APPLICATIONS OF THE MULTIPLICATION FORMULA FOR THE GAMMA FUNCTION TO E-FUNCTION SERIES

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1. Introductory. In two recent papers [**1**, **2**] the Barnes integral for the E-functions was employed to sum a number of infinite series of E-functions. In §2 of this paper, by making use of the multiplication formula for the gamma function, the method is extended to series of E-functions of a different type.

The Barnes formula is

$$E(p\,;\,\alpha_r : q\,;\,\rho_s : z) = \frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\Pi\Gamma(\alpha_r-\zeta)}{\Pi\Gamma(\rho_s-\zeta)}\,z^{\zeta}\,d\zeta, \qquad \text{.....................(1)}$$

where $|\,\text{amp}\,z\,| < \pi$, and the integral is taken up the η-axis, with loops, if necessary, to ensure that the origin lies to the left and the points $\alpha_1, \alpha_2, \ldots, \alpha_p$ to the right of the contour. Zero and negative integral values of the α's and ρ's are excluded, and the α's must not differ by integral values. When $p < q+1$ the contour is bent to the left at each end. When $p > q+1$ the formula is valid for $|\,\text{amp}\,z\,| < \frac{1}{2}(p-q+1)\pi$.

The gamma function formula is

$$\Gamma(z)\Gamma\left(z+\frac{1}{m}\right)\ldots\Gamma\left(z+\frac{m-1}{m}\right) = (2\pi)^{\frac{1}{2}m-\frac{1}{2}}m^{\frac{1}{2}-mz}\Gamma(mz), \qquad \text{..................(2)}$$

where m is a positive integer.

From (2) it can be deduced that, if r is a positive integer,

$$\Gamma\left(\frac{\alpha+r}{m}\right)\Gamma\left(\frac{\alpha+1+r}{m}\right)\ldots\Gamma\left(\frac{\alpha+m-1+r}{m}\right) = \Gamma\left(\frac{\alpha}{m}\right)\Gamma\left(\frac{\alpha+1}{m}\right)\ldots\Gamma\left(\frac{\alpha+m-1}{m}\right)m^{-r}(\alpha\,;\,r), \qquad \text{......(3)}$$

where

$$(\alpha\,;\,r) = \alpha(\alpha+1)\ldots(\alpha+r-1)\quad (r = 1, 2, 3, \ldots)\,;\ (\alpha\,;\,0) = 1. \qquad \text{..................(4)}$$

For, from (2), each side of (3) is equal to

$$(2\pi)^{\frac{1}{2}m-\frac{1}{2}}m^{\frac{1}{2}-\alpha-r}\,\Gamma(\alpha)(\alpha\,;\,r).$$

The formula

$$E\left(\frac{n+m+1}{2}, \frac{n+m+2}{2} : n+\tfrac{3}{2} : z^2\right) = \frac{\Gamma(\frac{1}{2})\Gamma(n+m+1)}{2^{m-\frac{1}{2}}}\frac{z^{n+m+1}}{(z^2+1)^{\frac{1}{2}m+\frac{1}{2}}}\,T_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}\left\{\frac{z}{\sqrt{(z^2+1)}}\right\} \qquad \text{.....................(5)}$$

will also be required. In proving it use will be made of the two following formulae [**3**, pp. 305, 307]:

$$Q_n^m(z) = \sqrt{\left\{\frac{\pi}{2\sqrt{(z^2-1)}}\right\}}\{z+\sqrt{(z^2-1)}\}^{-n-\frac{1}{2}}\frac{\Gamma(n+m+1)}{\Gamma(n+\frac{3}{2})}\,F\left\{\begin{matrix}\frac{1}{2}+m, \frac{1}{2}-m\\ n+\frac{3}{2}\end{matrix}\,;\,\frac{\sqrt{(z^2-1)}-z}{2\sqrt{(z^2-1)}}\right\}; \qquad \text{......(6)}$$

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and

$$T_n^{-m}(z) = \frac{1}{\Gamma(m+1)}\left(\frac{1-z}{1+z}\right)^{\frac{1}{2}m} F\left(\begin{matrix}-n, n+1 \\ m+1\end{matrix}; \frac{1-z}{2}\right) \quad \text{.....................(7)}$$

Now, from (6),

$$Q_n^m(iz) = i^{-n-1}\sqrt{\left\{\frac{\pi}{2\sqrt{(z^2+1)}}\right\}}\{\sqrt{(z^2+1)}+z\}^{-n-\frac{1}{2}}\frac{\Gamma(n+m+1)}{\Gamma(n+\frac{3}{2})} F\left\{\begin{matrix}\frac{1}{2}+m, \frac{1}{2}-m \\ n+\frac{3}{2}\end{matrix}; \frac{\sqrt{(z^2+1)}-z}{2\sqrt{(z^2+1)}}\right\},$$

and, from (7),

$$T_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}\left\{\frac{z}{\sqrt{(z^2+1)}}\right\} = \frac{1}{\Gamma(n+\frac{3}{2})}\{\sqrt{(z^2+1)}+z\}^{-n-\frac{1}{2}} F\left\{\begin{matrix}\frac{1}{2}+m, \frac{1}{2}-m \\ n+\frac{3}{2}\end{matrix}; \frac{\sqrt{(z^2+1)}-z}{2\sqrt{(z^2+1)}}\right\}.$$

Hence

$$Q_n^m(iz) = i^{-n-1}\sqrt{\left(\frac{\pi}{2}\right)}\frac{\Gamma(n+m+1)}{(z^2+1)^{1/4}} T_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}\left\{\frac{z}{\sqrt{(z^2+1)}}\right\}. \quad \text{..................(8)}$$

But [3, p. 304]

$$Q_n^m(iz) = i^{-n-1}\frac{\Gamma\left(\frac{n+m+1}{2}\right)\Gamma\left(\frac{n+m+2}{2}\right)}{2^{1-m}\Gamma(n+\frac{3}{2})}\frac{(z^2+1)^{\frac{1}{2}m}}{z^{n+m+1}} F\left(\frac{n+m+1}{2}, \frac{n+m+2}{2}; n+\tfrac{3}{2}; -z^{-2}\right)$$

$$= i^{-n-1}2^{m-1}z^{-n-m-1}(z^2+1)^{\frac{1}{2}m} E\left(\frac{n+m+1}{2}, \frac{n+m+2}{2} : n+\tfrac{3}{2} : z^2\right);$$

hence, on comparing this with (8), formula (5) is obtained.

2. Infinite series. The first series is

$$\sum_{n=0}^{\infty}\frac{(-k)^n}{n!} E\left(\frac{\alpha+n}{m}, \frac{\alpha+1+n}{m}, \ldots, \frac{\alpha+m-1+n}{m}, \alpha_1, \ldots, \alpha_p : q ; \rho_s : z\right)$$

$$= \left(1+\frac{k}{m}\right)^{-\alpha} E\left\{\frac{\alpha}{m}, \frac{\alpha+1}{m}, \ldots, \frac{\alpha+m-1}{m}, \alpha_1, \ldots, \alpha_p : q ; \rho_s : z\left(1+\frac{k}{m}\right)^m\right\}, \quad \text{.........(9)}$$

where m is a positive integer, $|\operatorname{amp} z| < \pi$, $|\operatorname{amp}\{z(1+k/m)^m\}| < \pi$, $|k/m| < 1$.

From (1) the *E*-function on the left is equal to

$$\frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\,\Gamma\left(\frac{\alpha+n}{m}-\zeta\right)\ldots\Gamma\left(\frac{\alpha+m-1+n}{m}-\zeta\right)\Pi\Gamma(\alpha_r-\zeta)}{\Pi\Gamma(\rho_s-\zeta)} z^{\zeta}\,d\zeta,$$

and from (3) this is equal to

$$\frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\Gamma\left(\frac{\alpha}{m}-\zeta\right)\ldots\Gamma\left(\frac{\alpha+m-1}{m}-\zeta\right)\Pi\Gamma(\alpha_r-\zeta)}{\Pi\Gamma(\rho_s-\zeta)} m^{-n}(\alpha-m\zeta; n) z^{\zeta}\,d\zeta.$$

Hence, on changing the order of integration and summation, the series is equal to

$$\frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\Gamma\left(\frac{\alpha}{m}-\zeta\right)\ldots\Gamma\left(\frac{\alpha+m-1}{m}-\zeta\right)\Pi\Gamma(\alpha_r-\zeta)}{\Pi\Gamma(\rho_s-\zeta)} z^{\zeta} F\left(\alpha-m\zeta;\,; -\frac{k}{m}\right)d\zeta,$$

and from this, since

$$F(\alpha-m\zeta;\,; -k/m) = (1+k/m)^{m\zeta-\alpha},$$

the result follows.

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In particular, when $m=2$, $p=0$, $q=1$, $\alpha=n+m+1$, $\rho_1=n+\frac{3}{2}$, if z is replaced by z^2 and k by $2k$, it follows from (5) that

$$\sum_{r=0}^{\infty}\frac{(n+m+1\,;\,r)}{r!}\left\{\frac{-kz}{\sqrt{(z^2+1)}}\right\}^r T_{m+\frac{3}{2}+r}^{-n-\frac{1}{2}}\left\{\frac{z}{\sqrt{(z^2+1)}}\right\}$$
$$=\left\{\frac{z^2+1}{z^2(1+k)^2+1}\right\}^{\frac{1}{2}m+\frac{1}{4}} T_{m+\frac{3}{2}}^{-n-\frac{1}{2}}\left[\frac{z(1+k)}{\sqrt{\{z^2(1+k)^2+1\}}}\right], \quad \ldots\ldots(10)$$

where $|\operatorname{amp} z|<\frac{1}{2}\pi$, $|\operatorname{amp}\{z(1+k)\}|<\frac{1}{2}\pi$, $|k|<1$.

The next summation is

$$\sum_{n=0}^{\infty}\frac{(\beta\,;\,n)}{n!}E\left(\begin{matrix}\frac{\alpha+n}{m},\ldots,\frac{\alpha+m-1+n}{m},\alpha_1,\ldots,\alpha_p : z\\ \frac{\rho+n}{m},\ldots,\frac{\rho+m-1+n}{m},\rho_1,\ldots,\rho_q\end{matrix}\right)$$
$$=\frac{\Gamma(\rho-\alpha-\beta)}{\Gamma(\rho-\alpha)}m^{\beta}E\left(\begin{matrix}\frac{\alpha}{m},\ldots,\frac{\alpha+m-1}{m},\alpha_1,\ldots,\alpha_p : \quad z\\ \frac{\rho-\beta}{m},\ldots,\frac{\rho-\beta+m-1}{m},\rho_1,\ldots,\rho_q\end{matrix}\right), \quad \ldots\ldots\ldots\ldots(11)$$

where $R(\rho-\alpha-\beta)>0$, $|\operatorname{amp} z|<\pi$.

For the series is equal to

$$\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\Gamma\left(\frac{\alpha}{m}-\zeta\right)\ldots\Gamma\left(\frac{\alpha+m-1}{m}-\zeta\right)\Pi\Gamma(\alpha_r-\zeta)}{\Gamma\left(\frac{\rho}{m}-\zeta\right)\ldots\Gamma\left(\frac{\rho+m-1}{m}-\zeta\right)\Pi\Gamma(\rho_s-\zeta)}z^{\zeta}F\left(\begin{matrix}\alpha-m\zeta,\beta\,;\,1\\ \rho-m\zeta\end{matrix}\right)d\zeta.$$

But, if $R(\rho-\alpha-\beta)>0$,

$$F\left(\begin{matrix}\alpha-m\zeta,\beta\,;\,1\\ \rho-m\zeta\end{matrix}\right)=\frac{\Gamma(\rho-m\zeta)\Gamma(\rho-\alpha-\beta)}{\Gamma(\rho-\alpha)\Gamma(\rho-\beta-m\zeta)};$$

hence, on applying (2) to $\Gamma(\rho-m\zeta)$ and $\Gamma(\rho-\beta-m\zeta)$, the result is obtained.

Again, if m and n are positive integers, $|\operatorname{amp} z|<\pi$, $R(\rho-\alpha-n)>0$,

$$\sum_{r=0}^{\infty}\frac{m^r(n\,;r)}{r!(\rho\,;\,r)}E\left(\begin{matrix}\frac{\alpha+r}{m},\ldots,\frac{\alpha+m-1+r}{m},\alpha_1,\ldots,\alpha_p : z\\ \rho_1,\ldots,\rho_q\end{matrix}\right)$$
$$=\frac{(1-\rho\,;\,n)}{m^n}E\left(\begin{matrix}\frac{\alpha}{m},\ldots,\frac{\alpha+m-1}{m},\frac{\alpha-\rho+1}{m},\ldots,\frac{\alpha-\rho+m}{m},\alpha_1,\ldots,\alpha_p : z\\ \frac{\alpha-\rho+n+1}{m},\ldots,\frac{\alpha-\rho+n+m}{m},\rho_1,\ldots,\rho_q\end{matrix}\right)\ldots.(12)$$

For the series is equal to

$$\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\Gamma\left(\frac{\alpha}{m}-\zeta\right)\ldots\Gamma\left(\frac{\alpha+m-1}{m}-\zeta\right)\Pi\Gamma(\alpha_r-\zeta)}{\Pi\Gamma(\rho_s-\zeta)}z^{\zeta}F\left(\begin{matrix}n,\alpha-m\zeta\,;\,1\\ \rho\end{matrix}\right)d\zeta.$$

Now, if $R(\rho-\alpha-n)>0$,

$$F\left(\begin{matrix}n,\alpha-m\zeta\,;\,1\\ \rho\end{matrix}\right)=\frac{\Gamma(\rho)\Gamma(\rho-\alpha-n+m\zeta)}{\Gamma(\rho-n)\Gamma(\rho-\alpha+m\zeta)}=(1-\rho\,;\,n)\frac{\Gamma(\alpha-\rho+1-m\zeta)}{\Gamma(\alpha-\rho+1+n-m\zeta)}.$$

Hence, on applying formula (2) to the gamma functions, the expression on the right of (12) is obtained.

Next, if m and n are positive integers such that $m > n$, if $R(\rho-\alpha-\beta) > 0$ and if $|\operatorname{amp} z| < \pi$,

$$\sum_{r=0}^{\infty} \frac{(\beta\,;\,r)}{r!}\left(\frac{n}{m}\right)^r E\left(\begin{matrix}\dfrac{\alpha+r}{n}, \dots, \dfrac{\alpha+n-1+r}{n}, \alpha_1, \dots, \alpha_p : z \\ \dfrac{\rho+r}{m}, \dots, \dfrac{\rho+m-1+r}{m}, \rho_1, \dots, \rho_q\end{matrix}\right)$$

$$= \left(\frac{m}{m-n}\right)^{\beta} E\left(\begin{matrix}\dfrac{\alpha}{n}, \dots, \dfrac{\alpha+n-1}{n}, \dfrac{\rho-\alpha-\beta}{m-n}, \dots, \dfrac{\rho-\alpha-\beta+m-n-1}{m-n}, \alpha_1, \dots, \alpha_p : z \\ \dfrac{\rho-\beta}{m}, \dots, \dfrac{\rho-\beta+m-1}{m}, \dfrac{\rho-\alpha}{m-n}, \dots, \dfrac{\rho-\alpha+m-n-1}{m-n}, \rho_1, \dots, \rho_q\end{matrix}\right). \quad \text{......(13)}$$

For the series is equal to

$$\frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\Gamma\left(\frac{\alpha}{n}-\zeta\right)\dots\Gamma\left(\frac{\alpha+n-1}{n}-\zeta\right)\Pi\Gamma(\alpha_r-\zeta)}{\Gamma\left(\frac{\rho}{m}-\zeta\right)\dots\Gamma\left(\frac{\rho+m-1}{m}-\zeta\right)\Pi\Gamma(\rho_s-\zeta)} z^{\zeta} F\left(\begin{matrix}\alpha-n\zeta, \beta\,;\,1 \\ \rho-m\zeta\end{matrix}\right) d\zeta.$$

Now, if $R(\rho-\alpha-\beta) > 0$,

$$F\left(\begin{matrix}\alpha-n\zeta, \beta\,;\,1 \\ \rho-m\zeta\end{matrix}\right) = \frac{\Gamma(\rho-m\zeta)\Gamma\{\rho-\alpha-\beta-(m-n)\zeta\}}{\Gamma\{\rho-\alpha-(m-n)\zeta\}\Gamma(\rho-\beta-m\zeta)},$$

and, from (2), this is equal to

$$\frac{\Gamma\left(\frac{\rho}{m}-\zeta\right)\dots\Gamma\left(\frac{\rho+m-1}{m}-\zeta\right)}{\Gamma\left(\frac{\rho-\beta}{m}-\zeta\right)\dots\Gamma\left(\frac{\rho-\beta+m-1}{m}-\zeta\right)}$$

$$\times \frac{\Gamma\left(\frac{\rho-\alpha-\beta}{m-n}-\zeta\right)\dots\Gamma\left(\frac{\rho-\alpha-\beta+m-n-1}{m-n}-\zeta\right)}{\Gamma\left(\frac{\rho-\alpha}{m-n}-\zeta\right)\dots\Gamma\left(\frac{\rho-\alpha+m-n-1}{m-n}-\zeta\right)}\left(\frac{m}{m-n}\right)^{\beta}.$$

From this the result follows.

Similar results can be obtained by using the formula

$$\Gamma\left(\frac{\alpha-r}{m}\right)\Gamma\left(\frac{\alpha+1-r}{m}\right)\dots\Gamma\left(\frac{\alpha+m-1-r}{m}\right)$$
$$= \Gamma\left(\frac{\alpha}{m}\right)\dots\Gamma\left(\frac{\alpha+m-1}{m}\right)(-m)^r/(1-\alpha\,;\,r), \quad \text{.........................(14)}$$

where r is a positive integer.

For example, if $|\operatorname{amp} z| < \pi$, $|\operatorname{amp}\{z(1+k/m)^{-m}\}| < \pi$, $|k/m| < 1$,

$$\sum_{n=0}^{\infty}\frac{k^n}{n!}E\left(p\,;\,\alpha_r : \frac{\rho-n}{m}, \dots, \frac{\rho+m-1-n}{m}, \rho_1, \dots, \rho_q : z\right)$$
$$= \left(1+\frac{k}{m}\right)^{\rho-1} E\left\{p\,;\,\alpha_r : \frac{\rho}{m}, \dots, \frac{\rho+m-1}{m}, \rho_1, \dots, \rho_q : z\left(1+\frac{k}{m}\right)^{-m}\right\}. \quad \text{......(15)}$$

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For the series is equal to

$$\frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\Pi\,\Gamma(\alpha_r-\zeta)}{\Gamma\left(\frac{\rho}{m}-\zeta\right)\dots\Gamma\left(\frac{\rho+m-1}{m}-\zeta\right)\Pi\,\Gamma(\rho_s-\zeta)}\,F\left(1-\rho+m\zeta\,;\,;\;-\frac{k}{m}\right)z^{\zeta}\,d\zeta,$$

and

$$F(1-\rho+m\zeta\,;\,;\;-k/m)=(1+k/m)^{\rho-1-m\zeta}.$$

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