Glasgow Mathematical Journal (2023), **65**, pp. 414–429
doi:10.1017/S0017089523000010

Glasgow
Mathematical
Journal

RESEARCH ARTICLE

On the Hilbert scheme of the moduli space of torsion-free sheaves on surfaces

O. Mata-Gutiérrez[1], L. Roa-Leguizamón[2,*] and H. Torres-López[3]

[1]Departamento De Matemáticas, Centro Universitario De Ciencias Exactas E Ingenierías, Universidad De Guadalajara, Avenida Revolución 1500, Guadalajara, Jalisco, México
[2]Departamento De Matemáticas, Universidad De Los Andes, Carrera 1 #18A-12, 111 711, Bogota, Colombia
[3]Conacyt - U. A. Matemáticas, U. Autónoma De Zacatecas, Calzada Solidaridad Entronque Paseo A La Bufa, C.P. 98000 Zacatecas, Zac., México
*Corresponding author. E-mail: leonardo.roa@cimat.mx

Received: 25 February 2022; **Revised:** 19 December 2022; **Accepted:** 4 January 2023;
First published online: 2 February 2023

Keywords: elementary transformation, moduli of vector bundles, moduli of sheaves, Hilbert scheme

2020 Mathematics Subject Classification: *Primary* - 14D20, *Secondary* - 14J60

Abstract
The aim of this paper is to determine a bound of the dimension of an irreducible component of the Hilbert scheme of the moduli space of torsion-free sheaves on surfaces. Let X be a nonsingular irreducible complex surface, and let E be a vector bundle of rank n on X. We use the m-elementary transformation of E at a point $x \in X$ to show that there exists an embedding from the Grassmannian variety $\mathbb{G}(E_x, m)$ into the moduli space of torsion-free sheaves $\mathfrak{M}_{X,H}(n; c_1, c_2 + m)$ which induces an injective morphism from $X \times M_{X,H}(n; c_1, c_2)$ to $Hilb_{\mathfrak{M}_{X,H}(n;c_1,c_2+m)}$.

1. Introduction

Let X be a nonsingular irreducible complex projective variety of dimension d. Let E be a vector bundle of rank n and fixed Chern classes $c_i \in H^{2i}(X, \mathbb{Z})$ on X. The m-elementary transformation E' of E at the point $x \in X$ is defined as the kernel of a surjection $\alpha : E \longrightarrow \mathcal{O}_x^m$ which fits the exact sequence

$$0 \to E' \to E \to \mathcal{O}_x^m \to 0. \tag{1.1}$$

It is not hard to check that the class of such extensions is parameterized by $\mathbb{G}(E_x, m)$. This elementary transformation coincides with those defined by Maruyama, when X is a curve (see, [17]) but differs when dim $X \geq 2$, because the point $x \in X$ is not a divisor anymore.

Maruyama used his definition of elementary transformation to construct vector bundles on nonsingular projective varieties. Since then these elementary transformations have been a powerful tool in order to get topological and geometric properties of the moduli space of sheaves, for instance:

When X is a curve and $m = 1$, the elementary transformation E' of E is a vector bundle. Moreover, if E is a general stable vector bundle then E' is stable, and under this condition, Narasimhan and Ramanan used elementary transformations to determine certain subvarieties (called Hecke cycles) in the moduli space of vector bundles on curves, see [20, 21]. These Hecke cycles are contained in a component of the Hilbert scheme of the moduli space of vector bundles on curves (called Hecke component). Hence, Narasimhan and Ramanan computed a bound for the dimension of the Hecke component and proved that is nonsingular in those points defined by Hecke cycles. Moreover, when X is a curve and $m \geq 2$, Brambila-Paz and Mata-Gutiérrez in [2] generalized the construction of Hecke cycles using Grassmannians and defined Hecke Grassmannians. They proved that the corresponding Hecke component is nonsingular and a bound for its dimension was given.

In case that X is a surface and $m = 1$, Coskun and Huizenga [3] used elementary transformations to determine a component of the moduli space of vector bundles of rank two and compute a bound for its dimension. Also, Costa and Miró-Roig used priority sheaves and elementary transformations in the sense of Maruyama in order to establish maps between certain moduli spaces over $\mathbb{P}^2$ with the same rank and different Chern classes (see [7]).

The aim of this paper is to consider the case when X is a surface and $m \geq 1$, we use m-elementary transformations to determine Hecke cycles in the moduli space of stable torsion-free sheaves and determine geometrical aspects of a component of its Hilbert scheme. Specifically, we prove the following result (see Theorem 3.10):

Theorem 1.1. *The Hilbert scheme* $\mathrm{Hilb}_{\mathfrak{M}_{X,H}(n;\,c_1,c_2+m)}$ *of the moduli space of stable torsion-free sheaves has an irreducible component of dimension at least* $2 + \dim\, M_{X,H}(n; c_1, c_2)$.

The proof of this Theorem follows some ideas and techniques of [2, 20]. For a fixed vector bundle E and a point $x \in X$, we determine a closed embedding $\phi_z : \mathbb{G}(E_x, m) \mapsto \mathfrak{M}_{X,H}(n; c_1, c_2 + m)$ (see Proposition 3.4). We use the closed embedding ϕ_z to define the injective morphism

$$\begin{aligned} \psi : X \times M_{X,H}(n; c_1, c_2) &\longrightarrow \mathrm{Hilb}_{\mathfrak{M}_{X,H}(n;\,c_1,c_2+m)} \\ z &\mapsto \phi_z(\mathbb{G}(E_x, m)). \end{aligned}$$

Additionally, we establish the following morphism

$$\Phi : \mathbb{G}(\mathcal{U}, m) \to \mathfrak{M}_{X,H}(n; c_1, c_2 + m)$$

where $\mathcal{U}$ denotes the universal family parameterized by $M_{X,H}(n; c_1, c_2)$. This morphism allows us to determine an irreducible projective variety of $\mathfrak{M}_{X,H}(n; c_1, c_2 + m) - M_{X,H}(n; c_1, c_2 + m)$ and we get the following result (see Theorem 3.6):

Theorem 1.2. *Let m, n natural integers with $1 \leq m < n$. Then $\mathfrak{M}_{X,H}(n; c_1, c_2) - M_{X,H}(n; c_1, c_2 + m)$ contains an irreducible projective variety Y of dimension $3 + \dim\, M_{X,H}(n; c_1, c_2)$ such that the general element $F \in Y$ fits into exact sequence*

$$0 \to F \to E \to \mathcal{O}_{X,x} \otimes W \to 0,$$

where $E \in M_{X,H}(n; c_1, c_2)$, $W \in \mathbb{G}(E_x, m)$ and $x \in X$. In particular, if $n = 2$ then Φ is injective and Y is a divisor.

As an application of the previous result, we compute the Hilbert polynomial of the Hilbert scheme $\mathrm{Hilb}^P_{\mathfrak{M}_{X,H}(n;\,c_1,c_2)}$ which contains the cycle $\phi_z(\mathbb{G}(E_x, m))$ when X is the projective plane. In particular, we prove the following (see Theorem 4.3);

Theorem 1.3. *Assume that $c_1 = -1$ (resp. $c_1 = 0$) and that $c_2 \geq 2$ (resp. $c_2 \geq 3$ is odd). Let $L = a\epsilon + b\delta$, (resp. $a\varphi + b\psi$) be an ample line bundle in $Pic(\mathfrak{M}_{\mathbb{P}^2}(2; c_1, c_2))$. Then, $\mathcal{HG}$ is the component of the Hilbert scheme* $\mathrm{Hilb}^P_{\mathfrak{M}_{\mathbb{P}^2}(2;\,c_1,c_2)}$ *where P is the Hilbert polynomial defined as;*

$$\begin{aligned} P(m) &= \chi(\mathbb{P}(E_x), \phi_z^*(a\epsilon + b\delta)) = \chi(\mathbb{P}(E_x), \mathcal{O}_{\mathbb{P}(E_x)}(mb)). \\ (resp. \quad P(m) &= \chi(\mathbb{P}(E_x), \phi_z^*(a\varphi + b\psi)) = \chi(\mathbb{P}(E_x), \mathcal{O}_{\mathbb{P}(E_x)}(m(c_2 - 1)b))). \end{aligned}$$

The paper is organized as follows: Section 2 contains a brief summary of the main results of Grassmannians of vector bundles, moduli space of torsion-free sheaves, and m-elementary transformations. In Section 3, we give some technical results which allow us to prove our main results: Theorems 1.1 and 1.2. In Section 4, an application of the previous results is indicated for the Hilbert scheme of moduli space of rank 2 sheaves on the projective plane.

https://doi.org/10.1017/S0017089523000010 Published online by Cambridge University Press

2. Preliminaries

Let X be a nonsingular irreducible complex projective algebraic surface. This section contains a brief summary about stable torsion-free sheaves on surfaces, and we recall some basic facts on Grasmannians of vector bundles and m-elementary transformations see [9, 10, 14] for more details.

2.1. Grassmannian

We will collect here the principal properties of Grassmannians of vector bundles necessary for our purpose. For a fuller treatment, we refer the reader to [8, 25].

Let E be a vector bundle of rank n on X. Let $p_E : \mathbb{G}(E, m) \to X$ be the Grassmannian bundle of rank m quotients of E whose fiber at $x \in X$ is the Grassmannian $\mathbb{G}(E_x, m)$ of m-dimensional quotients of E_x, that is

$$\mathbb{G}(E, m) = \{(x, W) \mid x \in X,\ E_x \to W \to 0\}.$$

Let

$$0 \to S_E \to p^*E \to Q_E \to 0$$

be the tautological exact sequence over $\mathbb{G}(E, m)$ where S_E and Q_E denote the universal subbundle of rank $n - m$ and universal quotient of rank m, respectively. The tangent bundle of $\mathbb{G}(E, m)$ is the vector bundle $T\mathbb{G}(E, m) = Hom(S_E, Q_E)$ and hence $T_x\mathbb{G}(E, m) = Hom(S_{E_x}, Q_{E_x})$. Moreover, we have the following exact sequence:

$$0 \to T_{p_E} \to T\mathbb{G}(E, m) \to p_E^*TX \to 0$$

where T_{p_E} is the relative tangent bundle to the fibers and $T_{p_E} = S_E^* \otimes Q_E$.

2.2. Torsion-Free sheaves

Let H be an ample divisor on X. For a torsion-free sheaf $\mathcal{E}$ on X with Chern classes $c_i \in H^{2i}(X, \mathbb{Z})$, $i = 1, 2$ one sets

$$\mu_H(\mathcal{E}) := \frac{\deg_H(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}, \qquad P_m(\mathcal{E}) := \frac{\chi(\mathcal{E} \otimes H^m)}{\mathrm{rk}\,(\mathcal{E})},$$

where $\deg_H(\mathcal{E})$ is the degree of $\mathcal{E}$ defined by $c_1(\mathcal{E}).H$ and $\chi(\mathcal{E} \otimes H^m)$ denotes the Hilbert polynomial defined by $\sum(-1)^i h^i(X, \mathcal{E} \otimes H^m)$.

Definition 2.1. *Let H be an ample divisor on X. A torsion-free sheaf $\mathcal{E}$ on X is H-stable (resp. stable) if for all nonzero subsheaf $\mathcal{F} \subset \mathcal{E}$*

$$\mu_H(\mathcal{F}) < \mu_H(\mathcal{E}) \;\; (resp.\;\; P_m(\mathcal{F}) < P_m(\mathcal{E})).$$

We want to emphasize that both notions of stability depend on the ample divisor we fix on the underlying surface X and it is easily seen that H-stability implies stability.[1]

Recall that any H-stable (resp. stable) torsion-free sheaf is simple, i.e. if $\mathcal{E}$ is H-stable (resp. stable), then $\dim Hom(\mathcal{E}, \mathcal{E}) = 1$. We will denote by $M_{X,H}(n; c_1, c_2)$ the moduli space of H-stable vector bundles on X of rank n and fixed Chern classes c_1, c_2 and by $\mathfrak{M}_{X,H}(n; c_1, c_2)$ the moduli space of stable torsion-free sheaves on X. Since locally free is an open property and H-stability implies stability, it follows that $M_{X,H}(n; c_1, c_2)$ is an open subset of $\mathfrak{M}_{X,H}(n; c_1, c_2)$. In general an universal family on $X \times M_{X,H}(n; c_1, c_2)$

[1] The H-stability is frequently called Mumford–Takemoto stability and the stability is called Gieseker–Maruyama stability.

(resp. on $X \times \mathfrak{M}_{X,H}(n; c_1, c_2)$) does not exist, the existence of such universal family is guaranteed by the following criterion.

Lemma 2.2. [14, Corollary 4.6.7] *Let X be a nonsingular surface and let H be an ample divisor on X. Let n, c_1, c_2 fixed values for the rank and Chern classes. If $gcd(n, c_1.H, \frac{1}{2}c_1.(c_1 - K_X) - c_2) = 1$, then there is an universal family on $X \times M_{X,H}(n; c_1, c_2)$ (resp. $X \times \mathfrak{M}_{X,H}(n; c_1, c_2)$).*

2.3. m-elementary transformations.

Definition 2.3. *Let E be a locally free sheaf on X of rank n and Chern classes c_1, c_2 and let*

$$0 \to E' \to E \to \mathcal{O}_x^m \to 0 \tag{2.1}$$

be an exact sequence of sheaves, where $\mathcal{O}_x^m = \oplus_{i=1}^m \mathcal{O}_x$ is the sum of skyscraper sheaf with support on $x \in X$. The coherent sheaf E' is called the m*-elementary transformation of E at $x \in X$.*

Notice that even though E is locally free, its elementary transformation E' is a torsion free sheaf not locally free. Moreover if E is H-stable then E' is also H-stable. However, if E is stable then E' is not necessarily stable (see for instance [6, Remark 1]).

The m-elementary transformations have been used for several authors to construct many vector bundles on a higher dimensional projective variety and to determine topological and geometric properties of the moduli space of sheaves. For instance, Maruyama did a general study of elementary transformations of sheaves in his master's and doctoral theses [16, 17]. In [20] Narasimhan and Ramanan used elementary transformations of vector bundles on curves to introduce certain subvarieties in the moduli space of vector bundles which they called Hecke cycles. Brambila-Paz and the first author also used m-elementary transformations to describe a nonsingular open set of the Hilbert scheme of the moduli space of vector bundles on a curve [2]. Coskun and Huizenga have used elementary transformations to study priority sheaves since that they are well-behaved under elementary modifications [3–5].

We now collect some other basic properties related with m-elementary transformations in the following result.

Proposition 2.4. *Let H be an ample divisor on X. Let E be a vector bundle on X of rank n and Chern classes c_1, c_2, and let E' be a m-elementary transformation of E at $x \in X$, i.e. we have*

$$0 \to E' \to E \to \mathcal{O}_x^m \to 0. \tag{2.2}$$

Then,

(i) $rk(E') = n$, $c_1(E') = c_1$, $c_2(E') = c_2 + m$ and $\chi(E') = \chi(E) - m$.
(ii) E' is a torsion-free sheaf not locally free.
(iii) If E is H-stable, then E' is H-stable. Hence, E' is stable.

Proof.

(i) The proof follows directly from the exact sequence and Riemann–Roch Theorem.
(ii) Clearly E' is torsion free since E is a vector bundle. Now, suppose that E' is locally free, by [10, Chapter 4, Lemma 3], it follows that $E = E'$ which is impossible because $c_2(E') = c_2 + m$. Therefore E' is a torsion-free sheaf not locally free.
(iii) Let F be subsheaf of E' and assume that E is H-stable. It is clear that F is a subsheaf of E and by item (i), it follows that

$$\mu_H(F) < \mu_H(E) = \mu_H(E').$$

Hence E' is H-stable and therefore stable. □

https://doi.org/10.1017/S0017089523000010 Published online by Cambridge University Press

Remark 2.5. *The class of extensions (2.2) are parameterized by $\mathbb{G}(E_x, m)$. Furthermore, any $W \in \mathbb{G}(E_x, m)$ defines a surjective linear transformation $\tilde{\alpha}_W : E_x \to W \to 0$ which determines a surjective morphism of sheaves $\alpha_W : E \to \mathcal{O}_x^m$. If E^W denotes* $\ker(\alpha_W)$ *then we have the exact sequence:*

$$0 \to E^W \to E \to \mathcal{O}_x^m \to 0. \tag{2.3}$$

The following result will be used in the next sections:

Lemma 2.6. *Let E be a vector bundle on X and let $\mathcal{O}_x$ be the skyscraper sheaf with support on $x \in X$. Then, for any integer $m \geq 1$ we have*

$$Ext^i\left(\mathcal{O}_x^m, E\right) = 0, \quad i \neq 2.$$

For a deeper discussion of m-elementary transformations, we refer to reader to [2, 3].

2.4. Hecke cycles on the moduli space of vector bundles on curves.

Let X be a smooth projective curve, and let $x \in X$ be a point. For any vector bundle E on X, the m-elementary transformation

$$0 \to E' \to E \to \mathcal{O}_x^m \to 0 \tag{2.4}$$

determines a vector bundle E', where $\deg(E') = \deg(E) - m$ and $\mathrm{rk}(E') = \mathrm{rk}(E)$. If E is general in the moduli space $M_X(n, d)$ of stable vector bundles of rank n and degree d, then E' is stable (see [2, Proposition 2.4]).

In [20] Narasimhan and Ramanan considered the m-elementary transformations of type

$$0 \to E' \to E \to \mathcal{O}_x \to 0$$

to prove that, for a general $E \in M_X(n, d)$ (for an explicit description of the general open set in $M_X(n, d)$ see [20, Lemma 5.5]), the pair (E, x) determines a closed embedding

$$\Phi_{(E,x)} : \mathbb{P}\left(E_x^*\right) \to M_X(n, d-1). \tag{2.5}$$

(see, [20, Lemma 5.8]) and therefore $\mathbb{P}\left(E_x^*\right)$ can be considered as a subscheme of the moduli space $M_X(n, d-1)$. These projective subschemes are called Hecke cycles. Every Hecke cycle determines a point in the Hilbert scheme $\mathrm{Hilb}_{M_X(n,d-1)}$. Narasimhan and Ramanan proved that there is an open subscheme in $M_X(n, d)$ which is isomorphic to an open subscheme of $\mathrm{Hilb}_{M_X(n,d-1)}$ (see, [20, Theorem 5.13]).

Later, in [2] the authors generalize the ideas of Narasimhan and Ramanan and they considered m-elementary transformations, $m > 1$ in order to prove that, if $E \in M_X(n, d)$ is general (for an explicit description of the general open set in $M_X(n, d)$ see [2, Proposition 2.4]), then E' is stable. Moreover, every pair (E, x) determines a closed embedding

$$\Phi_{(E,x)} : \mathbb{G}(E_x, m) \to M_X(n, d-m) \tag{2.6}$$

(see [2, Proposition 3.1]) and therefore $\mathbb{G}(E_x, m)$ can be considered as a Grassmannian subvariety in the moduli space $M_X(n, d-m)$ which is called m-Hecke cycles. Hence, they concluded that $\mathrm{Hilb}_{M(n,d-m)}$ has an irreducible component $\mathcal{HG}$ of dimension $(n^2 - 1)(g - 1) + 1$ where every m-Hecke cycle determines a smooth point (see, [2, Theorem 1.1]).

The principal significance of [20, Lemma 5.8] and [2, Proposition 3.1] is that the morphisms (2.5) and (2.6) are closed embeddings. It allows determine m-Hecke cycles and geometric and topological properties of the Hilbert scheme $\mathrm{Hilb}_{M_X(n,d-m)}$.

https://doi.org/10.1017/S0017089523000010 Published online by Cambridge University Press

3. On the moduli space of torsion free sheaves

The aim of this section is to define an embedding from $\mathbb{G}(E_x, m)$ into the moduli space $\mathfrak{M}_{X,H}(n; c_1, c_2 + m)$ of torsion-free sheaves. Generalizing some techniques of [2, 20] we establish a closed embedding $\phi_z : \mathbb{G}(E_x, m) \to \mathfrak{M}_{X,H}(n; c_1, c_2 + m)$ and an injective algebraic morphism $\Psi : X \times M_{X,H}(n; c_1, c_2) \to \mathrm{Hilb}_{\mathfrak{M}_{X,H}(n;c_1,c_2+m)}$, where $z = (x, E) \in X \times M_{X,H}(n; c_1, c_2)$ and $Hilb_{\mathfrak{M}_{X,H}(n;c_1,c_2+m)}$ denotes the Hilbert scheme of the moduli space $\mathfrak{M}_{X,H}(n; c_1, c_2)$. Moreover, we construct an irreducible variety properly contained in $\mathfrak{M}_{X,H}(n; c_1, c_2 + m) - M_{X,H}(n; c_1, c_2 + m)$.

The following Lemma deals with m-elementary transformations, specifically we compute the dimension of the morphisms of a m-elementary transformation E' of E. The important point to note here is that E is a vector bundle. Here and subsequently, E denotes a vector bundle on X.

Lemma 3.1. *Let H be an ample divisor on X. Let E' be a torsion-free sheaf of rank n and let E be an H-stable vector bundle of rank n. If $c_1(E') = c_1(E)$, then* $\dim\ Hom(E', E) \leq 1$.

Proof. Let $f : E' \to E$ be a not zero homomorphism. By [10, Proposition 7, Chapter 4] the morphism f is injective and hence we have the sequence

$$0 \to E' \to E \to E/E' \to 0.$$

By [12, Proposition 6.4.], we have the following long exact sequence

$$\begin{aligned} 0 \to &\mathrm{Hom}(E/E', E) \to \mathrm{Hom}(E, E) \to \mathrm{Hom}(E', E) \to \\ &\mathrm{Ext}^1(E/E', E) \to \mathrm{Ext}^1(E, E) \to \mathrm{Ext}^1(E', E) \to \cdots \end{aligned}$$

Note that E/E' has support in a finite number of points because $c_1(E) = c_1(E')$, hence $\mathrm{Hom}(E/E', E) = 0$. On the other hand Lemma 2.6, implies that $\mathrm{Ext}^1(E/E', E) = 0$. Since E is a H-stable vector bundle, it follows that

$$\dim\ \mathrm{Hom}(E, E) = \dim\ \mathrm{Hom}(E', E) = 1$$

as we desired. □

Set $z := (x, E) \in X \times M_{X,H}(n; c_1, c_2)$ and let m be a fixed natural number with $m < n$. Let $\pi_E : \mathbb{G}(E, m) \to X$ be the Grassmannian bundle associated to E and for any $x \in X$ denote by $\mathbb{G}(E_x, m)$ the Grassmannian of m-quotients of E_x. On $\mathbb{G}(E, m)$, we have the tautological exact sequence

$$0 \to S_E \to \pi_E^* E \to Q_E \to 0, \tag{3.1}$$

where S_E is the universal subbundle and Q_E is the universal quotient bundle. Note that for any $x \in X$, if we restrict (3.1) to $\mathbb{G}(E_x, m)$ then we obtain

$$0 \to S_{E_x} \to \mathcal{O}_{\mathbb{G}} \times E_x \to Q_{E_x} \to 0. \tag{3.2}$$

Let us denote by $\mathbb{G}(z) := \mathbb{G}(E_x, m)$. Consider on $X \times \mathbb{G}(z)$, the surjective morphism $\alpha : p_1^* E \longrightarrow p_1^* \mathcal{O}_x \otimes p_2^* Q_{E_x}$ associated to the canonical surjective morphism $\alpha_x : \mathcal{O}_{\mathbb{G}} \times E_x \to Q_{E_x}$ in (3.2) under the isomorphism:

$$\begin{aligned} H^0\left(X \times \mathbb{G}(z), p_1^* E^* \otimes p_1^* \mathcal{O}_x \otimes p_2^* Q_{E_x}\right) &\cong H^0\left(\mathbb{G}(z), p_{2_*}(p_1^* E^* \otimes p_1^* \mathcal{O}_x) \otimes Q_{E_x}\right) \\ &\cong H^0\left(\mathbb{G}(z), p_{2_*} p_1^*(E_x^*) \otimes Q_{E_x}\right) \\ &\cong H^0\left(\mathbb{G}(z), \left(\mathcal{O}_{\mathbb{G}} \times E_x^*\right) \otimes Q_{E_x}\right) \\ &\cong H^0\left(\mathbb{G}(z), \mathcal{H}om\left(\mathcal{O}_{\mathbb{G}} \times E_x, Q_{E_x}\right)\right), \end{aligned}$$

where the second isomorphism is given by projection formula (see, [19], p. 76). Here, taking the kernel of the surjective morphism $\alpha : p_1^* E \longrightarrow p_1^* \mathcal{O}_x \otimes p_2^* Q_{E_x}$, we get the exact sequence

$$0 \longrightarrow \mathcal{F}_z \longrightarrow p_1^* E \longrightarrow p_1^* \mathcal{O}_x \otimes p_2^* Q_{E_x} \longrightarrow 0 \tag{3.3}$$

on $X \times \mathbb{G}(z)$.

https://doi.org/10.1017/S0017089523000010 Published online by Cambridge University Press

Lemma 3.2. *Let $z=(x,E)\in X\times M_{X,H}(n;c_1,c_2)$ and $W\in\mathbb{G}(z)$, then*

$$\mathcal{T}or^1\left(\mathcal{O}_{\{x\}\times\mathbb{G}},\mathcal{O}_{X\times\{W\}}\right)=0.$$

Proof. Restricting the exact sequence

$$0\to I_{\{x\}\times\mathbb{G}}\to\mathcal{O}_{X\times\mathbb{G}}\to\mathcal{O}_{\{x\}\times\mathbb{G}}\to 0$$

to $X\times\{W\}$, we get

$$0\to\mathcal{T}or^1\left(\mathcal{O}_{\{x\}\times\mathbb{G}},\mathcal{O}_{X\times\{W\}}\right)\to I_{\{x\}\times\mathbb{G}}|_{X\times\{W\}}\to\mathcal{O}_X\to\mathcal{O}_x\to 0$$

As is well-known $p_1^*I_x\cong I_{\{x\}\times\mathbb{G}}$ and $I_{\{x\}\times\mathbb{G}}|_{X\times\{W\}}\cong I_x$. Then it follows that

$$\mathcal{T}or^1\left(\mathcal{O}_{\{x\}\times\mathbb{G}},\mathcal{O}_{X\times\{W\}}\right)=0.$$

□

With the above notation and as consequence of Lemma 3.2, we have the following result.

Proposition 3.3. *If E is H-stable, then $\mathcal{F}_z$ is a family of stable torsion-free sheaves parameterized by $\mathbb{G}(z)$.*

Proof. Let $W\in\mathbb{G}(z)$. Restricting the exact sequence (3.3) to $X\times\{W\}$, we get the exact sequence

$$0\longrightarrow E^W\longrightarrow E\longrightarrow\mathcal{O}_x\otimes W\longrightarrow 0 \tag{3.4}$$

over X. Hence, E^W is a torsion-free sheaf of rank n called the m-elementary transformation of E in x defined by W. Since $c_1(\mathcal{O}_x\otimes W)=0$ and E is H-stable, it follows that E^W is H-stable and therefore stable with $c_1(E^W)=c_1(E)$ (see Proposition 2.4). Moreover, by Whitney sum and $c_2(\mathcal{O}_x\otimes W)=-\dim(W)=-m$ we get $c_2(E^W)=c_2(E)+m$ which completes the proof. □

The classification map of $\mathcal{F}_z$ is given by

$$\phi_z:\mathbb{G}(z)\to\mathfrak{M}_{X,H}(n;c_1,c_2+m)$$
$$W\mapsto E^W,$$

where E^W was defined in the above Proposition. The following result shows that the morphism ϕ_z is a closed embedding. For the proof of the proposition, we follow the techniques and ideas of [20, Lemma 5.10], and [2, Proposition 3.1] who proved a similar result for vector bundles on curves.

Proposition 3.4. *For any point $z=(x,E)\in X\times M_{X,H}(n;c_1,c_2)$, the morphism $\phi_z:\mathbb{G}(z)\to\mathfrak{M}_{X,H}(n;c_1,c_2+m)$ is a closed embedding.*

Proof. We first prove that the morphism ϕ_z is injective. Assume that there exist $W_1,W_2\in\mathbb{G}(z)$ such that $\psi:E^{W_1}\to E^{W_2}$ is an isomorphism, we claim that $W_1=W_2$. Recall that for any $i=1,2$, we have the following exact sequence

$$0\longrightarrow E^{W_i}\xrightarrow{f_i}E\xrightarrow{\alpha_i}\mathcal{O}_x\otimes W_i\longrightarrow 0\ .$$

By Lemma 3.1 we have $\dim\operatorname{Hom}(E^{W_1},E)=1$, it follows that there exist $\lambda\in\mathbb{C}^*$ such that $\lambda f_1=f_2\circ\psi$. Hence, $\operatorname{Im}f_{1,x}=\operatorname{Im}f_{2,x}$ which implies $W_1=W_2$. Therefore, ϕ_z is injective.

We now proceed to show the injectivity of the differential map $d\phi_z:T_W\mathbb{G}(z)\to\mathfrak{M}_{X,H}(n;c_1,c_2+m)$. By [20, Lemma 5.10], its infinitesimal deformation map in $W\in\mathbb{G}(z)$ is, up to the sign, the composition of the natural map $T_W\mathbb{G}(z)\to\operatorname{Hom}\left(E^W,\mathcal{O}_x\otimes W\right)$ with the boundary map $\operatorname{Hom}\left(E^W,\mathcal{O}_x\otimes W\right)\to\operatorname{Ext}^1(X,E^W,E^W)$ given by the long exact sequence

$$0\to\operatorname{Hom}\left(E^W,E^W\right)\to\operatorname{Hom}(E^W,E)\to\operatorname{Hom}\left(E^W,\mathcal{O}_x\otimes W\right)\to\operatorname{Ext}^1\left(E^W,E^W\right)\to\cdots$$

https://doi.org/10.1017/S0017089523000010 Published online by Cambridge University Press

obtained from (3.4). Notice that $\mathrm{Hom}\left(E^W, E^W\right) \cong \mathbb{C}$ because E^W is an H-stable free torsion sheaf. Moreover, $\mathrm{Hom}(E^W, E) \cong \mathbb{C}$ by Lemma 3.1. Therefore, the coboundary morphism

$$\delta : \mathrm{Hom}\left(E^W, \mathcal{O}_x \otimes W\right) \to \mathrm{Ext}^1\left(E^W, E^W\right)$$

is injective. □

As in [2, 20], a consequence of the above result is that we determine a collection of closed subschemes in $\mathfrak{M}_{X,H}(n; c_1, c_2 + m)$ and a collection of points in its Hilbert scheme (see, [20, Definition 5.12]). From a stable vector bundle E on X, we constructed the family $\mathcal{F}_z$ of stable torsion-free sheaves. Analogously, if we start with a family $\mathcal{E}$ of stable vector bundles on X parameterized by T, then we can construct a family of of stable torsion-free sheaves $\mathcal{F}$. In the next paragraphs, we describe the construction when $\mathcal{E}$ is the universal family of stable vector bundles parameterized by $M_{X,H}(n; c_1, c_2)$.

Let H be an ample divisor on X. As is well-known if $\gcd\left(n, c_1.H, \frac{1}{2}c_1.(c_1 - K_X) - c_2\right) = 1$, then there exists a universal family $\mathcal{U}$ of vector bundles parameterized by $M_{X,H}(n; c_1, c_2)$ (see Lemma 2.2). Under this conditions, we will determine a family $\mathcal{F}$ of stable torsion-free sheaves parameterized by $\mathbb{G}(\mathcal{U}, m)$ which extends to $\mathcal{F}_z$ (see Proposition 3.3).

Let $\mathcal{U}$ be the universal family of vector bundles parameterized by $M_{X,H}(n; c_1, c_2)$, hence $p : \mathcal{U} \to X \times M_{X,H}(n; c_1, c_2)$ is a vector bundle. We denote by $\pi_{\mathcal{U}} : \mathbb{G}(\mathcal{U}, m) \to X \times M_{X,H}(n; c_1, c_2)$ the Grassmannian bundle of quotients associated to $\mathcal{U}$. An element of $\mathbb{G}(\mathcal{U}, m)$ is a pair $((x, E), W)$, where $(x, E) \in X \times M_{X,H}(n; c_1, c_2)$ and $W \in \mathbb{G}(E_x, m)$. The tautological exact sequence over $\mathbb{G}(\mathcal{U}, m)$ is

$$0 \to S_{\mathcal{U}} \to \pi_{\mathcal{U}}^* \mathcal{U} \xrightarrow{\alpha} Q_{\mathcal{U}} \to 0, \tag{3.5}$$

where $Q_{\mathcal{U}}$ denotes the universal quotient bundle of rank m over $\mathbb{G}(\mathcal{U}, m)$. We now consider the graph of the following composition

$$\mathbb{G}(\mathcal{U}, m) \xrightarrow{\pi_{\mathcal{U}}} X \times M_{X,H}(n; c_1, c_2) \xrightarrow{p_1} X,$$

$\Gamma := \Gamma_{p_1 \circ \pi_U}$ as a subvariety of $X \times \mathbb{G}(\mathcal{U}, m)$. Then we have the following result.

Lemma 3.5. *Let $g \in \mathbb{G}(\mathcal{U}, m)$. Then*

(a) $\mathcal{T}or^1(I_{X\times\{g\}}, \mathcal{O}_\Gamma) = 0$.

(b) There exists a canonical surjective morphism of sheaves

$$(id \times p_2 \circ \pi_{\mathcal{U}})^* \mathcal{U} \to \mathcal{O}_\Gamma \otimes p^*_{\mathbb{G}(\mathcal{U})} Q_{\mathcal{U}} \to 0, \tag{3.6}$$

over $X \times \mathbb{G}(\mathcal{U}, m)$, determined by α, where $p_{\mathbb{G}(\mathcal{U})} : X \times \mathbb{G}(\mathcal{U}, m) \to \mathbb{G}(\mathcal{U}, m)$ and $p_2 : X \times M_{X,H}(n; c_1, c_2) \to M_{X,H}(n; c_1, c_2)$ are the respective second projections.

Proof. Taking $\beta := p_{\mathbb{G}(\mathcal{U})}|_\Gamma$ as the restriction of the projection, we have the following commutative diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\ i\ } & X \times \mathbb{G}(\mathcal{U}) \\ & {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle p_{\mathbb{G}(U)}} & \\ & \mathbb{G}(\mathcal{U}) & \end{array}$$

where $i : \Gamma \to X \times \mathbb{G}(\mathcal{U})$ is the inclusion map, hence $I_{X\times g}|_\Gamma = i^* p^*_{\mathbb{G}(\mathcal{U})}(I_g) = \beta^*(I_g)$.

From the exact sequence

$$0 \to I_g \to \mathcal{O}_{\mathbb{G}(\mathcal{U})} \to \mathcal{O}_g \to 0,$$

we get

$$0 \to \beta^*(I_g) \to \beta^*(\mathcal{O}_{\mathbb{G}(\mathcal{U})}) \to \beta^*(\mathcal{O}_g) \to 0,$$

Therefore, $\mathcal{T}or^1(I_{X\times\{g\}}, \mathcal{O}_\Gamma) = 0$ and this prove (a).

https://doi.org/10.1017/S0017089523000010 Published online by Cambridge University Press

Now, to prove (b) consider the surjective map $\alpha : \pi_{\mathcal{U}}^* \mathcal{U} \to Q_{\mathcal{U}}$ given in (3.5) and notice that $\beta^*\alpha : \beta^*\pi_{\mathcal{U}}^* \mathcal{U} \to \beta^* Q_{\mathcal{U}}$ is also surjective. Since $\beta^*\pi_{\mathcal{U}}^*(\mathcal{U}) \cong (id \times p_2 \circ \pi_{\mathcal{U}})^* (\mathcal{U})|_\Gamma$ and $\beta^* Q_{\mathcal{U}} \cong p_{\mathbb{G}(\mathcal{U})}^*(Q_{\mathcal{U}})|_\Gamma$, we get a surjective morphism

$$(id \times p_2 \circ \pi_{\mathcal{U}})^* (\mathcal{U})|_\Gamma \to \mathcal{O}_\Gamma \otimes p_{\mathbb{G}(\mathcal{U})}^* Q_{\mathcal{U}}. \tag{3.7}$$

Hence, from the exact sequence

$$0 \to (id \times p_2 \circ \pi_{\mathcal{U}})^* \mathcal{U} \otimes I_\Gamma \to (id \times p_2 \circ \pi_{\mathcal{U}})^* \mathcal{U} \to (id \times p_2 \circ \pi_{\mathcal{U}})^* \mathcal{U}\,|_\Gamma \to 0$$

and the morphism (3.7) we get the surjective map $(id \times p_2 \circ \pi_{\mathcal{U}})^* \mathcal{U} \to \mathcal{O}_\Gamma \otimes p_{\mathbb{G}(\mathcal{U})}^* Q_{\mathcal{U}}$ which completes the proof. □

According to the above Lemma, let us denote by $\mathcal{F}$ the kernel of the surjective morphism (3.6). Hence, we get the exact sequence

$$0 \to \mathcal{F} \to (id \times p_2 \circ \pi_{\mathcal{U}})^* \mathcal{U} \to \mathcal{O}_\Gamma \otimes p_{\mathbb{G}(\mathcal{U})}^* Q_{\mathcal{U}} \to 0. \tag{3.8}$$

Note that $(id \times p_2 \circ \pi_{\mathcal{U}})^* (\mathcal{U})|_{X \times ((x,E),W)} = E$ and $\mathcal{O}_\Gamma \otimes p_{\mathbb{G}(\mathcal{U})}^* Q_{\mathcal{U}}|_{X \times ((x,E),W)} = \mathcal{O}_x \otimes W$. Since $p_{\mathbb{G}(\mathcal{U})}^* Q_{\mathcal{U}}$ is a vector bundle and $\mathcal{T}or^1(I_{X \times \{g\}}, \mathcal{O}_\Gamma) = 0$, it follows that $\mathcal{T}or^1(I_{X \times \{g\}}, \mathcal{O}_\Gamma \otimes p_{\mathbb{G}(\mathcal{U})}^* Q_{\mathcal{U}}) = p_{\mathbb{G}(\mathcal{U})}^* Q_{\mathcal{U}} \otimes \mathcal{T}or^1(I_{X \times \{g\}}, \mathcal{O}_\Gamma) = 0$. Therefore, restricting the exact sequence (3.8) to $X \times \{((x, E), W)\}$, we get the exact sequence

$$0 \longrightarrow E^W \longrightarrow E \longrightarrow \mathcal{O}_x \otimes W \longrightarrow 0$$

over X. Moreover, if we restrict (3.8) to $X \times \mathbb{G}(z)$, we obtain (3.3).

Hence by similar arguments to Proposition 3.3, we have that $\mathcal{F}$ is a family of stable torsion-free sheaves of rank n of type $(c_1, c_2 + m)$ which determines a morphism

$$\begin{aligned} \Phi : \mathbb{G}(\mathcal{U}, m) &\rightarrow \mathfrak{M}_{X,H}(n; c_1, c_2 + m) \\ ((x, E), W) &\mapsto E^W. \end{aligned}$$

Note that Im Φ lies in $\mathfrak{M}_{X,H}(n; c_1, c_2 + m) - M_{X,H}(n; c_1, c_2 + m)$. In the following theorem, we compute the dimension of Im Φ.

Theorem 3.6. *Let m, n natural integers with* $1 \le m < n$. *Then* $\mathfrak{M}_{X,H}(n; c_1, c_2 + m) - M_{X,H}(n; c_1, c_2 + m)$ *contains an irreducible projective variety Y of dimension* $3 + \dim M_{X,H}(n; c_1, c_2)$ *such that the general element* $F \in Y$ *fits into exact sequence*

$$0 \to F \to E \to \mathcal{O}_{X,x} \otimes W \to 0,$$

where $E \in M_{X,H}(n; c_1, c_2)$, $W \in \mathbb{G}(E_x, m)$ *and* $x \in X$. *In particular, if* $n = 2$ *then* Φ *is injective and Y is a divisor.*

Proof. We will prove that image of Φ is an irreducible variety of dimension $3 + \dim M_{X,H}(n; c_1, c_2)$. For this, it will thus be sufficient to compute the dimension of the fibers of Φ. Let $F \in \text{Im}\,\Phi$, then there exists $((x, E), W) \in \mathbb{G}(\mathcal{U}, m)$ such that F fits into the following exact sequence

$$0 \to F \to E \to \mathcal{O}_{X,x} \otimes W \to 0, \tag{3.10}$$

where E is a vector bundle and $W \in \mathbb{G}(E_x, m)$. We claim $\dim \text{Ext}^1(\mathcal{O}_{X,x} \otimes W, F) = m^2$.

From the exact sequence (3.10), we get the long exact sequence

$$\begin{gathered} 0 \to \text{Hom}(\mathcal{O}_{X,x}, F) \to \text{Hom}(\mathcal{O}_{X,x}, E) \to \text{Hom}(\mathcal{O}_{X,x}, \mathcal{O}_{X,x} \otimes W) \to \\ \text{Ext}^1(\mathcal{O}_{X,x}, F) \to \text{Ext}^1(\mathcal{O}_{X,x}, E) \to \text{Ext}^1(\mathcal{O}_{X,x}, \mathcal{O}_{X,x} \otimes W) \to \dots \end{gathered}$$

Since $\text{Hom}(\mathcal{O}_{X,x}, E) = 0$ and by Lemma 2.6 $\text{Ext}^1(\mathcal{O}_{X,x}, E) = 0$, it follows that

$$\dim \text{Ext}^1(\mathcal{O}_{X,x}, F) = \dim \text{Hom}(\mathcal{O}_{X,x}, \mathcal{O}_{X,x} \otimes W) = m.$$

Thus, $\dim \text{Ext}^1(\mathcal{O}_{X,x} \otimes W, F) = m^2$.

https://doi.org/10.1017/S0017089523000010 Published online by Cambridge University Press

We now proceed to compute the dimension of Im Φ. Let p_i be denote the canonical projection of $X \times \mathbb{G}(E_x, m)$ for $i = 1, 2$ and consider the sheaf $\mathcal{H}om(p_1^*\mathcal{O}_x \otimes p_2^*\mathcal{Q}_{E_x}, p_1^*F)$. Taking higher direct image, we obtain on $\mathbb{G}(E_x, m)$ the sheaf:

$$\Lambda := R^1_{p_{2_*}} \mathcal{H}om\left(p_1^*\mathcal{O}_x \otimes p_2^*\mathcal{Q}_{E_x}, p_1^*F\right).$$

This Λ is locally free over $\mathbb{G}(E_x, m)$ because

$$H^0(\mathcal{H}om(\mathcal{O}_{X,x} \otimes W, F)) \cong \mathrm{Hom}(\mathcal{O}_{X,x} \otimes W, F) = 0,$$

for any $W \in \mathbb{G}(E_x, m)$. Hence, the fiber of Λ at $W \in \mathbb{G}(E_x, m)$ is $\mathrm{Ext}^1(\mathcal{O}_{X,x} \otimes W, F)$.

Let $\pi : \mathbb{P}\Lambda \to \mathbb{G}(E_x, m)$ denote the projectivization of the sheaf Λ. By [11, Lemma 3.2] there exists an exact sequence:

$$0 \to (id \times \pi)^* p_1^*F \otimes \mathcal{O}_{X\times\mathbb{P}\Lambda}(1) \to \mathcal{E} \to (id \times \pi)^*(p_1^*\mathcal{O}_{X,x} \otimes p_2^*\mathcal{Q}_{E_x}) \to 0 \tag{3.11}$$

on $X \times \mathbb{P}\Lambda$ such that, for each $p \in \mathbb{P}\Lambda$, its restriction to $X \times \{p\}$ is the extension

$$0 \longrightarrow F \longrightarrow \mathcal{E}_{|_p} \longrightarrow \mathcal{O}_{X,x} \otimes W \longrightarrow 0$$

where $\mathcal{E}_{|_p} := \mathcal{E}_{|_{X\times\{p\}}}$.

The set

$$U := \{p \in \mathbb{P}\Lambda \sim|\sim \mathcal{E}_{|_p} \text{ is locally free and stable}\}$$

is irreducible open set of dimension $m(n-m) + m^2 - 1 = mn - 1$. Therefore, the dimension of the fiber of Φ is $mn - 1 - m^2 = m(n-m) - 1$ and then we have

$$\begin{aligned}\dim\ \mathrm{Im}\ \Phi &= m(n-m) + 2 + \dim\ M_{X,H}(n; c_1, c_2) - m(n-m) + 1\\ &= 3 + \dim\ M_{X,H}(n; c_1, c_2).\end{aligned}$$

Note that for rank two case, the morphism ϕ is injective because the dimension of $\mathbb{P}\mathrm{Ext}^1(\mathcal{O}_{X,x} \otimes W, F) = 0$ and $\mathbb{P}\mathrm{Ext}^1(\mathcal{O}_{X,x} \otimes W, F)$ is irreducible. □

By functorial construction, we also have the following algebraic morphism

$$\begin{aligned}\Psi : X \times M_{X,H}(n; c_1, c_2) &\to \mathrm{Hilb}_{\mathfrak{M}_{X,H}(n;c_1,c_2+m)}\\ z = (x, E) &\mapsto \mathbb{G}(z)\end{aligned}$$

with $\mathbb{G}(z) := \phi_z(\mathbb{G}(E_x, m))$. This construction is essentially the same as the one carried out in [2, 20].

The injectivity of the function $\Psi : X \times M_{X,H}(n; c_1, c_2) \to \mathrm{Hilb}_{\mathfrak{M}_{X,H}(n;c_1,c_2+m)}$ is established in the next proposition. The proof proceeds as [2, Proposition 3.2] and we use the following two lemmas.

Lemma 3.7. *Let X be an irreducible variety and let*

$$0 \to F \to E \to G \to 0$$

be an exact sequence of sheaves over X. If E and G are locally free sheaves, then F is locally free.

Proof. Let H be a sheaf on X. We claim that for any locally free sheaf E on X $\mathcal{E}xt^i(E, H) = 0$. By [12, Proposition 6.8], we have

$$\mathcal{E}xt^i(E, H)_x \cong \mathrm{Ext}^i(E_x, H_x)$$

which is zero for any $x \in X$ because [10, Theorem 17]. Consider the exact sequence

$$0 \to F \to E \to G \to 0 \tag{3.12}$$

https://doi.org/10.1017/S0017089523000010 Published online by Cambridge University Press

where E and G are locally free sheaves. Applying the functor $\mathcal{H}om(-,H)$ to the exact sequence (3.12), we get

$$0 \to \mathcal{H}om(G,H) \to \mathcal{H}om(E,H) \to \mathcal{H}om(F,H) \to$$
$$\mathcal{E}xt^1(G,H) \to \mathcal{E}xt^1(E,H) \to \mathcal{E}xt^1(F,H) \to \mathcal{E}xt^2(G,H) \to \cdots$$

Note that $\mathcal{E}xt^i(G,H) = \mathcal{E}xt^i(E,H) = 0$ for $i > 0$. Therefore, $\mathcal{E}xt^1(F,H) = 0$ from which we conclude that F is locally free as we desired. □

Lemma 3.8 ([14], Lemma 8.2.12). *Let F_1 and F_2 be μ-semistable sheaves on X. If a is sufficiently large integer and $C \in |aH|$ a general nonsingular curve, then $F_1|_C$ and $F_2|_C$ are S-equivalent if and only if $F_1^{**} \cong F_2^{**}$*

Proposition 3.9. *The morphism $\Psi : X \times M_{X,H}(n;c_1,c_2) \to \mathrm{Hilb}_{\mathfrak{M}_{X,H}(n;c_1,c_2+m)}$ defined as above is injective.*

Proof. Assume that for $i = 1,2$, there exist $z_i = (x_i, E_i) \in X \times M_{X,H}(n;c_1,c_2)$ such that $\mathbb{G}(z_1) = \mathbb{G}(z_2)$, we want to prove that $E_1 \cong E_2$ and $x_1 = x_2$. We recall that for any $z_i = (x_i, E_i)$ there exists a family $\mathcal{F}_{z_i}$ of stable torsion-free sheaves parameterized by $\mathbb{G}(z_i)$, and $\mathcal{F}_{z_i}$ fits into the following exact sequence

$$0 \longrightarrow \mathcal{F}_{z_i} \longrightarrow p_1^*E_i \longrightarrow p_1^*\mathcal{O}_{x_i} \otimes p_2^*Q_{E_{x_i}} \longrightarrow 0 \qquad (3.13)$$

of sheaves over $X \times \mathbb{G}(z_i)$, where p_j denotes the j-projection over $X \times \mathbb{G}(z_i)$. From universal properties of moduli space $\mathfrak{M}_{X,H}(n;c_1,c_2+m)$, there exists an isomorphism $\beta : \mathbb{G}(z_1) \to \mathbb{G}(z_2)$ that induces the following commutative diagrams

$$\begin{array}{ccc} \mathbb{G}(z_1) & \xrightarrow{\beta} & \mathbb{G}(z_2) \\ & {\scriptstyle \phi_{z_1}} \searrow \quad \swarrow {\scriptstyle \phi_{z_2}} & \\ & \mathfrak{M}_{X,H}(n;c_1,c_2+m) & \end{array}$$

and

$$\begin{array}{ccc} X \times \mathbb{G}(z_1) & \xrightarrow{id_X \times \beta} & X \times \mathbb{G}(z_2) \\ & {\scriptstyle p_1} \searrow \quad \swarrow {\scriptstyle p_1'} & \\ & X & \end{array}$$

i.e. $\phi_{z_1} = \phi_{z_2} \circ \beta$ and $p_1 = p_1' \circ (id_X \times \beta)$. By the universal property of $\mathfrak{M}_{X,H}(n;c_1,c_2+m)$, we have

$$\mathcal{F}_{z_1} \cong (id_X \times \beta)^* \mathcal{F}_{z_2} \otimes p_2^*(L)$$

for some line bundle L on $\mathbb{G}(z_1)$. The following properties are satisfied:

(1) L is trivial.
(2) $R^1 p_{1*}(\mathcal{F}_{z_1}) = R^1 p'_{1*}(\mathcal{F}_{z_2}) = 0$.
(3) $p_{1*}\mathcal{F}_{z_1} = p'_{1*}\mathcal{F}_{z_2}$.

First we proved that $\mathcal{F}_{z_i}|_{\{y\} \times \mathbb{G}(z_i)} \cong E_y \otimes \mathcal{O}_{\mathbb{G}(z_i)}$ is trivial for any $y \neq x_i$. Restricting the exact sequence (3.13), we obtain

$$0 \to \mathcal{T}or^1\left(\mathcal{O}_{\mathbb{G}}, p_1^*\mathcal{O}_{x_i} \otimes p_2^*Q_{E_{x_i}}\right) \to \mathcal{F}_{z_i}|_{y \times \mathbb{G}(z_i)} \to p_1^*(E_i)|_{y \times \mathbb{G}(z_i)} \to 0.$$

https://doi.org/10.1017/S0017089523000010 Published online by Cambridge University Press

Note that $p_1^*(E_i)|_{y\times\mathbb{G}(z_i)} \cong E_y \otimes \mathcal{O}_{\mathbb{G}(z_i)}$ and $\mathcal{F}_{z_i}|_{y\times\mathbb{G}(z_i)}$ are vector bundle of the same rank, then by Lemma 3.7 we have $\mathcal{T}or^1\left(\mathcal{O}_{\mathbb{G}}, p_1^*\mathcal{O}_{x_i} \otimes p_2^*Q_{E_{x_i}}\right) = 0$ and $\mathcal{F}_{z_i}|_{y\times\mathbb{G}(z_i)} \cong E_y \otimes \mathcal{O}_{\mathbb{G}(z_i)}$. On the other hand

$$(id_X \times \beta)^*\left(\mathcal{F}_{z_2}\right)|_{y\times\mathbb{G}(z_1)} = \beta^*\left(\mathcal{F}_{z_2}|_{y\times\mathbb{G}(z_2)}\right) = \beta^*\left(E_y \otimes \mathcal{O}_{G(z_2)}\right) = E_y \otimes \mathcal{O}_{G(z_1)}.$$

Therefore,

$$E_y \otimes \mathcal{O}_{G(z_1)} = \mathcal{F}_{z_1}|_{y\times\mathbb{G}(z_1)} \cong \left((id_X \times \beta)^*\mathcal{F}_{z_2} \otimes p_2^*(L)\right)|_{y\times\mathbb{G}(z_1)} = E_y \otimes \mathcal{O}_{G(z_1)} \otimes L.$$

Thus, L is trivial [22, p. 12] and this prove (1). Moreover

$$\mathcal{F}_{z_1}|_{x_1\times\mathbb{G}(z_1)} \cong \left((id_X \times \beta)^*\mathcal{F}_{z_2}\right)|_{x_1\times\mathbb{G}(z_1)} = \beta^*\left(E_{x_1} \otimes \mathcal{O}_{G(z_2)}\right) = E_{x_1} \otimes \mathcal{O}_{\mathbb{G}(z_1)}.$$

And for any $y \in X$ we have

$$R^1p_{1_*}\left(\mathcal{F}_{z_1}\right)_y = H^1\left(\mathcal{F}_{z_1}|_{y\times\mathbb{G}(z_1)}\right) = H^1\left(E_y \otimes \mathcal{O}_{\mathbb{G}(z_1)}\right) = 0.$$

Similarly, we can prove that $\mathcal{F}_{z_2}|_{x_2\times\mathbb{G}(z_2)} \cong E_{x_2} \otimes \mathcal{O}_{\mathbb{G}(z_2)}$ and
$R^1p'_{1_*}\left(\mathcal{F}_{z_2}\right) = 0$ and this prove (2). Since $p_1 = p'_1 \circ (id \times \beta)$ and $(id_X \times \beta)$ is an isomorphism, we get

$$\begin{aligned} p_{1*}\left(\mathcal{F}_{z_1}\right) = p_{1*}(id\times\beta)^*\left(\mathcal{F}_{z_2}\right) &= \left(p'_1 \circ (id\times\beta)\right)_*(id\times\beta)^*\mathcal{F}_{z_2}\right) \\ &= p'_{1*}((id\times\beta)_*(id\times\beta)^*\left(\mathcal{F}_{z_2}\right)) \\ &= p'_{1_*}\left(\mathcal{F}_{z_2}\right), \end{aligned}$$

and this proves (3). We now proceed to show that $E_1 \cong E_2$ and $x_1 = x_2$. The proof will be divided into three steps:

Step 1: We will show that $E_1 \otimes I_{x_1} \cong E_2 \otimes I_{x_2}$.
Taking the direct image of (3.13) by p_1 we obtain the following exact sequence:

$$0 \to p_{1_*}\left(\mathcal{F}_{z_1}\right) \to p_{1_*}\left(p_1^*E_1\right) \to p_{1_*}\left(p_1^*\mathcal{O}_{x_1} \otimes p_2^*Q_{E_{1,x_1}}\right) \to 0$$

because $R^1p_{1_*}\left(\mathcal{F}_{z_1}\right) = 0$. And we can complete the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 \otimes I_{x_1} & \longrightarrow & E_1 & \longrightarrow & E_1 \otimes \mathcal{O}_{x_1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & p_{1*}(\mathcal{F}_{z_1}) & \longrightarrow & p_{1*}(p_1^*E_1) & \longrightarrow & p_{1*}(p_1^*\mathcal{O}_{x_1} \otimes p_2^*Q_{E_{1,x_1}}) & \longrightarrow & 0. \end{array}$$

Since $p_{1_*}p_1^*(E_1) \cong E_1$ and $p_{1_*}\left(p_1^*\mathcal{O}_{x_1} \otimes p_2^*Q_{E_{1,x_1}}\right) \cong E_1 \otimes \mathcal{O}_{x_1}$ by projection formula, it follows that $p_{1_*}\mathcal{F}_{z_1} \cong E_1 \otimes I_{x_1}$. We can now proceed analogously to obtain $p'_{1_*}\mathcal{F}_{z_2} \cong E_2 \otimes I_{x_2}$. Therefore,

$$E_1 \otimes I_{x_1} \cong p_{1_*}\mathcal{F}_{z_1} \cong p'_{1_*}\mathcal{F}_{z_2} \cong E_2 \otimes I_{x_2}.$$

Step 2: We will show that $E_1 \cong E_2$;
Note that the general curve on X does not goes through the points x_1 and x_2, hence $E_1|_C \cong (E_1 \otimes I_{x_1})|_C \cong (E_2 \otimes I_{x_1})|_C \cong E_2|_C$ for the general curve $C \in |aH|$. From Lemma 3.8, we conclude that $E_1 \cong E_2$ which is the desired conclusion.

Step 3: We show will that $x_1 = x_2$;
Notice that by step 1 there exists an isomorphism $\lambda : E_1 \otimes I_{x_1} \to E_2 \otimes I_{x_2}$. On the other hand, step 2 provided us an isomorphism $\phi : E_1 \to E_2$. Considering the exact sequence

$$0 \longrightarrow E_i \otimes I_{x_i} \xrightarrow{f_i} E_i \xrightarrow{\alpha_i} E_i \otimes \mathcal{O}_{x_i} \longrightarrow 0$$

https://doi.org/10.1017/S0017089523000010 Published online by Cambridge University Press

for $i=1,2$. Moreover $\phi\circ f_1$, $f_2\circ\lambda\in \mathrm{Hom}(E_1\otimes I_{x_1}, E_2)$, and hence by Lemma 3.1, $\phi\circ f_1 = t(f_2\circ\lambda)$ for some $t\in\mathbb{C}^*$. Without loss of generality, we suppose that $t=1$ therefore we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1\otimes I_{x_1} & \xrightarrow{f_1} & E_1 & \xrightarrow{\alpha_1} & E_1\otimes \mathcal{O}_{x_1} & \longrightarrow & 0 \\ & & \downarrow \lambda & & \downarrow \phi & & \downarrow \alpha & & \\ 0 & \longrightarrow & E_2\otimes I_{x_2} & \xrightarrow[f_2]{} & E_2 & \xrightarrow[\alpha_2]{} & E_2\otimes \mathcal{O}_{x_2} & \longrightarrow & 0, \end{array}$$

where α is an isomorphism of skyscraper sheaves supported at x_1 and x_2, respectively. Hence $x_1=x_2$. Therefore, Ψ is injective which establishes the proposition. □

We can now state our main result. The theorem computes a bound of the dimension of an irreducible subvariety of the Hilbert scheme $\mathrm{Hilb}_{\mathfrak{M}_{X,H}(n;\,c_1,c_2+m)}$.

Theorem 3.10. *The Hilbert scheme* $Hilb_{\mathfrak{M}_{X,H}(n;\,c_1,c_2+m)}$ *of the moduli space of stable vector bundles has an irreducible component of dimension at least* $2+\dim\, M_{X,H}(n;c_1,c_2)$.

Proof. The proof follows from Proposition 3.9. □

4. Application to the moduli space of sheaves on the projective plane

Let us denote by $\mathfrak{M}_{\mathbb{P}^2}(2;c_1,c_2)$ the moduli space of rank 2 stable sheaves on the projective plane $\mathbb{P}^2$ with respect to the ample line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$. By Proposition 3.4, the image $\phi_z(\mathbb{P}(z))$ defines a cycle in the Hilbert scheme of $\mathfrak{M}_{\mathbb{P}^2}(2;c_1,c_2)$

In this section, we will describe the component of the Hilbert scheme which contains the cycles $\phi_z(\mathbb{P}(E_x))$. Our computations use some results and techniques of [13, 24].

Definition 4.1. *Let E be a normalized rank 2 sheaf on $\mathbb{P}^2$. A line L (resp. a conic C) $\subset\mathbb{P}^2$ is jumping line (resp. jumping conic) if $h^1(L, E(-c_1-1)|_L)\neq 0$ (resp. $h^1(C, E|_C)\neq 0$).*

The following theorem was proved in [24]

Theorem 4.2. *Assume that $c_1=-1$ (resp. $c_1=0$) and that $c_2=n\geq 2$ (resp. $c_2=n\geq 3$ is odd). Then*

(i) $Pic(\mathfrak{M}_{\mathbb{P}^2}(2;c_1,c_2))$ is freely generated by two generators denoted by ϵ and δ (resp. φ and ψ).
(ii) An integral linear combination $a\epsilon+b\delta$ (resp. $a\varphi+b\psi$) is ample if and only if $a>0$ and $b>0$.
(iii) Consider the following sets in $\mathfrak{M}_{\mathbb{P}^2}(2;c_1,c_2)$:

$$D_1 = \{sheaves\ with\ a\ given\ jumping\ conic\ (resp-line)\}.$$

$$D_2 = \{sheaves\ with\ a\ given\ jumping\ line\ (resp.conic)\ passing\ through\ 1\ (resp.3)\ given\ points\}.$$

Then D_1 is the support of a reduced divisor in the linear system $|\epsilon|$ (resp. $|\varphi|$) and D_2 is the support of a reduced divisor in the linear system $|\delta|$ (resp. $|\frac{1}{2}(n-1)\psi|$).

Following the construction given in Section 3, if $z=(x,E)\in\mathbb{P}^2\times M_{\mathbb{P}^2}(2;c_1,c_2-1)$ then, Proposition 3.3, we have a family $\mathcal{F}_z$ of H-stable torsion-free sheaves rank two on $\mathbb{P}^2$ parameterized by $\mathbb{P}(E_x)$ or $\mathbb{P}(z)$ for short. Such family fits in the following exact sequence

$$0 \longrightarrow \mathcal{F}_z \longrightarrow p_1^*E \longrightarrow p_1^*\mathcal{O}_x \otimes p_2^*Q_{E_x} \longrightarrow 0, \tag{4.1}$$

defined on $\mathbb{P}^2 \times \mathbb{P}(z)$. The classification map of $\mathcal{F}_z$ is the morphism

$$\phi_z : \mathbb{P}(z) \to \mathfrak{M}_{\mathbb{P}^2}(2; c_1, c_2) \tag{4.2}$$

defined as $\phi_z(W) = E^W$.

We now use the exact sequence (4.1) and the morphism (4.2) to determine the irreducible component of the Hilbert scheme $\mathrm{Hilb}_{\mathfrak{M}_{\mathbb{P}^2}(2;c_1,c_2)}$ of the moduli space $\mathfrak{M}_{\mathbb{P}^2}(2; c_1, c_2)$, $c_1 = 0$ or -1 which contains the cycles $\phi_z(\mathbb{P}(z))$. This component is denoted by $\mathcal{HG}$.

For the proof of the theorem, we first establish the result for two particular cases: $c_1 = -1$ and $c_1 = 0$.

Theorem 4.3. *Under the notation of Theorem 4.2*

(1) *Assume that $c_1 = -1$ and let $c_2 \geq 2$. Let $\mathfrak{L} := a\epsilon + b\delta$ be an ample line bundle in Pic$(\mathfrak{M}_{\mathbb{P}^2}(2; c_1, c_2))$. Then, $\mathcal{HG}$ is the component of the Hilbert scheme $\mathrm{Hilb}^P_{\mathfrak{M}_{\mathbb{P}^2}(2;c_1,c_2)}$ where P is the Hilbert polynomial defined as;*

$$P(m) = \chi\left(\mathbb{P}(z), \phi_z^*(\mathfrak{L})\right) = \chi\left(\mathbb{P}(z), \mathcal{O}_{\mathbb{P}(z)}(mb)\right).$$

(2) *Assume that $c_1 = 0$ and let $c_2 \geq 3$ odd number. Let $\mathfrak{L} := a\varphi + b\psi$ be an ample line bundle in Pic$(\mathfrak{M}_{\mathbb{P}^2}(2; c_1, c_2))$. Then, $\mathcal{HG}$ is the component of the Hilbert scheme $\mathrm{Hilb}^P_{\mathfrak{M}_{\mathbb{P}^2}(2;c_1,c_2)}$ where P is the Hilbert polynomial defined as;*

$$P(m) = \chi\left(\mathbb{P}(z), \phi_z^*(\mathfrak{L})\right) = \chi\left(\mathbb{P}(z), \mathcal{O}_{\mathbb{P}(z)}\left(m\left(c_2 - 1\right)b\right)\right).$$

Proof.

(1) Let $z = (x, E) \in \mathbb{P}^2 \times M_{\mathbb{P}^2}(2; c_1, r)$, $c_1 = -1$ and $r \geq 1$. Consider the family $\mathcal{F}_z$ of stable sheaves of rank two given by the exact sequence (4.1). Then, $\mathcal{F}_{z_t} := (\mathcal{F}_z)|_{\mathbb{P}^2\times\{t\}}$ is stable for any $t \in \mathbb{P}(z)$ and by Proposition 2.4 its Chern classes are $c_1(\mathcal{F}_{z_t}) = -1$ and $c_2 := c_2(\mathcal{F}_{z_t}) = r + 1 \geq 2$. Therefore, we have the morphism

$$\phi_z : \mathbb{P}(E_x) \longrightarrow \mathfrak{M}_{\mathbb{P}^2}(2; c_1, c_2), \quad t \mapsto \mathcal{F}_z|_t$$

and set $\tau = p_1^*(\mathcal{O}_{\mathbb{P}^2}(1))$.

Now we will compute $\phi_z^*\epsilon$ and $\phi_z^*\delta$.

Let $l \geq 0$, from the exact sequence (4.1) we have

$$\begin{aligned} 0 \to & p_{2_*}\mathcal{F}(-l\tau) \to p_{2_*}p_1^*E(-l\tau) \to p_{2_*}p_1^*\mathcal{O}_x(-l\tau) \otimes p_2^*Q_{E_x} \to \\ & R^1p_{2_*}\mathcal{F}(-l\tau) \to R^1p_{2_*}p_1^*E(-l\tau) \to R^1p_{2_*}\left(p_1^*\mathcal{O}_x(-l\tau) \otimes p_2^*Q_{E_x}\right) \to 0. \end{aligned}$$

Using the projection formula, we get

$$R^ip_{2_*}p_1^*E(-l\tau) = \mathcal{O}_{\mathbb{P}(E_x)} \otimes H^i(\mathbb{P}^2, E(-l)).$$

Since $E(-l)$ is a stable vector bundle on $\mathbb{P}^2$ with $c_1 \leq 0$, it follows that $p_{2_*}p_1^*E(-l\tau) = 0$ and $R^ip_{2_*}p_1^*E(-l\tau)$ is a trivial bundle. Moreover, by similar arguments we have

$$R^ip_{2_*}\left(p_1^*\mathcal{O}_x(-l\tau) \otimes p_2^*Q_{E_x}\right) \cong Q_{E_x} \otimes p_{2_*}p_1^*\mathcal{O}_x(-l\tau) \cong Q_{E_x} \otimes H^i\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-l)_x\right).$$

Hence $R^1p_{2_*}p_1^*\mathcal{O}_x(-l\tau) \otimes p_2^*Q_{E_x} = 0$ and $p_{2_*}p_1^*\mathcal{O}_x(-l\tau) \otimes p_2^*Q_{E_x} = Q_{E_x}$. Therefore, we have the exact sequence

$$0 \to Q_{E_x} \to R^1p_{2_*}\mathcal{F}(-l\tau) \to R^1p_{2_*}p_1^*E(-l\tau) \to 0$$

where we conclude that $c_1(R^1p_{2_*}\mathcal{F}(-l\tau)) = 1$ for any $l \geq 0$.

According to [13, Lemmas 3.3 and 3.4], it follows that

$$\phi_z^*(\epsilon) = c_1\left(R^1p_{2_*}\mathcal{F}\right) - c_1(R^1p_{2_*}\mathcal{F}(-2\tau) = 0$$

and

$$\phi_z^*(\delta) = (r+1)c_1\left(R^1p_{2_*}\mathcal{F}\right) - rc_1\left(R^1p_{2_*}\mathcal{F}(-\tau)\right) = 1.$$

https://doi.org/10.1017/S0017089523000010 Published online by Cambridge University Press

Hence, we conclude that

$$P(m) = \chi(\mathbb{P}(z), \phi_z^*(a\epsilon + b\delta)) = \chi(\mathbb{P}(z), \mathcal{O}_{\mathbb{P}(E_x)}(mb))$$

as we desired.

(2) For the case, $c_1 = 0$ and $c_2 \geq 3$ odd. Consider $z = (x, E) \in \mathbb{P}^2 \times M_{\mathbb{P}^2}(2; c_1, r)$, $c_1 = 0$ and $r \geq 2$ even. From the exact sequence (4.1), we get $\mathcal{F}_{z_t} := \mathcal{F}_{z|_{\mathbb{P}^2 \times \{t\}}}$ is stable for all $t \in \mathbb{P}(E_x)$ and $c_1(\mathcal{F}_{z_t}) = 0$, $c_2 := c_2(\mathcal{F}_{z_t}) = r + 1 \geq 3$ odd. By [13, Lemmas 3.3 and 3.4] we have that

$$\phi_z^*(\varphi) = c_1\left(R^1 p_{2_*}\mathcal{F}(-\tau)\right) - c_1(R^1 p_{2_*}\mathcal{F}(-2\tau)) = 0,$$

and

$$\phi_z^*(\psi) = \frac{1}{2} r\left((r+1)c_1\left(R^1 p_{2_*}\mathcal{F}\right) - (r-1)c_1\left(R^1 p_{2_*}\mathcal{F}(-\tau)\right)\right) = c_2 - 1.$$

which implies

$$P(m) = \chi(\mathbb{P}(z), \phi_z^*(a\varphi + b\psi)) = \chi(\mathbb{P}(z), \mathcal{O}_{\mathbb{P}(E_x)}(m(c_2 - 1)b)))$$

and the proof is complete. □

Acknowledgment. The first author acknowledges the financial support of Universidad de Guadalajara via PROSNI programme.

References

[1] W. P. Barth, K. Hulek, C. A. M. Peters and A. Van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 2nd edition (Springer-Verlag, Berlin, 2004), xii+436.

[2] L. Brambila-Paz and O. Mata-Gutiérrez, On the Hilbert Scheme of the moduli space of vector bundles over an algebraic curve, *Manuscripta Math.* **142** (2013), 525–544.

[3] I. Coskun and J. Huizenga, The ample cone of moduli spaces of sheaves on surfaces and the Brill-Noether Problem, *Manuscripta Math.* **142** (2013), 525–544.

[4] I. Coskun and J. Huizenga, Brill-Noether theorems and globally generated vector bundles on Hirzebruch surfaces, *Nagoya Math. J.* **238** (2020), 1–36.

[5] I. Coskun and J. Huizenga, The birational geometry of the moduli spaces of sheaves on $\mathbb{P}^2$, in *Proceedings of the Gökova Geometry-Topology Conference 2014, Gökova Geometry/Topology Conference (GGT)*, Gökova (2015), 114–155.

[6] I. Coskun and J. Huizenga, The moduli spaces of sheaves on surfaces, pathologies and Brill-Noether problems, in *Geometry of moduli*, Abel Symposia, vol. 14 (Springer, Cham, 2018), 75–105.

[7] L. Costa and R. M. Miró-roig, Elementary transformations and the rationality of the Moduli Spaces of Vector Bundles on $\mathbb{P}^2$, *Manuscripta Math.* **113**(1) (2004), 69–84.

[8] D. Eisenbud and J. Harris, *3264 and all that a second course on algebraic geometry* (Cambridge University Press, Cambridge, 2016).

[9] B. Fantechi, et al., *Fundamental algebraic geometry: Grothendieck's FGA explained*, *Mathematical Surveys and Monographs*, vol. **123** (2005).

[10] R. Friedman, *Algebraic surfaces and holomorphic vector bundles*. Universitext (Springer-Verlag, New York, 1998), x+328.

[11] L. Gottsche, Change of polarization and Hodge numbers of moduli spaces of torsion free sheaves on surfaces, *Mathematische Zeitschrift* **223** (1996), 247–260.

[12] R. Hartshorne, *Algebraic geometry*. *Graduate Texts in Mathematics*, vol. **52** (Springer-Verlag, New York-Heidelberg, 1977), xvi+496.

[13] A. Hirshowitz and K. Hulek, Complete families of stable vector bundles over $\mathbb{P}_2$, in *Complex Analysis and Algebraic Geometry*. Lecture Notes in Mathematics (Springer Verlag, 1985).

[14] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves,* Aspects of Mathematics, vol. E31 (Friedr. Vieweg Sohn, Braunschweig, 1997), xiv+269.

[15] J. Le Potier, *Lectures on vector bundles*. Translated by A. Maciocia. Cambridge Studies in Advanced Mathematics, vol. 54 (Cambridge University Press, Cambridge, 1997), viii+251.

[16] M. Maruyama, *Elementary transformations in the theory of algebraic vector bundles*, *Lecture Notes in Mathematics*, vol. 961 (2006), 241–266.

[17] M. Maruyama, On a family of algebraic vector bundles, in *Number theory, algebraic geometry and commutative algebra, in honor of Y. Akizuki* (Kinokuniya, Tokyo, 1973), 95–146.

https://doi.org/10.1017/S0017089523000010 Published online by Cambridge University Press

[18] M. Maruyama, Openness of a family of torsion free sheaves, *J. Math Kyoto Univ.* **16**(3) (1976), 627–637.
[19] D. Mumford, *Lectures on curves on an algebraic surface*. With a section by G. M. Bergman. Annals of Mathematics *Studies*, vol. **59** (Princeton University Press, Princeton, N.J., 1966), xi+200.
[20] M. S. Narasimhan and S. Ramanan, *Geometry of Hecke Cycles I*. C. P. Ramanujam-a tribute, Tata Institute of *Fundamental Research*, vol. **8** (Springer, Berlin-New York, 1978), 291–345.
[21] M. S. Narasimhan and S. Ramanan, Deformations of the moduli space of vector bundles over an algebraic curve, *Ann. Math.* **101**(3) (1975), 391–417. https://doi.org/10.2307/1970933.
[22] P. E. Newstead, *Introduction to moduli problems and orbit spaces,* Tata Institute of Fundamental Research Lectures on Mathematics and Physics, vol. 51 (Tata Institute of Fundamental Research, Bombay; by the Narosa Publishing House, New Delhi, 1978), vi+183.
[23] C. Okonek, M. Schneider and H. Spindler, *Vector bundles on complex projective spaces*, *Progress in Mathematics*, vol. **3** (Birkhäuser, Boston, MA, 1980), vii+389.
[24] S. A. Stromme, Ample divisors on fine moduli spaces on the projective plane, *Math. Z.* **187**(3) (1984), 405–423.
[25] A. Tyurin, Vector bundles, in *Collected works, volume I* (Bogomolov F., Gorodentsev A., Pidstrigach V., Reid M. and Tyurin N., Editors) (Universitätsverlag Göttingen, 2008), vii+389.

https://doi.org/10.1017/S0017089523000010 Published online by Cambridge University Press