DYNAMICAL MODELS FOR AXISYMMETRIC AND TRIAXIAL GALAXIES

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ABSTRACT. Non–spherical dynamical models for galaxies, and the methods for their construction, are reviewed. The theory for two–integral axisymmetric models is reasonably well developed. Stäckel models give considerable insight in the structure of both three–integral axisymmetric models and non–rotating triaxial systems. Triaxial galaxies with appreciable figure rotation require much further study. Applications to elliptical galaxies and the bulges of disk galaxies are discussed.

1. THE FUNDAMENTAL PROBLEM OF STELLAR DYNAMICS

The structure and dynamics of a collisionless stellar system are determined completely by specification of the distribution function $f(\mathbf{r}, \mathbf{v}, t)$, which gives the distribution of the stars in the system over position $\mathbf{r}$ and velocity $\mathbf{v}$ as function of the time t. The distribution function satisfies the collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla V \cdot \frac{\partial f}{\partial \mathbf{v}} = 0, \tag{1}$$

where V is the potential in which the stars move, and we have used Newton's equations of motion $\dot{\mathbf{v}} = -\nabla V$. If the stellar system is in a steady state then $\partial f / \partial t = 0$. The density $\rho(\mathbf{r})$ of the system is the integral of f over the velocities

$$\rho(\mathbf{r}) = \int\int\int f(\mathbf{r}, \mathbf{v})\, d^3\mathbf{v}. \tag{2}$$

In a self–gravitating system, V is the gravitational potential of the density itself, and is connected to ρ via Poisson's equation

$$\nabla^2 V = 4\pi G \rho. \tag{3}$$

In order to obtain a *dynamical model* for a self–gravitating stellar system in equilibrium, equations (1), (2) and (3) have to be solved simultaneously. A solution corresponds to a dynamical model only if $f \geq 0$. The problem of finding f for a stellar system in equilibrium is the fundamental problem of stellar dynamics (Chandrasekhar 1942). It is often referred to as the *self–consistent* problem.

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In many cases, Jeans' theorem can be used in order to reduce the number of variables in the problem. It states that f depends on the phase–space coordinates $(\mathbf{r}, \mathbf{v})$ only through the isolating integrals of motion admitted by the potential (Jeans 1915; Lynden–Bell 1962*b*). Any such function automatically satisfies equation (1). Hence one is left with equations (2) and (3).

Two main approaches towards a solution of the self–consistent problem may be identified. In the first method one specifies f as a function of the integrals of motion. Then equations (2) and (3) combined form an integro–differential equation for V. The alternative is to specify the density and/or the potential, and then to solve the integral equation (2) for f. Non–consistent models are obtained in a similar way. Since in this case Poisson's equation does not have to be satisfied, one usually chooses a potential and then either gives f and calculates ρ from equation (2), or one specifies ρ and solves equation (2) for f.

As an example, consider spherical galaxies. A spherical potential admits four isolating integrals. In addition to the energy E, the three components L_x, L_y and L_z of the angular momentum vector $\mathbf{L}$ are conserved as well. As a result, the general distribution function of a spherical galaxy must be of the form $f(E, \mathbf{L})$. If f is spherical in all its properties, it can depend only on the magnitude of $\mathbf{L}$, but not on its direction, so that $f = f(E, L^2)$. Such models have anisotropic velocity distributions. For $f = f(E)$ the velocity distribution is isotropic. Eddington showed in 1915 that, for a given $\rho(r)$, it is always possible to invert equation (2) explicitly in order to obtain a unique $f(E)$. If $\rho(r)$ falls off with radius sufficiently rapidly (cf. Hunter 1974) this $f(E)$ is nowhere negative, and represents the unique isotropic solution. Many anisotropic solutions $f(E, L^2)$ exist. They can sometimes be found by analytic inversion techniques (e.g., Dejonghe 1987*a*), but they are usually constructed by assumption of a special functional form for f, or by numerical techniques. For recent reviews, see Binney (1982*a*), Binney & Tremaine (1987), and Richstone (1987).

In this paper, we review dynamical models that are not spherically symmetric. In §2 we discuss axisymmetric models with distribution functions that depend on two integrals. It has recently become clear that such models apply at most to the spheroidal bulges of disk galaxies only. Elliptical galaxies and box–shaped bulges require three–integral models. These may be axisymmetric, but most likely they are triaxial and have slow figure rotation. Due to lack of knowledge regarding extra integrals of motion, few such models exist to date. They are described in §§3 and 4. §5 is devoted to the so–called Stäckel models, for which three exact integrals are known in closed form. They can be used to construct realistic three–integral models of axisymmetric galaxies, and also of non–rotating triaxial systems.

2. AXISYMMETRIC MODELS WITH $f = f(E, L_z)$

Let (ϖ, ϕ, z) be cylindrical coordinates. An axisymmetric potential $V(\varpi, z)$ always admits two exact isolating integrals of motion, the energy E and the component of the angular momentum vector that is parallel to the symmetry axis, L_z. It is therefore natural to consider dynamical models with $f = f(E, L_z)$. Although such models are not as widely applicable to galaxies as was once believed, it is nevertheless useful to review the various methods for their construction. We shall mainly discuss three–dimensional models; for a review of circular disks, see Kalnajs (1976).

2.1 Inversion

Fricke (1952) showed that in a potential $V(\varpi, z)$, a distribution function f of the form

$$f(E, L_z) = \sum a_{ij} E^{i-j-3/2} L_z^{2j}, \tag{4}$$

corresponds to a density ρ that is given by

$$\rho(\varpi, z) = \sum b_{ij} V^i \varpi^{2j}, \tag{5}$$

where the summation is over i and j, and the a_{ij} and b_{ij} are constants that are related one to one. Thus, if a given density $\rho(\varpi, z)$ in a potential $V(\varpi, z)$ can be expressed as $\rho(\varpi, V)$, and can be expanded in the form (5), then the unique distribution function $f(E, L_z)$ that is consistent with it follows from equation (4). The corresponding velocity moments can be given as series expansions in a form similar to equation (5). However, the velocity dispersions can often be derived more rapidly by direct integration of the stellar hydrodynamical equations (see §2.3). A major drawback of Fricke's method is that the convergence of the series (4) is not guaranteed for all values of E and L_z.

An early application of Fricke's method was given by Kuzmin & Kutuzov (1962), who calculated $f(E, L_z)$ for a family of oblate mass models found previously by Kuzmin (1956). The models are nearly spheroidal, and form a continuous sequence between Hénon's (1960) spherical isochrone and Kuzmin's (1953) disk.

Miyamoto & Nagai (1975) presented a remarkable (ρ, V)–pair that describes a sequence of flattened models that connects the spherical Plummer (1911) model with Kuzmin's disk. At large radii, the density falls off as ϖ^{-3} in the equatorial plane, but it decreases as z^{-5} everywhere else. As a result, the models appear to have both a bulge and a disk. Nagai & Miyamoto (1976) derived $f(E, L_z)$ for their models by Fricke's method, and delineated the kinematic properties.

Lynden–Bell (1962*a*) generalized Eddington's (1915*b*) isotropic spherical inversion formula to axisymmetric models. His method for the actual calculation of f is not easy to apply in practice. It requires taking a Laplace transform, and then two inverse Laplace transforms, and hence is restricted to very special densities. Alternative formulations in terms of Stieltjes and Mellin transforms (Hunter 1975*b*; Kalnajs 1976) avoid this problem. However, just as in Fricke's method, these different versions of the inversion method all require that ρ is given explicitly as $\rho(\varpi, V)$. Only very few (ρ, V)–pairs with this property are known. Consequently, only a small number of exact models have been constructed.

The inversion method has been applied to various modifications of the Plummer model, both oblate and prolate (Lynden–Bell 1962*a*; Hunter 1975*b*; Lake 1981*a*). Lynden–Bell's generalized Plummer models have a finite total mass; their distribution function is the sum of two terms of the form (4), so that it could have been calculated easily with Fricke's method as well.

Hunter (1975*a*) calculated $f(E, L_z)$ for a homogeneous spheroid of finite radius by Lynden–Bell's method, and showed that it is not everywhere positive. He also derived an approximate form for the distribution function of a mildly inhomogeneous spheroid, in the limit of small flattening, and found that $f \geq 0$.

Recently, Dejonghe (1986) rediscussed the inversion method in detail, with emphasis on the use of Mellin transforms. He gives a very useful list of explicit distribution functions $f(E, L_z)$ that correspond to a variety of fairly general functions

$\rho(\varpi, V)$. These can be used as convenient building blocks for dynamical models. Dejonghe's list includes all previously known cases, and many new ones. In particular, he derived $f(E, L_z)$ for a density of the form $\rho = V^a(1 - \varpi^2 V^2)^b$, with a and b constants. This allowed him to give the distribution function for the Miyamoto & Nagai (1975) model in a more convenient form, and to show that it is positive everywhere. By using the same building block, Dejonghe & de Zeeuw (1987) found the exact distribution function for the Kuzmin & Kutuzov (1962) model.

Dejonghe discusses the connection between the inversion problem for axisymmetric systems and the corresponding problem for anisotropic spherical models, and shows explicitly that the direct inversion method is numerically unstable. As a result, in practice the method is limited to *analytic* (ρ, V)-pairs for which, as we have seen, it must be possible to write $\rho = \rho(\varpi, V)$ in closed form. In addition to the six cases already mentioned, not counting the homogeneous spheroid, only one further (ρ, V)-pair with this property is known. These are Satoh's (1980) $n = \infty$ models, which form a continuous sequence connecting the Kuzmin disk with the point mass. The models resemble those of Miyamoto & Nagai (1975), but have a weaker disk component. Satoh calculated the kinematic properties of these models (see also §2.3), but he did not derive $f(E, L_z)$. This can be done by Fricke's method, and most likely also by direct inversion.

Although the above results seem to indicate that the calculation of $f(E, L_z)$ for a given density $\rho(\varpi, z)$ is limited to a few special cases, which have been found in a haphazard way, the situation is not quite desperate. Dejonghe (1986) has developed a method to generate many (ρ, V)-pairs with the desired property in a systematic fashion. In an earlier paper the same method was used to generate new exact spherical models (Dejonghe 1984). Construction of axisymmetric models via this technique promises to yield many new and useful models.

Finally, we remark that ρ constrains only the part of the distribution function that is even in the velocities, i.e., it can be used to determine $f(E, L_z^2)$. The odd part of f can be found by exactly the same techniques as the even part, by inversion of $\varpi\rho\langle v_\phi\rangle$ (cf. §2.3), instead of ρ (Hunter 1975*a*; Dejonghe 1986).

2.2 Assumed Form for f.

The alternative to direct inversion is to assume a functional form for $f(E, L_z)$, and then to find the potential–density pair (ρ, V) that is consistent with it. This approach has produced realistic spherical models (e.g., Michie 1963; King 1966; Binney 1982*b*). Prendergast & Tomer (1970) developed an efficient numerical technique to do this for axisymmetric models. After adopting a simple form for $f(E, L_z)$, they calculate ρ by equation (2), and substitute the result in equation (3). This produces a non-linear equation for the potential V, which is solved by iteration. The authors then employed this *self-consistent field method* for the construction of models with a variety of flattenings and radial density profiles.

Wilson (1975) extended the work of Prendergast & Tomer, with a smoother form for f, and constructed a small number of oblate models designed to fit the observations of the elliptical galay NGC 3379. He was able to reproduce the radial density profile, the isophote shape and even the inner part of the rotation curve. Further exploration showed that the models have a variation of eccentricity with radius that is always peaked, and that models flatter than about E4 are unrealistic. Lake (1981*b*) used another form for f, and was able to find prolate models flatter than E4. His models have eccentricity profiles which are decreasing with radius, and can have streaming, i.e., rotation, around the long axis only.

The most recent, and succesful, application of the self-consistent field method is by Jarvis & Freeman (1985*a*, *b*). They constructed models for the bulges of disk galaxies. A new aspect of their models is that the potential is taken as the sum of a bulge potential and a disk potential. During the iteration, the latter is kept fixed, and is set equal to the potential of a Miyamoto & Nagai model. Thus, the resulting bulge models are not self-consistent, which—for this application—is exactly as it should be. Jarvis & Freeman find that, with only a small contribution from the disk, their models accurately reproduce all available photometric and kinematic properties of spheroidal bulges, but not of the box-shaped ones.

A difficulty with the self-consistent field method is that the shape of a dynamical model is not related to the form of f in a very transparent way. This is evident from the differences between the above mentioned models, and also from the work of Ruiz & Schwarzschild (1976), who constrained the form of $f(E, L_z)$ and the shape of the density distribution simultaneously, and found that their conditions overspecified the model. This problem was studied in detail by Hunter (1977), who derived the conditions that f must satisfy in order to produce density distributions with roughly constant eccentricity profiles.

A special case where a particular choice of $f(E, L_z)$ leads to an analytic solution of the self-consistent problem was given by Toomre (1982). He considered scale-free axisymmetric density distributions with, in spherical coordinates, $\rho(r, \theta) = S(\theta)/r^2$, so that $V \propto \ln[r + P(\theta)]$. Here $S(\theta)$ and $P(\theta)$ describe the shape of the model, and are still to be determined. Toomre assumed that f is given by

$$f_n = c_n L_z^{2n} e^{-E}, \tag{6}$$

with c_n a constant. Combination of equations (2) and (3) then produced a nonlinear equation for $P(\theta)$ which, remarkably, can be solved in closed form for all $n \geq 0$. $S(\theta)$ then follows by differentiation. Thus, the form (6) describes a one-parameter family of exact scale-free models. Two special cases are the standard isothermal halo ($n = 0$) and Mestel's (1963) disk of infinite extent ($n = \infty$).

The scale-free models are not realistic, since they have infinite total mass, central density, and central potential. Also, for $n > 0$ the surfaces of constant density are tori with vanishingly small central holes. However, this last defect can be remedied by considering two-component models. Toomre showed that the structure of a model consisting of an infinite Mestel disk and a halo with distribution function f_n can also be found analytically. The special case $n = 0$ had already been discussed in detail by Monet, Richstone & Schechter (1981). Furthermore, two-component models with $f = f_n + f_m$ and $0 \leq n \leq m < \infty$ can have a bulge-disk structure, and can be given in closed form for $m = 2n + 1$. The particular case $n = 0$, $m = 1$ is the model used by Richstone in his extensive study of scale-free models with three integrals of motion (Miller 1982; §3.3 below).

2.3 Stellar Hydrodynamics

The kinematic properties of a dynamical model can be derived from the velocity moments of f. Let v_ϖ, v_ϕ and v_z be the velocity components in the cylindrical coordinates (ϖ, ϕ, z), and denote an average over all velocities by $\langle\ \ \rangle$. By symmetry, it follows that $\langle v_\varpi \rangle = \langle v_z \rangle \equiv 0$, so that there can be mean streaming around the symmetry axis only, with velocity $\langle v_\phi \rangle$ (which is a function of ϖ and z).

The second order moments of f are related to ρ and V by the Jeans equations, also referred to as the equations of stellar hydrodynamics (Jeans 1922; Chandrasekhar 1942). By symmetry, $\langle v_\varpi v_\phi \rangle = \langle v_\phi v_z \rangle = 0$. For $f = f(E, L_z)$, we also have $\langle v_\varpi v_z \rangle = 0$, $\langle v_\varpi^2 \rangle = \langle v_z^2 \rangle$, and the Jeans equations reduce to

$$\begin{aligned} \frac{1}{\rho}\frac{\partial \rho \langle v_\varpi^2 \rangle}{\partial \varpi} + \frac{\langle v_\varpi^2 \rangle - \langle v_\phi^2 \rangle}{\varpi} &= -\frac{\partial V}{\partial \varpi}, \\ \frac{1}{\rho}\frac{\partial \rho \langle v_z^2 \rangle}{\partial z} &= -\frac{\partial V}{\partial z}. \end{aligned} \tag{7}$$

Given a density ρ in a potential V, the moments $\langle v_\varpi^2 \rangle = \langle v_z^2 \rangle$ and $\langle v_\phi^2 \rangle$ can be determined by direct solution of equations (7), without knowing the explicit form of the corresponding f. In fact, even if f is known, it is often easier to calculate the moments in this way, rather than by averaging of v_ϖ^2 and v_ϕ^2 over velocity space. However, a solution of the Jeans equations may give velocity dispersions that are unphysical, since the implied f is not guaranteed to be non–negative.

As already mentioned, specification of ρ does not fix the part of f that is odd in the velocities. Hence $\langle v_\phi \rangle$ is undetermined. In many cases $\langle v_\phi \rangle$ is chosen such that the local velocity dispersions are equal in all three directions, i.e., $\langle v_\varpi^2 \rangle = \langle v_z^2 \rangle = \langle v_\phi^2 \rangle - \langle v_\phi \rangle^2$. These solutions are referred to as *isotropic*. It is also possible to determine $\langle v_\phi \rangle$ via an entropy argument (Dejonghe 1986).

The Jeans equations have played a central role in Galactic Dynamics (e.g., Oort 1965). They have also been put to good use for spherical galaxies (Binney & Mamon 1982), but few applications to general axisymmetric models exist. Bagin (1972) considered rather special density distributions. A very interesting set of solutions was given by Nagai & Miyamoto (1976). They derived an infinite set of axisymmetric (ρ, V)–pairs, each of which connects the Plummer model with one of Toomre's (1963) disks. The Miyamoto & Nagai (1975) models mentioned earlier are given by the first of these pairs, since Toomre's $n = 1$ disk is identical to Kuzmin's disk. The authors were able to integrate equations (7) explicitly for all these generalized Toomre models, so that the velocity dispersions could be given in closed form. In the same fashion, Satoh (1980) presented another infinite set of three–dimensional models, connecting the Kuzmin disk with various generalizations of the Plummer model. He solved the Jeans equations for his $n = \infty$ model only, and compared the results with observations of NGC 4697.

Hunter (1977) used the Jeans equations in his investigation of the relations between the functional form of $f(E, L_z)$, the anisotropy of the velocity dispersions, and the shape of the density distribution in self–consistent models (cf. §2.2). He also showed that equations (7) can be solved by simple quadrature in case $\rho = \rho(\varpi, V)$ explicitly. Since both the Nagai & Miyamoto $n = 1$ models and the Satoh $n = \infty$ models have densities with this property, it is not surprising that for them the Jeans equations can be solved exactly.

2.4 Applications

The standard example of an axisymmetric system is our own Galaxy. However, it has long been known that $\langle v_\varpi^2 \rangle \neq \langle v_z^2 \rangle$ in the solar neighbourhood (e.g., Oort 1928). It was concluded that the distribution function of the Galaxy must depend on a third argument, i.e., there must be a third isolating integral of motion, I_3. Although special potentials which admit a third integral were considered by various authors (Oort 1932; Lindblad 1933; Kuzmin 1953, 1956; Hori 1962) it was established only through the analytic work of Contopoulos (1960) and the numerical work of Ollongren (1962) that in realistic galactic potentials indeed most stellar orbits have an effective third integral (see also Martinet & Mayer 1975). No simple expression is known for I_3, and to date no satisfactory dynamical model for the Galaxy exists.

Although a two-integral model was known to be inadequate for the Galaxy, it was thought for a long time that elliptical galaxies have a simpler dynamical structure, and are oblate axisymmetric systems with $f = f(E, L_z)$ and an isotropic velocity distribution, so that their flattening is caused by rotation (Freeman 1975). This premise generated most of the work described in the previous sections. However, at about the time that Wilson (1975) produced his two-integral models, spectroscopic observations indicated that large elliptical galaxies rotate much slower than expected (Bertola & Capaccioli 1975; Illingworth 1977).

All $L_z \neq 0$ orbits in an oblate galaxy have a definite sense of rotation around the symmetry axis, but both clockwise and counter-clockwise motion may occur. By reversing the direction of motion for an arbitrary fraction of stars in each orbit, and hence making the velocity distribution anisotropic (§2.3), we may obtain as small a mean streaming velocity $\langle v_\phi \rangle$ (i.e., observed rotation) as desired, even with $f = f(E, L_z)$. However, by using the tensor virial theorem to connect the ratio of the observed maximum rotational velocity and the central velocity dispersion with the apparent flattening, Binney (1978*a*) concluded that the then available kinematic observations of elliptical galaxies were best represented by models with $\langle v_\varpi^2 \rangle \neq \langle v_z^2 \rangle$. This is supported by more recent studies (§3.4). As a result, it is generally accepted that the most natural models for elliptical galaxies have all three velocity dispersion components unequal, and hence must have distribution functions that depend on three integrals.

In smaller elliptical galaxies the velocity anisotropy decreases, and rotation becomes more important (Davies *et al.* 1983). The still smaller bulges of disk galaxies, which resemble elliptical galaxies in many respects, all rotate nearly as fast as the oblate isotropic models (Kormendy & Illingworth 1982). Jarvis & Freeman (1985*b*) have shown that, by inclusion of the effect of the disk, the *spheroidal* bulges are completely consistent with $f(E, L_z)$-models. However, the *box*-shaped bulges have velocity fields that need three-integral models.

3. AXISYMMETRIC MODELS WITH $f = f(E, L_z, I_3)$

Most orbits in realistic axisymmetric potentials are tubes around the symmetry axis, and have an effective third integral. The remainder is generally made up of a host of minor orbit families and irregular orbits. The latter do not have a third integral of motion. Binney (1982*c*) has argued that Jeans' theorem is not valid for potentials that support irregular orbits (see also Pfenniger 1986). Hence, no true equilibrium solutions may exist for such systems. In many cases of interest, however, the fraction of irregular orbits is small. On time scales of the order of a Hubble time these orbits are nearly indistinguishable from regular ones (e.g., Goodman & Schwarzschild 1981). For practical purposes, one may probably still use Jeans' theorem for these systems, and construct approximate equilibrium models. Since a given $\rho(\varpi, z)$ determines a unique $f(E, L_z)$, generally many different distribution functions $f(E, L_z, I_3)$ are consistent with it.

3.1 Exact Models

An exact third isolating integral of motion is known for the special axisymmetric potentials for which the Hamilton–Jacobi equation separates in spheroidal coordinates (Stäckel 1890; Lynden–Bell 1962*c*). In a classic paper, Kuzmin (1956) showed that many such Stäckel potentials correspond to realistic axisymmetric mass models. For such models Jeans' theorem is strictly valid. We shall consider exact solutions based on these potentials in §5.

3.2 Approximate Distribution Functions

For nearly spherical systems, the third integral is related closely to the square of the total angular momentum. This is suggested by the form of I_3 in the Stäckel potentials, most clearly in the limiting case where $V(r,\theta) = F(r) + G(\theta)/r^2$, with $F(r)$ and $G(\theta)$ arbitrary functions. This is the Eddington (1915*a*) potential, which admits $I_3 = L^2 - 2G(\theta)$ as exact integral. Numerical orbit calculations in more realistic potentials show that L^2 indeed does not vary much along an orbit (e.g., Saaf 1968; Innanen & Papp 1977). This fact can be used to include the dependence of f on I_3 in an approximate way.

Lupton (1985) simply used L^2 as third integral, and constructed realistic models of globular clusters with an assumed form for $f(E, L_z, I_3)$. A more elaborate treatment was given by Petrou (1983*a*). She used an approximation of the form $I_3 = L^2 - 2G(r,\theta)$, with $G(r,\theta)$ a simple function of the potential. This I_3 turned out to be constant along individual orbits to better than a few percent. She then modified the lowered Maxwellian form of the distribution function $f(E, L_z)$ used by Prendergast & Tomer (1970), by inclusion of a factor that depends on I_3, chosen such that the orbits that do not come close to the center are all depopulated, independent of their inclination. The potential and density that correspond to this f were then determined by the self–consistent field method (§2.2). This produced quite realistic models for nearly round elliptical galaxies. Due mainly to the chosen L_z–dependence of f, the models all have peaked rotation curves, and cannot become much flatter than E2.3. In order to remedy this defect, Petrou (1983*b*) modified f in such a way that the non–radial orbits are depopulated predominantly outside the equatorial plane, which results in flatter models. She presented detailed results for an E3.5 model. It has a nearly flat rotation curve, a velocity ellipsoid that becomes radially aligned at large distances from the center, a realistic density profile, and elliptical isophotes.

A somewhat different approach was taken by Binney & Petrou (1985). They proposed a particular form for the three–integral distribution function for box–shaped bulges, and showed that the corresponding density in the potential of the spherical isochrone has the correct photometric and kinematic properties. It is likely that more realistic models can be obtained by taking their distribution function, and applying the self–consistent field method after inclusion of a disk potential, just as was done for spheroidal bulges by Jarvis & Freeman (1985*a, b*).

3.3 Scale–Free Models

As we have seen in §2.2, scale–free models suffer from some defects. They can, however, provide considerable insight in the structure of more realistic models. Richstone (1980, 1982, 1984) made a detailed study of the possible three–integral models that are consistent with a particular axisymmetric scale-free model. He took the case where the potential is logarithmic, and constant on similar oblate spheroids. The associated density has dimples near the symmetry axis. Richstone studied a particular example with a flattening similar to an E6 galaxy. The great majority of orbits in this potential belongs to one family, that of tubes circling the short axis. He did not encounter any stochastic orbits in his survey. Thus, all orbits have an effective third integral.

Richstone used Schwarzschild's method (§4.2) to compute numerical distribution functions by reconstructing the given density with individual orbit densities. Because the model is scale-free, this reconstruction needs to be done at one radius only, so that the problem is effectively one-dimensional. By optimizing the values of the total angular momentum and the total second velocity moments, Richstone found a large variety of different dynamical models.

Subsequently, Levison & Richstone (1985*a, b*) produced non-consistent models in the same potential, by using the orbits to reproduce a density $\rho(r) \sim r^{-3}$, instead of $\rho(r) \sim r^{-2}$, so that the mass-to-light ratio $M/L \propto r$. They obtained the somewhat unexpected result that the kinematics of these different models are very similar. If this is true for more realistic models as well, then the chances of finding evidence for dark matter in elliptical galaxies when only observations of stellar kinematics are available, are very slim.

3.4 Stellar Hydrodynamics

Solutions of the Jeans equations for anisotropic axisymmetric models relevant for elliptical galaxies have been given only recently. Bacon, Simien & Monnet (1983), and Bacon (1985) considered the case where $\langle v_{\varpi}^2 \rangle \neq \langle v_z^2 \rangle$, but with the restriction that the velocity ellipsoid is radially aligned everywhere, so that $\langle v_r v_\theta \rangle = 0$. By analogy with the spherical case (Binney & Mamon 1982), they derived the formal solution of the Jeans equations for an assumed functional form of the anisotropy parameter $\beta = 1 - \langle v_\theta^2 \rangle / \langle v_r^2 \rangle$. They considered the case $\beta = \beta(r) = kr/(1+ar)$, with k and a constants, in detail. This corresponds to a velocity distribution that is isotropic in the center, and, for $0 < k \leq 1$, predominantly radial at large radii, as suggested by N-body simulations (van Albada 1982). The authors evaluated their solutions numerically for density distributions that are stratified on similar oblate spheroids and have, in projection, a de Vaucouleurs profile. Comparison of the results with observations of a small number of well-observed elliptical galaxies revealed that the larger ones are indeed best fit with anisotropic models.

A similar investigation was done by Fillmore (1986), who solved the Jeans equations by an iterative numerical technique for three different assumptions for the shape of the velocity ellipsoid. He employed a mass model identical to the one used by Bacon, and showed that observations of both major and minor axis dispersion profiles can put strong constraints on the form of the velocity distribution.

4. TRIAXIAL MODELS

Since models with $f = f(E, L_z, I_3)$, and hence with anisotropic velocity distributions, seem to be required for the majority of elliptical galaxies, the assumption that these galaxies are oblate, i.e., with two of the three axes exactly equal, becomes rather artificial. It is much more natural to assume that these galaxies are triaxial

(Binney 1978*a, b*). This hypothesis is supported by a number of observational lines of evidence (e.g., Schechter 1987).

A general triaxial potential admits only one isolating integral, the orbital energy E. Although there are three planes of reflection symmetry, there is no symmetry axis, and no component of the angular momentum is conserved. Schwarzschild (1979, 1982) has shown by numerical means that in realistic triaxial potentials most stellar orbits possess two effective integrals I_2 and I_3 in addition to E. Just as for the third integral in axisymmetric systems, in the general case no simple expressions are known for these extra integrals.

4.1 Exact Solutions

There exists a large class of triaxial mass models with *non-rotating* figures that have a potential of Stäckel form, and admit three exact integrals of motion. Self-consistent models of this kind will be discussed in §5. For triaxial systems with *rotating* figures, only two analytic solutions exist, one exact, the other approximate.

Freeman (1966) constructed dynamical models for homogeneous triaxial ellipsoids of arbitrary axis ratios, rotating at a critical frequency such that the centrifugal force exactly balances the gravitational attraction on the long axis. The distribution function f of his models depends on the energy integral in the corotating coordinate system only. However, shape and rotation are related in a unique way, which is in contradiction with the observations of elliptical galaxies. Hunter (1974) proved that for general axis ratios, and a general figure rotation, no solutions exist. In the special case of a homogeneous spheroid, solutions exist that depend on more integrals (Bisnovatyi-Kogan & Zeldovich 1970), although the model with $f(E, L_z)$ is unphysical (§2.1).

Vandervoort (1980*a, b*) constructed approximate solutions for rotating polytropic models in which the density is a power of the potential, and is only mildly concentrated towards the center. The distribution functions found by Vandervoort depend on the energy only, and the velocity dispersions are isotropic. This illustrates that even triaxial models can have an isotropic velocity distribution. However, these models again require a unique rotation speed. Vandervoort & Welty (1981, 1982) developed an analytic—and approximate—version of the self-consistent field method (§2.2), in which the iterative solution is terminated after the first one-half iteration. They used this method to construct a more general set of polytropic models, with a variety of rotation speeds. They considered distribution functions that depend on an approximate second integral, which is only valid in nearly homogeneous systems. The resulting anisotropic models can be considered as the stellar dynamical counterparts of the S-type Riemann fluid ellipsoids (Chandrasekhar 1969). Bohn (1983) used the same method to construct prolate models.

4.2 Schwarzschild's Method

A completely new approach to the self-consistent problem, which sidesteps our ignorance of two of the three arguments of f, was introduced by Schwarzschild (1979). He specified a mass model and derived the gravitational potential by integration of Poisson's equation. Then he calculated a large number of stellar orbits by numerical means, and computed their individual density distributions on a grid of cells, by determining the average time spent in each cell by each orbit. He then used linear programming to find a combination of orbital densities that reproduces the original density distribution, with all occupation numbers non-negative.

If this procedure is succesful, one has obtained a numerical representation of a distribution function that is consistent with the assumed density. In each cell the number of stars going through it are known. Since the orbits have been computed, the velocities of the stars are known at each position. Thus, the orbits and their occupation numbers give the distribution of the stars over position and velocities. This is, by definition, the distribution function (§1). Schwarzschild (1979) applied this method to a nearly ellipsoidal triaxial mass model with axis ratios 1 : 5/8 : 1/2, and a realistic density profile, and was able to find a solution for f. This established the existence of realistic triaxial equilibrium models.

Most orbits in Schwarzschild's models have three effective integrals of motion, and belong to four major families: box orbits, which have no net average angular momentum, tube orbits around the short axis and two kinds of tubes circling the long axis. Each star in a tube orbit has a definite sense of rotation around the appropriate axis, but both clockwise and counter-clockwise motion may occur on the same orbit. The orbital shapes and their individual density distributions are determined by the values of the integrals. As a result, the derived occupation numbers depend on the integrals. Thus, without knowledge of the explicit forms of two of the integrals, still a distribution function f is found that depends on all three of them. We remark that instead of linear programming, one can also use Lucy's (1974) method (Newton & Binney 1984), non-negative least squares (Pfenniger 1984), or a maximum-entropy method (Tremaine & Richstone 1987).

Schwarzschild constructed his model with boxes and short axis tubes only. Its internal structure, and the observable properties, were studied by Merritt (1980). Later work showed that long axis tubes can be included in the model as well, indicating that different equilibrium models can exist for the same density distribution. Levison & Richstone (1987) used Schwarzschild's method to survey the solutions for two different triaxial scale-free models, one nearly oblate, and the other nearly prolate. These models contain the same major orbit families as found in Schwarzschild's model (Levison, priv. comm.). The equilibrium models can have mean streaming about the long axis and about the short axis (but see below), and show a large variety in observable properties.

4.3 Figure Rotation

Schwarzschild (1982) constructed two distinct equilibrium solutions for his original model, but now with a slow rotation of the figure around its short axis. This showed that realistic triaxial dynamical models exist whose shape is not determined by the (figure) rotation.

In the rotating models, the Coriolis force distinguishes direct and retrograde motion, and each of the tube orbit families mentioned above splits up in two branches, with different shapes. The long axis tubes tip out of the plane that contains the intermediate and short axes, with the direct and retrograde branches tipping in opposite directions. In order to obtain a model with triaxial symmetry, each branch of this family has to be populated equally, so that no net streaming about the long axis can occur. The streaming about the short axis can be quite complicated, since in addition to the short axis tubes, also the boxes and the tipped long axis tubes have a net average angular momentum with respect to this axis.

Vietri (1986) used Schwarzschild's method to construct rotating triaxial models for the bulge of the Galaxy: He assumed that the bulge counter-rotates with respect to the disk, and heavily populated the tube orbits that are retrograde in a frame that corotates with the bulge. If this counter-streaming is large enough, then the bulge will seem to rotate in the normal sense to an external observer. This unorthodox approach was inspired by the possibility that in this way the Liszt & Burton (1980) tilted HI disk can be explained as a stable dynamical phenomenon. Unfortunately, Vietri found that the counter-streaming is so small that the net rotation of the bulge (figure rotation minus counter-streaming) would be contrary to Galactic rotation, which is not observed.

Vietri's result does not rule out the existence of slowly rotating triaxial elliptical galaxies with counter-streaming in their central parts, even although van Albada (1987) could not simulate them in N-body experiments.

5. STÄCKEL MODELS

It is evident from the two preceding sections that the collection of realistic three-integral models for galaxies, both axisymmetric and triaxial, is still limited. Many questions regarding the structure of these systems remain. What are the extra integrals? How many f's are consistent with a given ρ? What is the full variety of equilibrium figures? What observations do we need in order to determine the intrinsic dynamical structure of an elliptical galaxy?

These questions, and many others, cannot be answered easily by numerical methods alone. Each model has to be constructed, or simulated, separately. This involves considerable labour and expense. We now show that many general aspects of the dynamics of triaxial and axisymmetric galaxies can be understood by analytic means, through a study of the Stäckel models.

5.1 Stäckel Potentials

Many triaxial mass models exist with a gravitational potential of Stäckel form, for which the Hamilton–Jacobi equation separates in ellipsoidal coordinates (Kuzmin 1973; de Zeeuw 1985*b*). Every orbit in such a model enjoys three exact isolating integrals of motion, E, I_2 and I_3, which are known explicitly, and which are quadratic in the velocities. I_2 and I_3 are related to the angular momentum integrals in the axisymmetric and spherical limits.

An individual orbit can be considered as the sum of three motions, one in each ellipsoidal coordinate. The stars are thus constrained—by the integrals of motion—to move between coordinate surfaces. Thus, all possible orbital shapes can be found by inspection of the ellipsoidal coordinate system in which the motion separates. It turns out that all centrally concentrated triaxial mass models of this kind have four families of orbits: boxes, short axis tubes, and two families of long axis tubes. These are exactly the four major orbit families that occur in Schwarzschild's (1979) nonrotating model, and also in Levison & Richstone's (1987) scale-free models. The orbital structure of the Stäckel models is *generic* for all moderately flattened triaxial systems without figure rotation (de Zeeuw 1985*a*; Gerhard 1985).

The prototypical triaxial Stäckel model is the perfect ellipsoid. It has a density distribution given by

$$\rho = \rho(m^2) = \frac{\rho_0}{(1+m^2)^2}, \qquad m^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, \tag{8}$$

with $a \geq b \geq c$. The oblate case, $a = b$, was discovered by Kuzmin (1956), in his classic study of separable models of the Galaxy. He also obtained the general case (Kuzmin 1973), and showed that it has four major orbit families. The perfect ellipsoid was rediscovered by de Zeeuw & Lynden-Bell (1985), who showed that it is the only inhomogeneous triaxial mass model with a Stäckel potential in which the density is stratified exactly on similar concentric ellipsoids. The orbital structure was delineated in detail by de Zeeuw (1985*b*).

The general form of a Stäckel potential contains a free function of one variable. The associated ellipsoidal coordinate system is determined by specification of two parameters. As a result, there is a large variety of mass models with a Stäckel potential. They have remarkable properties.

Kuzmin (1956, 1973) showed that in a Stäckel model the density $\rho(x, y, z)$ at a general point is related to that on the short axis, $\rho(0, 0, z)$, by a very simple formula, and that $\rho(x, y, z) \geq 0$ if and only if $\rho(0, 0, z) \geq 0$. This makes it possible to choose a short-axis density profile, and the values of the central axis ratios of the model, and find the complete mass model that has a Stäckel potential and this density profile, by one integration in one variable. A second integration gives this potential explicitly (de Zeeuw 1985*c*). This means that Poisson's equation can be integrated in closed form for the whole class of Stäckel models.

De Zeeuw, Peletier & Franx (1986) constructed many different mass models, and delineated their general properties. Models with a singular density in the centre only do not exist. The density cannot fall off more rapidly than r^{-4} as $r \to \infty$, except on the short axis. Models in which ρ falls off less rapidly than r^{-4} become spherical as $r \to \infty$. The only models that have surfaces of constant density that approach a finite flattening at large radii are those with $\rho \sim r^{-4}$.

With the exception of the perfect ellipsoid, on projection the triaxial Stäckel models have isophotes that are not exact ellipses. Their ellipticity changes with radius, but they do not show twisting isophotes (Franx 1987).

5.2 Equilibrium Models

The individual orbit densities in a Stäckel model are known in analytic form, and hence are evaluated easily. This makes it straightforward to construct self-consistent models by means of Schwarzschild's method, while avoiding laborious numerical orbit integrations. This is true not only for the triaxial models, but also for the various limiting cases with more symmetry, and a simpler orbital structure. These cases are depicted schematically in Figure 1, which shows the axis ratio plane for the perfect ellipsoid, and hence gives the layout of "Ellipsoid Land". Elliptical galaxies are no flatter than at most E6, and hence occupy the upper part of the triangle. This coincides with the area where the Stäckel models provide an adequate description of the orbital structure.

Bishop (1986) considered the perfect oblate spheroids. All orbits in them are short axis tubes. He first constructed one-dimensional continua of orbits, effectively specifying the dependence of f on one integral, and solved for the remaining unknown part of f by an algebraic technique (Vandervoort 1984). Prolate models have not yet been constructed. Since they can have streaming about the long axis only, they are probably less relevant for galaxies.

For $c = 0$ the models reduce to elliptic disks. These contain flat box orbits, and flat short axis tubes. Teuben (1987) used Schwarzschild's method to construct equilibrium models for nine different axis ratios, both with minimum and with

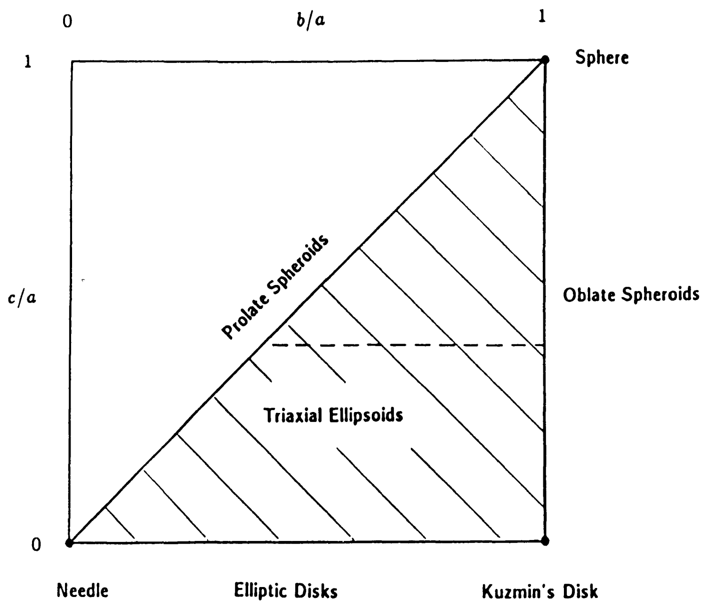

Figure 1. Plane of axis ratios for the perfect ellipsoid. The area above the dashed line corresponds to axis ratios relevant for elliptical galaxies.

maximum possible streaming, and discussed the kinematic properties. De Zeeuw, Hunter & Schwarzschild (1987) used analytic methods to prove rigorously that exact equilibrium models exist, and showed that there is a two–dimensional continuum of different f's that all are consistent with the same ρ. They also constructed the model with maximum possible streaming in nearly explicit form, by using all the box orbits, but only the thinnest tubes, i.e., the elliptic closed orbits. The fundamental reason for the non–uniqueness of the dynamical solutions is the existence of more than one major orbit family, so that orbits can be exchanged while keeping the density fixed and $f \geq 0$. Schwarzschild (1986) showed that when the perfect elliptic disk is truncated inside the foci of the elliptic coordinate system in which the motion separates, and hence only box orbits can be used, the dynamical solution is unique. The circular limit of the perfect elliptic disk is Kuzmin's disk, for which many authors have derived self–consistent dynamical models (cf. Kalnajs 1976). The opposite limit is a one–dimensional needle. Its curious properties are discussed by Tremaine & de Zeeuw (1987).

The triaxial case was investigated by Statler (1987). He considered 21 different perfect ellipsoidal mass models covering a regular grid in Figure 1. For each of these he constructed one solution by Lucy's method, thus establishing existence, and 15–20 distinct solutions by means of linear programming. In these, he optimized either the streaming around the short axis, or the streaming around the long axis, or a combination of the two. The properties of the models all vary smoothly with the axis ratios. Statler delineated the kinematic properties of the resulting models in detail, and discussed how observations might distinguish them.

The fundamental result of Statler's study is that many different dynamical models can be constructed with the same density distribution, as already suggested by the earlier work discussed in §4.2. By analogy with the above mentioned results

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for the perfect elliptic disk, this freedom is expected to be that of an arbitrary function of three variables (see also Dejonghe 1987*b*).

5.3 Exact Solutions

Many properties of the Stäckel models can be given in analytic form. It is therefore natural to ask whether it is possible to give exact distribution functions $f(E, I_2, I_3)$. This is generally not to be expected since, after all, every spherical potential is of Stäckel form, but only few analytic spherical models exist. However, in the last year some special solutions have been obtained, and more are likely to follow soon. A formal solution of the inversion problem has been obtained by Dejonghe (1987).

For any oblate Stäckel potential the special equilibrium model that contains the thinnest short axis tubes only (i.e., tubes without radial epicyclic motion) can be found by simple inversion of a one-dimensional Abel equation (de Zeeuw 1987). As a result, the corresponding distribution function can be given explicitly, and the kinematic properties can be calculated very easily. These models have the maximum possible streaming around the symmetry axis. Numerical solutions of this kind have been given by Bishop (1987). The similar prolate models require more work, due to the presence of the two types of long axis tubes. Exact distribution functions have recently been found by Park, de Zeeuw & Schwarzschild (1987).

It is likely that triaxial models with maximum streaming, containing thin tube orbits of the three families as well as boxes, can also be constructed in nearly explicit form, by generalizing the analysis of the perfect elliptic disk by de Zeeuw, Hunter & Schwarzschild (1987). This would give insight in the relative importance of the different orbit families, and in the detailed behaviour of the solutions near the focal curves of the ellipsoidal coordinates. This would resolve questions that Statler's models cannot answer, due to lack of numerical accuracy that results from the finite grid of cells.

Dejonghe & de Zeeuw (1987) have succeeded in generalizing Fricke's method (cf. §2.1) to axisymmetric three-integral models, by writing $f(E, L_z, I_3)$ as a triple series in powers of E, L_z and I_3. For each term the corresponding density in a given Stäckel potential can be calculated. Expansion of a given density in a series of these terms then gives f by comparison of coefficients. In practice, this last step may have to be done by numerical means, analogous to Schwarzschild's method, but now using these density components instead of individual orbit densities. The same method can be applied to triaxial systems also, and should give smooth solutions.

In order to obtain fully analytic solutions, Dejonghe & de Zeeuw took a slightly different approach. The Kuzmin & Kutuzov (1962) model (cf. §2.1), has a potential that is of Stäckel form. The authors reconstruct part of the density distribution of this model with three-integral components, and represent the remaining density by an $f(E, L_z)$ found via the two-integral inversion method.

All the above efforts have been directed at finding f for a given ρ. We have seen that for axisymmetric models often the approach that assumes a form for f was taken. This was motivated by the fact that in these cases many solutions exist for a given density, so that a plausible choice for f is likely to lead to an equilibrium model. When it became evident that elliptical galaxies are probably triaxial, virtually nothing was known concerning the existence of non-axisymmetric equilibrium models, let alone regarding their non-uniqueness. Hence, guessing a plausible distribution function seemed rather difficult. Therefore, the second approach was adopted, notably by Schwarzschild.

Now we know that many equilibrium solutions are likely to exist for triaxial models as well, it should not be too difficult to guess a plausible form for f. It seems very useful to attempt to construct dynamical models with this first method as well, in particular for the Stäckel models. A first step in this direction has been taken recently by Stiavelli and Bertin (1985). It should be noted that in order to obtain self-consistent models, f should not be chosen to obey the Ellipsoidal Hypothesis, i.e., $f = f(Q)$, where Q is any linear combination of the three integrals E, I_2 and I_3 (e.g., Chandrasekhar 1942). Although such an assumed form for f has produced interesting spherical models (Eddington 1914; Osipkov 1979; Merritt 1985), no self-gravitating axisymmetric of triaxial models of this kind exist (Eddington 1915*a*; Camm 1941; Fricke 1952).

5.4 Stellar Hydrodynamics

Eddington (1915*a*) already knew that in a Stäckel potential, the principal axes of the velocity ellipsoid are locally always aligned with the coordinate system in which the equations of motion separate. For triaxial models this means that the velocity ellipsoid is always aligned with the principal axes of the model in the central regions, and always aligned nearly radially at large distances. This is exactly what is also seen in the non-separable models (Merritt 1980). Furthermore, the Jeans equations have a simple form in ellipsoidal coordinates (Lynden-Bell 1960). Solution of them gives the three non trivial second velocity moments for any density in a given Stäckel potential. Since many different distribution functions are consistent with the same density, the danger of finding non-physical solutions of the equations is not very severe. For axisymmetric models it turns out that the equations are formally equivalent to those already solved by Bacon (1985) in a slightly different context (§2.3). Thus, by using his solutions, kinematic properties of axisymmetric Stäckel models can be derived easily. The triaxial case is under study by Wyn-Evans (1987, priv. comm.).

6. CONCLUDING REMARKS

It is fair to say that in the last decade considerable progress has been made in the construction of equilibrium models. However, Hunter's (1977) remark that *determining appropriate distribution functions for elliptical galaxies [is] a problem that deserves more study from fundamental stellar dynamic considerations than it has yet received*, has not lost its value.

For moderately flattened axisymmetric and non-rotating triaxial galaxies, the Stäckel models provide an adequate description of the internal dynamical structure. Construction of realistic three-integral models of this kind should evidently be pursued. Methods for doing this are already available. These models are well suited to establish which observations would have to be done in order to determine the intrinsic shape and structure of elliptical galaxies.

Much work remains to be done for triaxial systems with rotating figures, not only for moderately flattened elliptical galaxies, but also for the rapidly rotating nearly flat bars. It is well possible, although not proven, that no realistic rotating potentials with exact integrals, in addition to the energy in the corotating frame, exist. As a result, construction of such systems may have to be done largely by numerical means.

It is evident that the family of triaxial equilibrium structures is very rich. The most important next step in the study of these systems is the delineation of the models that are dynamically stable. If we then can establish which of the models are favored by the elliptical galaxies, we will have a much better understanding of the formation of these systems.

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DISCUSSION

Whitmore: I would like to add a note of caution about making the firm conclusion that bulges are completely explained by the oblate rotator models. In a paper by Whitmore, Rubin, and Ford two years ago (1984, *Ap. J.*, **287**, 66), we reanalysed Kormendy and Illingworth's observations along with some of our own. After taking into account several effects, the most important of which is contamination in the spectrum by light from the disk, we find that the spiral bulges fall about 30–40% below the oblate spheroid line. It has been stated during this conference that taking into account the flattening of the bulge by the disk potential makes the agreement nearly perfect. This is only about a 10% effect at best, so I would urge people to keep an open mind about this question.

Illingworth: In response to Brad Whitmore's comment that the bulges of disk galaxies are not fully consistent with being oblate rotators, I would like to note that the detailed comparisons made by Jarvis and Freeman do not support that contention. These authors compared the Kormendy/Illingworth bulge kinematical data with their isotropic dispersion oblate rotationally–flattened models (which include a superimposed disk potential) and found excellent agreement.

Jarvis: In relation to Brad's comment I would like to point out that at least in the case of one of the galaxies that we modeled, NGC7814, there is a negligibly small *luminous* disk. This means that contamination of bulge light by disk light would have been insignificant, leading us to believe that the kinematic observations are reliable and truly reflect the kinematics of the bulge *alone*.

Vietri: I would like to add my voice to the cautionary note rung by Whitmore on the nature of bulges (and small ellipticals). In fact, the only bulge for which we have a good de–projection, M31, has been known for $\sim$ 30 years to be inconsistent with being oblate (Lindblad 1956, *Stockholm Obs. Ann.*, **19**, No. 2). It can only be triaxial, or at most prolate. Furthermore, Zaritsky & Lo (1986, *Ap. J.*, **303**, 66) found variations of ellipticities and major axis position angles in all 12 bulges they observed.

Binney: A comment on the structure of Ellipsoid–Land. I believe there are two non–trivial state holders in Ellipsoid–Land: Statler & Bishop. Though nobody has yet proved this, I believe that if you choose a distribution function at random, you have a finite chance of landing on Bishop's frontier province, and a finite chance of landing in Statler's interior. By contrast, there is no need to obtain a visa for any of Ellipsoid–Land's lower frontier provinces.

de Zeeuw: In the "low countries" the orbital structure that is found in the Stäckel potentials is not generic; a small perturbation will produce a markedly different phase–space structure. Hence it is very likely that this part of the diagram is populated by unstable equilibrium models that are of minor interest only. I would think that the chance of landing on the oblate axisymmetric models is considerably smaller than ending up in the triaxial province.