A SPECTRAL APPROACH TO AN INTEGRAL EQUATION

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1. Introduction. In a recent paper [**7**], Rooney used a technique involving the Mellin transform to obtain solutions in certain spaces $\mathscr{L}_{\mu,p}$ of an integral equation which had been studied previously by Šub-Sizonenko [**9**]. The integral equation in question can be written as

$$(I+G^{0,1/2})\phi(x)=\psi(x) \qquad (x>0), \tag{1.1}$$

where I denotes the identity operator and $G^{0,1/2}$ is given by

$$(G^{0,1/2}\phi)(x)=\pi^{-1/2}\int_x^\infty (\log t/x)^{-1/2}\phi(t)\,dt/t,$$

with the inversion formula obtained by Rooney taking the form

$$\phi(x)=\int_x^\infty ((t/x)\operatorname{erfc}((\log t/x)^{1/2})-\pi^{-1/2}(\log t/x)^{-1/2})\psi(t)\,dt/t + \psi(x) \qquad (x>0). \tag{1.2}$$

Rooney verified that (1.1) and (1.2) formed an inversion pair in $\mathscr{L}_{\mu,p}$ for $1\leqslant p<\infty$ and $\mu>0$.

In this paper, we shall extend Rooney's result by obtaining inversion formulae for the integral equations

$$(\lambda I+G^{\eta,1/2})\phi(x)=\psi(x) \qquad (x>0) \tag{1.3}$$

and

$$(\lambda I+H^{\eta,1/2})\phi(x)=\psi(x) \qquad (x>0) \tag{1.4}$$

where $\lambda>0$ and $G^{\eta,1/2}$ and $H^{\eta,1/2}$ are particular cases of the operators $G^{\eta,\alpha}$ and $H^{\eta,\alpha}$ defined, for $\operatorname{Re}\alpha>0$ and $\eta\in\mathbb{C}$, by

$$(G^{\eta,\alpha}\phi)(x)=[\Gamma(\alpha)]^{-1}\int_x^\infty (x/t)^\eta(\log t/x)^{\alpha-1}\phi(t)\,dt/t \qquad (x>0), \tag{1.5}$$

$$(H^{\eta,\alpha}\phi)(x)=[\Gamma(\alpha)]^{-1}\int_0^x (t/x)^{\eta+1}(\log x/t)^{\alpha-1}\phi(t)\,dt/t \qquad (x>0). \tag{1.6}$$

Note that equation (1.3) reduces to equation (1.1) when $\lambda=1$ and $\eta=0$ and therefore in deriving inversion formulae for (1.3) and (1.4), we shall also obtain an inversion formula for (1.1).

Working within the framework of the Banach spaces L^p_μ (where $L^p_\mu=\mathscr{L}_{1/p-\mu,p}$ when μ is real), we shall first determine properties of $H^{\eta,\alpha}$ and $G^{\eta,\alpha}$ and shall establish that, under certain conditions, $H^{\eta,1/2}=(H^{\eta,1})^{1/2}$, $G^{\eta,1/2}=(G^{\eta,1})^{1/2}$ where $(H^{\eta,1})^{1/2}$ and $(G^{\eta,1})^{1/2}$

denote fractional powers of order 1/2 of $H^{\eta,1}$ and $G^{\eta,1}$ respectively. Then, by applying a result concerning the resolvent of a fractional power of an operator, we shall obtain the required inversion formulae which will be shown to include (1.2) as a special case.

2. Preliminaries. Let X denote a complex Banach space with norm $\|\ \|$ and let $L(X)$ denote the Banach space of bounded linear operators $A: X \to X$. We say that A is in the class $P(X)$ if

(a) $A \in L(X)$;

(b) $R(\lambda; A) \equiv (\lambda I - A)^{-1} \in L(X)$ for each $\lambda > 0$;

(c) $\|\lambda R(\lambda; A)\phi\| \leq M\|\phi\|$ for all $\lambda > 0$ and $\phi \in X$ where M, a positive constant, is independent of both $\lambda > 0$ and $\phi \in X$.

If A is an operator in $P(X)$, then a family of operators $\{(-A)^\alpha; \operatorname{Re}\alpha > 0\}$ can be generated by means of the formulae

$$(-A)^\alpha \phi = \pi^{-1}\sin(\pi\alpha)\int_0^\infty \lambda^{\alpha-1}[R(\lambda; A) - \lambda/(1+\lambda^2)](-A\phi)\,d\lambda$$
$$-\sin(\pi\alpha/2)A\phi, \qquad 0 < \operatorname{Re}\alpha < 2, \qquad \phi \in X, \tag{2.1}$$

$$(-A)^\alpha\phi = (-A)^{\alpha-n}(-A)^n\phi, \qquad n < \operatorname{Re}\alpha < n+2, \qquad n = 1, 2, \ldots, \phi \in X. \tag{2.2}$$

By appealing to conditions (b) and (c) above, we can readily show that the integral in (2.1) exists, for each $\phi \in X$, as a Bochner integral in X (see [**4**, p. 34 and pp. 118–119]). The main properties of the operators $(-A)^\alpha$ are summarised below.

THEOREM 2.1. *Let A be an operator in $P(X)$ and let $(-A)^\alpha$ be defined via* (2.1) *and* (2.2). *Then*

(a) $(-A)^\alpha \in L(X)$ *for each α such that* $\operatorname{Re}\alpha > 0$;

(b) $(-A)^\alpha(-A)^\beta = (-A)^{\alpha+\beta}$ *for* $\operatorname{Re}\alpha$, $\operatorname{Re}\beta > 0$;

(c) $[(-A)^\alpha]^\beta = (-A)^{\alpha\beta}$ *for* $0 < \alpha < 1$ *and* $\operatorname{Re}\beta > 0$;

(d) *for each $\lambda > 0$ and $\alpha \in (0, 1)$, the resolvent operator $R(\lambda; -(-A)^\alpha)$ exists in $L(X)$ and is given by the Bochner integral*

$$R(\lambda; -(-A)^\alpha)\phi = \int_0^\infty g_{\lambda,\alpha}(u)R(u; A)\phi\,du \qquad (\phi \in X) \tag{2.3}$$

where

$$g_{\lambda,\alpha}(u) = \pi^{-1}\sin(\pi\alpha)u^\alpha[\lambda^2 + 2\lambda u^\alpha\cos(\pi\alpha) + u^{2\alpha}]^{-1}. \tag{2.4}$$

Proof. These results can be found in [**1**] and [**3**] and can also be deduced as a special case of the theory presented in [**4**] and [**5**].

Motivated by the properties possessed by the operators $(-A)^\alpha$, we shall henceforth refer to $(-A)^\alpha$ as the αth power of $-A$.

3. The operators $G^{\eta,\alpha}$ and $H^{\eta,\alpha}$ on the spaces L^p_μ. In this section, we shall determine certain properties of the integral operators $G^{\eta,\alpha}$ and $H^{\eta,\alpha}$ given by (1.5) and (1.6) respectively. In particular, we shall examine the behaviour of these operators on the spaces L^p_μ of (equivalence classes of) functions ϕ such that $\int_0^\infty |x^{-\mu}\phi(x)|^p\,dx<\infty$. Here, and in the sequel, μ is any complex number and $1\le p<\infty$. Equipped with the norm $\|\ \|_{p,\mu}$ defined by

$$\|\phi\|_{p,\mu}=\left(\int_0^\infty |x^{-\mu}\phi(x)\,|^p\,dx\right)^{1/p}$$

the space L^p_μ is a Banach space and, for μ real, coincides with the space $\mathscr{L}_{1/p-\mu,\,p}$ of Rooney [**8**].

LEMMA 3.1. *Let* $\operatorname{Re}\alpha>0$ *and* $\phi\in L^p_\mu$.
(a) *If* $\operatorname{Re}(\eta+\mu)+1>1/p$, *then* $H^{\eta,\alpha}$ *is a bounded operator on* L^p_μ *with*

$$\|H^{\eta,\alpha}\phi\|_{p,\mu}\le(\Gamma(\operatorname{Re}\alpha)/|\Gamma(\alpha)|)(\operatorname{Re}(\eta+\mu)+1-1/p)^{-\operatorname{Re}\alpha}\,\|\phi\|_{p,\mu}. \tag{3.1}$$

(b) *If* $\operatorname{Re}(\eta-\mu)>-1/p$, *then* $G^{\eta,\alpha}$ *is a bounded operator on* L^p_μ *with*

$$\|G^{\eta,\alpha}\phi\|_{p,\mu}\le(\Gamma(\operatorname{Re}\alpha)/|\Gamma(\alpha)|)(\operatorname{Re}(\eta-\mu)+1/p)^{-\operatorname{Re}\alpha}\,\|\phi\|_{p,\mu}. \tag{3.2}$$

Proof. This can be proved in a routine manner by using a generalisation of an inequality of Hardy [**6**].

THEOREM 3.2. *Let* $\operatorname{Re}\alpha>0$.
(a) *If* $\operatorname{Re}(\eta+\mu)+1>1/p$, *then* $-H^{\eta,1}\in P(L^p_\mu)$ *and* $(H^{\eta,1})^\alpha=H^{\eta,\alpha}$.
(b) *If* $\operatorname{Re}(\eta-\mu)>-1/p$, *then* $-G^{\eta,1}\in P(L^p_\mu)$ *and* $(G^{\eta,1})^\alpha=G^{\eta,\alpha}$.

Proof. We shall prove (a), the proof of (b) being similar. Firstly, we remark that, under the given conditions, $-H^{\eta,1}\in L(L^p_\mu)$. Secondly, a routine calculation (see [**4**, p. 67)]) can be used to show that, for $\lambda>0$,

$$(\lambda I+H^{\eta,1})^{-1}\phi=(1/\lambda)\phi-(1/\lambda)^2H^{\eta+1/\lambda,1}\phi,\qquad \phi\in L^p_\mu. \tag{3.3}$$

Hence, it follows that

$$\begin{aligned}&\|\lambda(\lambda I+H^{\eta,1})^{-1}\phi\|_{p,\mu}\\ &\quad=\|\phi-(1/\lambda)H^{\eta+1/\lambda,1}\phi\|_{p,\mu}\\ &\quad\le\|\phi\|_{p,\mu}+(1/\lambda)(\operatorname{Re}(\eta+\mu+1/\lambda)+1-1/p)^{-1}\,\|\phi\|_{p,\mu}\quad\text{(from (3.1))}\\ &\quad<2\,\|\phi\|_{p,\mu}\end{aligned}$$

and this holds for all $\lambda>0$. Consequently, $-H^{\eta,1}$ belongs to the class $P(L^p_\mu)$ and so, from Theorem 2.1(a), $(H^{\eta,1})^\alpha$ exists as a bounded operator on L^p_μ. Now let ψ belong to the space $C_0^\infty(0,\infty)$ of smooth functions having compact support in $(0,\infty)$ and suppose that $0<\operatorname{Re}\alpha<1$. In this case, formula (2.1) can be replaced by

$$(-A)^\alpha\phi=\pi^{-1}\sin(\pi\alpha)\int_0^\infty\lambda^{\alpha-1}R(\lambda;A)(-A)\phi\,d\lambda\qquad\text{(see [1] and [5])},$$

and therefore

$$\begin{aligned}(H^{\eta,1})^{\alpha}\psi(x) &= \left(\pi^{-1}\sin(\pi\alpha)\int_0^{\infty}\lambda^{\alpha-2}H^{\eta+1/\lambda,1}\psi\,d\lambda\right)(x) \qquad (x>0)\\ &= \pi^{-1}\sin(\pi\alpha)\int_0^{\infty}\lambda^{\alpha-2}H^{\eta+1/\lambda,1}\psi(x)\,d\lambda\\ &= \pi^{-1}\sin(\pi\alpha)\int_0^{\infty}\lambda^{\alpha-2}\int_0^{x}(t/x)^{\eta+1/\lambda+1}\psi(t)\,dt/t\,d\lambda.\end{aligned}$$

The justification for transferring the point x inside the Bochner integral in the above analysis is provided by [**4**, Theorems 4.19 and 4.24]. If we now apply Fubini's theorem to interchange the order of integration, we obtain

$$\begin{aligned}(H^{\eta,1})^{\alpha}\psi(x) &= \pi^{-1}\sin(\pi\alpha)\int_0^{x}(t/x)^{\eta+1}\psi(t)\,dt/t\int_0^{\infty}\lambda^{\alpha-2}\exp[-\lambda^{-1}\log(x/t)]\,d\lambda\\ &= [\Gamma(\alpha)]^{-1}\int_0^{x}(t/x)^{\eta+1}(\log x/t)^{\alpha-1}\psi(t)\,dt/t\\ &= (H^{\eta,\alpha}\psi)(x).\end{aligned}$$

In a similar fashion, we can prove that $(H^{\eta,1})^{\alpha}\psi = H^{\eta,\alpha}\psi$ for $n<\operatorname{Re}\alpha<n+1$, $n=1,2,\dots,$ and $\psi\in C_0^{\infty}(0,\infty)$ while, for $\alpha=n+i\xi$, we have

$$\begin{aligned}(H^{\eta,1})^{n+i\xi}\psi &= [(H^{\eta,1})^{(n+i\xi)/2n}]^{2n}\psi\\ &= [H^{\eta,(n+i\xi)/2n}]^{2n}\psi\\ &= H^{\eta,n+i\xi}\psi,\end{aligned}$$

where the last step can be verified by direct calculation. This proves that $(H^{\eta,1})^{\alpha} = H^{\eta,\alpha}$, $\operatorname{Re}\alpha>0$, as operators on $C_0^{\infty}(0,\infty)$ and the general result follows from the continuity of the operators on L_{μ}^{p} in conjunction with the denseness of $C_0^{\infty}(0,\infty)$ in L_{μ}^{p}. This completes the proof.

Using the properties of fractional powers listed in Theorem 2.1, we can now write down the properties of $H^{\eta,\alpha}$ and $G^{\eta,\alpha}$ on L_{μ}^{p}.

THEOREM 3.3. *Let* $\operatorname{Re}(\eta+\mu)+1>1/p$ *and* $\phi\in L_{\mu}^{p}$. *Then*
(a) $H^{\eta,\alpha}H^{\eta,\beta}\phi = H^{\eta,\alpha+\beta}\phi$ *for* $\operatorname{Re}\alpha, \operatorname{Re}\beta>0$;
(b) $(H^{\eta,\alpha})^{\beta}\phi = H^{\eta,\alpha\beta}\phi$ *for* $0<\alpha<1$ *and* $\operatorname{Re}\beta>0$.

THEOREM 3.4. *Let* $\operatorname{Re}(\eta-\mu)>-1/p$ *and* $\phi\in L_{\mu}^{p}$. *Then*
(a) $G^{\eta,\alpha}G^{\eta,\beta}\phi = G^{\eta,\alpha+\beta}\phi$ *for* $\operatorname{Re}\alpha, \operatorname{Re}\beta>0$;
(b) $(G^{\eta,\alpha})^{\beta}\phi = G^{\eta,\alpha\beta}\phi$ *for* $0<\alpha<1$, $\operatorname{Re}\beta>0$.

4. The resolvent operators $R(\lambda;-G^{\eta,1/2})$ and $R(\lambda;-H^{\eta,1/2})$. In this final section, we apply the results of Sections 2 and 3 to obtain solutions of equations (1.3) and (1.4) when the right-hand side ψ belongs to L^p_μ. We begin by stating the following results concerning the resolvent operators $R(\lambda; -G^{\eta,\alpha})$ and $R(\lambda; -H^{\eta,\alpha})$.

THEOREM 4.1. *Let* $\lambda>0$, $0<\alpha<1$, $\eta\in\mathbb{C}$ *and let* $g_{\lambda,\alpha}$ *be the function defined by* (2.4). *Then*

(a) *if* $\mathrm{Re}(\eta+\mu)+1>1/p$, *the resolvent operator* $R(\lambda; -H^{\eta,\alpha})$ *exists in* $L(L^p_\mu)$ *and is given by*

$$R(\lambda; -H^{\eta,\alpha})\phi=(1/\lambda)\phi-\int_0^\infty g_{\lambda,\alpha}(u)u^{-2}H^{\eta+1/u,1}\phi\,du \qquad (\phi\in L^p_\mu); \tag{4.1}$$

(b) *if* $\mathrm{Re}(\eta-\mu)>-1/p$, *the resolvent operator* $R(\lambda; -G^{\eta,\alpha})$ *exists in* $L(L^p_\mu)$ *and is given by*

$$R(\lambda; -G^{\eta,\alpha})\phi=(1/\lambda)\phi-\int_0^\infty g_{\lambda,\alpha}(u)u^{-2}G^{\eta+1/u,1}\phi\,du \qquad (\phi\in L^p_\mu). \tag{4.2}$$

The integrals which appear in (4.1) *and* (4.2) *exist as Bochner integrals in* L^p_μ.

Proof. Formula (4.1) follows immediately from Theorem 2.1(d), (3.3) and the fact that $\int_0^\infty u^{-1}g_{\lambda,\alpha}(u)\,du=\lambda^{-1}$ for $0<\alpha<1$ and $\lambda>0$ (see [**3**]). The derivation of (4.2) is similar.

From Theorem 4.1, we can deduce immediately that the equations

$$(\lambda I+H^{\eta,\alpha})\phi=\psi;\ (\lambda I+G^{\eta,\alpha})\phi=\psi, \qquad (\psi\in L^p_\mu,\ 0<\alpha<1,\ \lambda>0)$$

have unique solutions in L^p_μ under appropriate restrictions on η, μ and p. For the particular case when $\alpha=1/2$ we can proceed as follows to determine these solutions more explicitly.

THEOREM 4.2. *Let* $\lambda>0$ *and* $\zeta\in L^p_\mu$.

(a) *If* $\mathrm{Re}(\eta+\mu)+1>1/p$, *then*

$$(\lambda I+H^{\eta,1/2})^{-1}\zeta(x)$$
$$=(1/\lambda)\zeta(x)-(\lambda^3\pi)^{-1}\Gamma(3/2)\int_0^x \Psi(3/2;3/2;\lambda^{-2}\log x/t)(t/x)^{\eta+1}\zeta(t)\,dt/t \qquad (x>0), \tag{4.3}$$

where Ψ *is as defined in* [**2**, p. 255].

(b) *If* $\mathrm{Re}(\eta-\mu)>-1/p$, *then*

$$(\lambda I+G^{\eta,1/2})^{-1}\zeta(x)$$
$$=(1/\lambda)\zeta(x)-(\lambda^3\pi)^{-1}\Gamma(3/2)\int_x^\infty \Psi(3/2;3/2;\lambda^{-2}\log t/x)(x/t)^{\eta}\zeta(t)\,dt/t \qquad (x>0). \tag{4.4}$$

Proof. (a). Let T be the operator defined on L^p_μ by

$$T\zeta=\int_0^\infty u^{-2}g_{\lambda,1/2}(u)H^{\eta+1/u,1}\zeta\,du.$$

By proceeding as in the proof of Theorem 3.2, we can verify that, for each $x>0$,

$$\begin{aligned}(T\zeta)(x) &= \int_0^\infty u^{-2} g_{\lambda,1/2}(u) H^{\eta+1/u,1}\zeta(x)\,du \\ &= \int_0^\infty u^{-2} g_{\lambda,1/2}(u) \int_0^x (t/x)^{\eta+1/u+1}\zeta(t)\,dt/t\,du \\ &= \int_0^x (t/x)^{\eta+1}\zeta(t)\,dt/t \int_0^\infty (t/x)^{1/u} u^{-2} g_{\lambda,1/2}(u)\,du \qquad \text{(by Fubini's theorem)}\end{aligned}$$

provided $\zeta \in C_0^\infty(0,\infty)$. The inner integral can be evaluated using [**2**, p. 255(2)] and we can state that, for $\zeta \in C_0^\infty(0,\infty)$,

$$(T\zeta)(x) = (\lambda^3\pi)^{-1}\Gamma(3/2)\int_0^x (t/x)^{\eta+1}\Psi(3/2;3/2;\lambda^{-2}\log x/t)\zeta(t)\,dt/t. \tag{4.5}$$

If we now apply the extended version of Hardy's inequality [**6**] in conjunction with the asymptotic expansion for Ψ given in [**2**, p. 278(1)], we can deduce that the operator defined by the right-hand side of (4.5) is in $L(L_\mu^p)$ under the stated conditions on η, μ and p. Consequently, from (4.1), the continuity of the operators and the denseness of $C_0^\infty(0,\infty)$ in L_μ^p, it follows that $(\lambda I + H^{\eta,1/2})^{-1}\zeta(x)$ is given by (4.3) for any $\zeta \in L_\mu^p$. This completes the proof of (a). The proof of (b) is similar.

COROLLARY 4.3. *Under the conditions stated in Theorem* 4.2, *we can write*

$$\begin{aligned}(\lambda I + H^{\eta,1/2})^{-1}\zeta(x) &= (1/\lambda)\zeta(x) - (\lambda^2\Gamma(1/2))^{-1}\int_0^x (\log x/t)^{-1/2}(t/x)^{\eta+1}\zeta(t)\,dt/t \\ &\quad + (1/\lambda)^3\int_0^x (t/x)^{\eta+1-1/\lambda^2}\operatorname{erfc}[\lambda^{-1}(\log x/t)^{1/2}]\zeta(t)\,dt/t;\end{aligned} \tag{4.6}$$

$$\begin{aligned}(\lambda I + G^{\eta,1/2})^{-1}\zeta(x) &= (1/\lambda)\zeta(x) - (\lambda^2\Gamma(1/2))^{-1}\int_x^\infty (\log t/x)^{-1/2}(x/t)^{\eta}\zeta(t)\,dt/t \\ &\quad + (1/\lambda)^3\int_x^\infty (x/t)^{\eta-1/\lambda^2}\operatorname{erfc}[\lambda^{-1}(\log t/x)^{1/2}]\zeta(t)\,dt/t.\end{aligned} \tag{4.7}$$

Proof. We note first that the function $\Psi(3/2;3/2;x)$ can be written as $e^x\Gamma(-1/2;x)$ [**2**, p. 266(21)] which in turn can be expressed as $-2e^x(\Gamma(1/2;x)-x^{-1/2}e^{-x})$.

Since $\Gamma(1/2;x)=\sqrt{\pi}\operatorname{erfc}(\sqrt{x})$ [**2**, p. 266(24)], substitution into (4.3) and (4.4) gives the stated formulae.

Finally, if we set $\eta=0$ and $\lambda=1$, then, from Theorem 4.2 and Corollary 4.3, we can state that (1.1) has a unique solution in L_μ^p, given by (1.2), whenever $\psi \in L_\mu^p$ and $\operatorname{Re}\mu < 1/p$. This agrees with the result obtained by Rooney.

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