J. Austral. Math. Soc. (*Series A*) **44** (1988), 252–258

LECS, LOCAL MIXERS, TOPOLOGICAL GROUPS AND SPECIAL PRODUCTS

CARLOS R. BORGES

(Received 18 December 1986)

Communicated by J. H. Rubinstein

Abstract

We prove that every (locally) contractible topological group is (L)EC and apply these results to homeomorphism groups, free topological groups, reduced products and symmetric products. Our main results are: The free topological group of a θ-contractible space is equiconnected. A paracompact and weakly locally contractible space is locally equiconnected if and only if it has a local mixer. There exist compact metric contractible spaces X whose reduced (symmetric) products are not retracts of the Graev free topological groups $F(X)$ $(A(X))$ (thus correcting results we published ibidem).

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 54 C 55; secondary 22 A 05, 57 S 05.

Keywords and phrases: θ-contractible, free topological group, reduced product, symmetric product, (L)EC, (local) mixers, homeomorphism groups.

Introduction

Let us recall that a space X is LEC (that is, locally equiconnected) if there exists a continuous function $\lambda: U \times I \to X$, where U is a neighbourhood of the diagonal of X, such that $\lambda(x,y,0) = x$, $\lambda(x,y,1) = y$ and $\lambda(x,x,t) = x$. X is EC (that is equiconnected) if $U = X \times X$.

Letting $\Delta^* = \{(x,x,y),(x,y,x),(y,x,x) | x,y \in X\}$ a *local* mixer for X is a continuous map $\mu: U \to X$ of a neighbourhood U of Δ^* in $X \times X \times X$ which satisfies the following condition: for each $x \in X$ and neighbourhood V of x, there

https://doi.org/10.1017/S1446788700029839 Published online by Cambridge University Press

exists a neighbourhood W of x such that

$$X \times W \times W \cup W \times X \times W \cup E \times W \times X \subset \mu^{-1}(V).$$

If $U = X \times X \times X$ then we call μ a *mixer* for X.

It is well known that ANR (metrizable) spaces are LEC (see [6]) and it is conjectured that (metrizable) LEC spaces are ANE (metrizable) (see [1] and [2] for partial answers to this conjecture).

In [13], a space X is called *semi-locally contractible* provided that it has an open cover $\{U_\alpha | \alpha \in \Lambda\}$, a family $\{\{a_\alpha\} | \alpha \in \Lambda\}$ of singleton subsets and a family $\{p_\alpha : U_\alpha \times I \to X | \alpha \in \Lambda\}$ of continuous functions such that each $p_\alpha(u, 0) = u$ and $p_\alpha(u, 1) = a_\alpha$, for all $u \in U_\alpha$. If each $a_\alpha \in U_\alpha$ then X is said to be weakly locally contractible. For convenience, let us say that each U_α is *contractible over* X (*to* a_α) and p_α *deforms* U_α (*to* a_α) *over* X. Fortunately, the concept of semi-local contractiblity is equivalent to that of weak local contractibility (see Lemma 6.1 in the Appendix).

Sakai [13] proves that a *metrizable space* is (L)EC if and only if it is (*semi-locally*) *contractible* and *has a* (*local*) mixer. Our Corollary 3.2 generalizes this result to paracompact spaces.

2. LEC groups

We prove that weak locally contractible topological groups are LEC. Let us recall that if X is a topological space, W is a neighbourhood of the diagonal of X and $z \in X$, then $W[z] = \{x \in X | (z, x) \in W\}$.

LEMMA 2.1. *Let H be a topological group and P an open symmetric neighbourhood of its unit element. Then there exists an open symmetric neighbourhood W of the diagonal of H such that $W[f] \subset fP$, for each $f \in H$.*

PROOF. Let $W = \{(f, g) \in H \times H | f^{-1}g \in P\}$ and note that W is an open symmetric neighbourhood of the diagonal of H (note that $\mu : H \times H \to H$, defined by $\mu(f, g) = f^{-1}g$, is continuous and $W = \mu^{-1}(P)$). We also get that, for each $f \in H$, $W[f] \subset fP$, because $(f, g) \in W$ if and only if $f^{-1}g \in P$ if and only if $g \in fP$. This completes the proof.

PROPOSITION 2.2. *Let H be a topological group, P an open symmetric neighbourhood of its unit element e and $\psi : P \times I \to H$ a continuous function such that $\psi(h, 0) = h$, $\psi(h, 1) = e$ and $\psi(e, t) = e$, for all $t \in I$ and $h \in P$. Then H is LEC.*

PROOF. By Lemma 2.1, let W be a symmetric neighbourhood of the diagonal such that $W[f] \subset fp$, for all $f \in H$. Define $\lambda: W \times I \to H$ by

$$\lambda(g, f, t) = f\psi(f^{-1}g, t).$$

(Note that λ is well defined, since $(g, f) \in W \Leftrightarrow (f, g) \in W \Rightarrow g \in W[f] \subset fP \Rightarrow f^{-1}g \in P$.) Then $\lambda(g, f, 0) = f\psi(f^{-1}g, 0) = ff^{-1}g = g$, $\lambda(g, f, 1) = f\psi(f^{-1}g, 1) = fe = f$, and $\lambda(g, g, t) = g\psi(g^{-1}g, t) = g\psi(e, t) = ge = g$. So we need only prove that λ is continuous, to complete the proof: Let $\{(g_\nu, f_\nu, t_\nu)\}$ be a net in $W \times I$ which converges to (g.f.t). Using the continuity of the inverse and multiplication maps for the topological group H, we then get that $\lim_\nu f_\nu^{-1} = f^{-1}$ and $\lim_\nu f_\nu^{-1}g_\nu = f^{-1}g$; so $\lim_\nu \psi(f_\nu^{-1}g_\nu, t_\nu) = \psi(f^{-1}g, t)$ and $\lim_\nu f_\nu\psi(f_\nu^{-1}g_\nu, t_\nu) = f\psi(f^{-1}g, t)$; equivalently, $\lim_\nu \lambda(g_\nu, f_\nu, t_\nu) = \lambda(g, f, t)$, which proves that λ is continuous.

THEOREM 2.3. *Let H be a topological group. H is LEC if and only if some nonempty open subset U of H is contractible over H to some $g \in H$.*

PROOF. The "if" part follows immediately from Proposition 6.2 in the Appendix and Proposition 2.2.

The "only if" part: Let D be a neighbourhood of the diagonal of X and $\lambda: D \times I \to X$ a continuous map such that $\lambda(f, g, 0) = f$, $\lambda(f, g, 1) = g$ and $\lambda(f, f, t) = f$. Pick a neighbourhood U of e such that $U \times U \subset D$ and define $\mu: U \times I \to H$ by $\mu(u, t) = \lambda(u, e, t)$; note that μ is continuous, $\mu(u, 0) = u$ and $\mu(u, 1) = e$, which shows that U is contractible over H. This completes the proof.

The preceding results prove that many homeomorphism groups are LEC. Henceforth, if M is a compact n-manifold, or a manifold which is homeomorphic to the interior of a compact n-manifold, and (M, N) is a proper manifold pair, let $\mathcal{H}(M)(\mathcal{H}(M, N);\ \mathcal{H}_1(M, N))$ denote the group of homeomorphisms of M (which are invariant on N; the identity on N) with the compact-open topology.

THEOREM 2.4. *The homeomorphism groups $\mathcal{H}(M)$, $\mathcal{H}(M, N)$ and $\mathcal{H}_1(M, N)$ are LEC.*

PROOF. From Corollaries 1.4 and 6.1 of [9] and Theorem 2.3, we get that $\mathcal{H}(M)$ is LEC. From Corollary 7.3 of [9] and Theorem 2.3, we get that $\mathcal{H}(M, N)$ and $\mathcal{H}_1(M, N)$ are LEC.

3. Equivalence of LEC and local mixers

The following results generalize Theorem I and the main Theorem of Sakai [13] to paracompact spaces.

THEOREM 3.1. *Let* $(X, \mathcal{T})$ *be a paracompact space. If* X *is* $(L)EC$ *then* X *has a (local) mixer.*

PROOF. (We only need to make some additions to the Proof of Theorem I of [13].) Let U be an open neighbourhood of the diagonal in X^2 and $\lambda: U \times I \to X$ a local equiconnecting function. For each $a \in X$, let U'_a and U''_a be open neighbourhoods of a in X such that $U'_a \times U'_a \subset U$ and $\lambda(U''_a \times U''_a \times I) \subset U'_a$. *Since* X *is paracompact, let* d *be a pseudometric for* X *such that* (the topology generated by d) $\mathcal{T}_d \subset \mathcal{T}$ *and* $\{\overline{B}(a,1) | a \in X\}$ *refines* $\{U''_a | a \in X\}$, where $\overline{B}(a,1) = \{x \in X | d(a,x) \leq 1\}$ (this can be done—see Lemma 38.1 of [14]). Next, *define a pseudometric* d^* on $X \times X \times X$ by $d^*((x,y,z),(x',y',z')) = \max\{d(x,x'), d(y,y'), d(z,z')\}$ *and note that* $\mathcal{T}_{d^*} \subset \mathcal{T}^3$ (the product topology on $X \times X \times X$ generated by $\mathcal{T}$).

The remainder of this proof is the same as that of Theorem I of [13], if one uses the topologies $\mathcal{T}_d$ and $\mathcal{T}_{d^*}$, *except for letting* W *be* a $\mathcal{T}$-neighbourhood of a *and picking* a $\mathcal{T}$-neighbourhood W'' of a *such that* $W'' \subset \overline{B}(a,1)$, keeping in mind that ($\mathcal{T}_d$-) $\mathcal{T}_{d^*}$-open or closed sets are also ($\mathcal{T}$-) $\mathcal{T}^3$-open or closed sets.

COROLLARY 3.2. *Let* X *be a paracompact space which is weak locally contractible. Then* X *is LEC if and only if* X *has a local mixer. If* X *is also contractible then* X *is EC if and only if* X *has a mixer.*

PROOF. Immediate from Theorem 1.3 and 1.5 of [3] and Theorem 3.1.

4. EC Graev groups

Concerning Graev free topological groups, throughout we will use the terminology of [4]. For the sake of brevity, we say that the space X is *θ-contractible* if $\theta \in X$ and there exsits a homotopy $h: X \times I \to X$ such that $h(x,0) = x$, $h(x,1) = \theta = h(\theta, t)$. (Let us call h a θ-homotopy.) It is possible for a compact metric space X to be contractible to a point $\theta \in X$ but not θ-contractible (see Example 6.4).

Even though we get very general conditions for the Graev free groups $(A(X), \mathcal{G})$ of a Tychonoff space X to be EC, we cannot find satisfactory conditions for those groups to be LEC.

THEOREM 4.1. *If X is a θ-contractible Tychonoff space then $(A(X), \mathcal{G})$ and $(F(X), \mathcal{G})$ are EC.*

PROOF. By the Proposition on page 2 of [12], we get that $(A(X), \mathcal{G})$ and $(F(X), \mathcal{G})$ are contractible. (The above mentioned proposition is proved for abelian groups $A(X), \mathcal{G})$, but it is clearly valid for $(F(X), \mathcal{G})$.) Therefore, by Theorem 2.3, $(A(X), \mathcal{G})$ and $(F(X), \mathcal{G})$ are contractible LEC spaces, which implies that they are EC.

QUESTION 4.2. If X is a Tychonoff locally contractible space are $(F(X), \mathcal{G})$ and $(A(X), \mathcal{G})$ LEC?

5. Symmetric and reduced products

In contrast to Theorem 4.1, there exists a compact subspace H of the euclidean plane whose reduced product H_∞ and symmetric product $SP^\infty H$ are not EC. Before establishing this fact, let us recall that a space X is an *h-space relative to $\theta \in X$* provided that there exists a continuous map $\mu: X \times X \to X$ such that $\mu(x, \theta) = \mu(\theta, x) = x$, for all $x \in X$. We will call X a *symmetric h*-space relative to $\theta \in X$ if also $\mu(a, b) = \mu(b, a)$, for all $a, b \in X$.

Let H be the "fan" subspace of the euclidean plane E^2 defined by $H = \{(0, y) | 0 \le y \le 1\} \cup \{(x, y) \in E^2 | \, |x| \le 1 \text{ and } y = kx \text{ for some } k = 0, 1, \dots\}$.

LEMMA 5.1. *The space H is compact metric and a symmetric h-space.*

PROOF. Clearly H is compact metric. So, letting $\overline{ab}$ denote the line segment joining a to b in E^2, let us define a function $\mu: H \times H \to H$ by

$$\mu(a, b) = \begin{cases} p \in \overline{\theta a}, & \text{with } |p| = |a| - |b|, \quad \text{if } |a| > |b|, \\ q \in \overline{\theta b}, & \text{with } |q| = |b| - |a|, \quad \text{if } |b| > |a|, \\ \theta, & \qquad\qquad\qquad\qquad\quad\; \text{if } |a| = |b|, \end{cases}$$

where θ denotes the origin of E^2. It is easily seen that μ is continuous. Clearly $\mu(a, \theta) = a = \mu(\theta, a)$; therefore H is a symmetric h-space.

PROPOSITION 5.2. *The reduced product H_∞ and the symmetric product $SP^\infty H$ are θ-contractible but they are not EC.* ∎

PROOF. (We deal only with H_∞, since the proof for $SP^\infty H$ is essentially the same.) Clearly H is θ-contractible (define $\mu: H \times I \to H$ by $\mu(x, t) = tx$). Therefore, we easily get that H_∞ is θ-contractible, by multiplying the homotopy μ. Therefore, $(F(H), \mathcal{G})$ is EC, by Theorem 4.1. Now, if H_∞ were a continuous

https://doi.org/10.1017/S1446788700029839 Published online by Cambridge University Press

retract of $(F(H), \mathcal{G})$, then H would also be a continuous retract of $(F(H), \mathcal{G})$, by Theorem 1.8 of [11] (clearly, this result remains valid for symmetric h-spaces and symmetric products); consequently, H would be EC, a contradiction (H is not even locally connected).

Proposition 5.2 shows that Theorems 2.6, 2.7, and 3.2 of [5] are false. (It is noteworthy that the error lies in the diagram of Theorem 2.6 of [5]. By starting with $(x_1, \ldots, x_i, x_i^{-1}, \ldots, x_n) \in (A_1(X))^n$, one immediately sees that it is *not commutative.*)

6. Appendix

The first two results should be folklore, but we cannot find them in the literature. The last result, which appears to be new, should prove very useful for a variety of problems on the extension of continuous functions.

LEMMA 6.1. *Let X be a space, $q \in U \subset X$ and $p \in X$. If there exists a continuous map $\mu: U \times I \to X$ such that $\mu(u, 0) = u$ and $\mu(u, 1)p$ then U is contractible over X to q.*

PROOF. Let $\alpha = \mu|\{q\} \times I$ and define $\psi: U \times I \to X$ by

$$\psi(u,t) = \begin{cases} \mu(u.2t), & 0 \le t \le 1/2, \\ \alpha(2-2t), & 1/2 \le t \le 1. \end{cases}$$

It is easily seen that ψ satisfies all requirements.

PROPOSITION 6.2. *Let H be a topological group and W a nonempty open subset of H which is contractible over H to some point $p \in H$. Then there exists a symmetric neighbourhood P of the unit element e of H and a continuous map $\psi: P \times I \to H$ such that $\psi(u, 0) = u$, $\psi(u, 1) = e = \psi(e, t)$, for all $u \in P$ and $t \in I$.*

PROOF. Let $\psi: W \times I \to H$ be a continuous map such that $\psi(w, 0) = w$, $\psi(w, 1) = p$. By Lemma 6.1, we assume that $p \in W$. Next, let $V = p^{-1}W$ and define $\psi': V \times I \to H$ by $\psi'(p^{-1}w, t) = p^{-1}\psi(w, t)$; clearly, ψ' is continuous and $\psi'(p^{-1}w, 0) = p^{-1}w$ and $\psi(p^{-1}w, 1) = e$. Finally, let $P = V \cap V^{-1}$ and define $\psi: P \times I \to H$ by $\psi(x, t) = (\psi'(e, t))^{-1}\psi'(x, t)$. It is easily seen that P and ψ satisfy all requirements.

PROPOSITION 6.3. *If a Hausdorff k-space Z is LEC then its cone CZ is EC.*

PROOF. Clearly CZ is LEC, by the Adjunction theorem on page 678 of [8], with $X = Z \times I$ (X is LEC, by Theorem II.2 of [8]), $A = Z \times \{0\}$, $Y = \{p\}$, for some $p \notin X$, and $f: A \to Y$ the constant function. (Note that the map $\pi_1: Z \times I \to A$, defined by $\pi_1(z,t) = (z,0)$, is a retraction. Also the map $\pi_2: Z \times I \to I$, defined by $\pi_2(z,t) = t$, is a halo for A in X.) Therefore, by Theorem 2.4 of [6], CZ is EC.

EXAMPLE 6.4. The space H in Lemma 5.1 is contractible to $p = (0, 1/2)$ but it is not p-contractible. H is also weakly locally contractible but is not locally contractible. (Note that if $h: H \times I \to H$ were a homotopy such that $h(p,t) = p$, for all $t \in I$, then, by compactness of I, there would exist $\delta > 0$ such that $h(B(p,\delta) \times I) \subset B(p, 1/4)$, where $B(x,s)$ denotes the ball with center x and radius s. This is impossible since there are no arcs in $B(p, 1/4)$ joining p to points with a nonzero abscissa. This also shows that H is not locally contractible.)

References

[1] C. R. Borges, 'A study of absolute extensor spaces', *Pacific J. Math.* **31** (1969), 609–617.
[2] ——, 'Absolute extensor spaces: A correction and an answer', *Pacific J. Math.* **50** (1974), 29–30.
[3] ——, '(Local) mixers and (L)EC-spaces', *Math. Japon.* **30** (1985), 85–88.
[4] ——, 'Free topological groups', *J. Austral. Math. Soc. Ser. A* **23** (1977), 360–365.
[5] ——, 'Free groups, symmetric and reduced products', *J. Austral. Math. Soc. Ser. A* **28** (1979), 174–178.
[6] J. Dugundji, 'Locally equiconnected spaces and absolute neighborhood retracts', *Fund. Math.* **57** (1965), 187–193.
[7] J. Dugundji, *Topology* (Allyn and Bacon, Boston, Mass., 1966).
[8] E. Dyer and S. Eilenberg, 'An adjunction theorem for locally equiconnected spaces', *Pacific J. Math.* **41** (1972), 669–685.
[9] R. Edwards and R. Kirby, 'Deformations of spaces and embeddings', *Ann. of Math.* (2) **93** (1971), 63–88.
[10] M. I. Graev, 'Free topological groups', *Amer. Math. Soc. Transl.* (Ser. 1) **8** (1962), 305–364.
[11] I. M. James, 'Reduced product spaces', *Ann. of Math.* (2) **62** (1955), 179–197.
[12] E. Katz, S. A. Morris and P. Nickolas, 'Free abelian topological groups and adjunction spaces' to appear.
[13] K. Sakai, 'A characterization of local equiconnectedness', *Pacific J. Math.* **111** (1984), 231–241.
[14] S. Willard, *General Topology* (Addison-Wesley, Reading, Mass., 1970).

Department of Mathematics
University of California
Davis, California 95616
U.S.A.