Question 1. For which prime essential rings R is $\widehat{l_R}$ a special atom or is $\overline{l_R}$ a supernilpotent atom?

Question 2. Does the special radical $\widehat{l_A}$ contain a nonzero $*$-ring for every nonzero prime essential ring A?

Note that a positive answer to the second question will mean that the lattice $\mathbb{S}$ is atomic and every special atom is generated by a nonzero $*$ ring. This would also give a positive answer to the question of Gardner and the question of Leavitt.

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