

SOME RESULTS ON DEFINING SETS OF t -DESIGNS

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We investigate how varying the parameters of t - (v, k, λ) designs affects the sizes of smallest defining sets. In particular, we consider the effect of varying each of the parameters t , v and λ . We establish a number of new bounds for the sizes of smallest defining sets and find the size of smallest defining sets for an infinite family of designs. We also show how one of our results can be applied to the problem of finding critical sets of Latin squares.

1. INTRODUCTION

Let V be a v -set, and suppose that \mathcal{B} is a collection of k -subsets of V with the property that each t -subset of V is in exactly λ of the elements of \mathcal{B} . Then the ordered pair (V, \mathcal{B}) is called a t - (v, k, λ) design. The elements of V are called *points*, and the elements of \mathcal{B} *blocks*. We often abbreviate t - (v, k, λ) design to t -design, or simply design. A design with $\lambda = 1$ is called a *Steiner* design.

To avoid trivialities, we assume throughout that $0 < t < k < v - t$. Although a block is formally defined as a set of elements, it is often simpler to avoid explicitly using set notation. For instance, the block $\{1, 3, a, b\}$ may instead be written as $\{13ab\}$ or $13ab$. Further, although designs may be multisets, we use set notation and terminology throughout.

The concept of a defining set of a t -design was introduced by K. Gray in the series of papers [6, 7, 8]; see also the survey papers by Street [23, 24].

DEFINITION 1.1: A set of blocks S which is a subset of a unique t - (v, k, λ) design D is a *defining set* of D . The *size* of S equals $|S|$ and S is said to be *smallest* if no other defining set of D has smaller size. A defining set is *minimal* if it does not properly contain a defining set.

EXAMPLE 1.2. Let $S = \{124, 235, 346\}$. Then it is easily verified that S is a smallest defining set of the 2 - $(7, 3, 1)$ design $D = (V, \mathcal{B})$ with $V = \{1, 2, 3, 4, 5, 6, 7\}$ and $\mathcal{B} = \{124, 235, 346, 457, 561, 672, 713\}$. The design D is known as a *Fano plane*.

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A number of authors have discussed the sizes of smallest defining sets of families of designs. Let $\{D_d\}_{d \in \Lambda}$ denote a family of designs with some indexing set Λ . Let μ_d denote the fraction of blocks in a smallest defining set of D_d . B. Gray, Hamilton and O’Keefe [5] have shown that if $D_d = PG(2, d)$, then $\mu_d \leq 1/2 + \varepsilon_d$, where $\varepsilon_d \rightarrow 0$ as $d \rightarrow \infty$. In contrast, B. Gray [4] has shown that if D_d is the symmetric design obtained from the points and hyperplanes of $PG(d, 2)$, then $\mu_d \rightarrow 1$ as $d \rightarrow \infty$. If d is an odd prime power and D_d is the Hadamard design cyclically generated from the quadratic residues of $GF(d)$, then Sarvate and Seberry [22] conjecture that $\mu_d \leq 1/2$.

In all of these families of designs the block size k varies. In this paper we investigate how varying each of the parameters t , λ and v , whilst k remains fixed, affects the sizes of smallest defining sets.

The background material we require will be reviewed in the next section. In the remainder of this section we summarise our results and the layout of the remaining sections.

The number of blocks, $|\mathcal{B}|$, in a design is denoted by b , and each point appears in exactly r blocks. Standard counting arguments yield the following result.

LEMMA 1.3. *For a t -(v, k, λ) design D , the following relations hold:*

$$rv = bk \quad \text{and} \quad \lambda \binom{v}{t} = b \binom{k}{t}.$$

Further, D is an s -(v, k, λ_s) design for $0 \leq s \leq t$, and the values of λ_s are given by

$$\lambda_s = \lambda \binom{v-s}{t-s} / \binom{k-s}{t-s}.$$

Note that $\lambda_0 = b$, $\lambda_1 = r$ and $\lambda_t = \lambda$.

We show in Section 3 that if s_t and s_s are the sizes of a smallest defining set of a design D considered as a t -design and as an s -design respectively, then $s_s \geq s_t + 2^{t-s} - 1$.

In Section 4, we present some new bounds for the size of a smallest defining set of a t -(v, k, λ) design, when $k = t + 1$ or $v = k + t + 1$. Using these results, we completely solve for the sizes of smallest defining sets of the infinite family of 2-(6, 3, λ) designs. We also investigate the infinite families of 2-(7, 3, λ) and 3-(8, 4, λ) designs.

DEFINITION 1.4: Let $D = (V, \mathcal{B})$ and suppose $S \subseteq \mathcal{B}$. If $S \subseteq \mathcal{C}$, where $E = (V, \mathcal{C})$ is a design with the same parameters as D , implies that:

1. E is isomorphic to D , then S is said to be a *class defining set* of D ;
2. $E = D$ or E is not isomorphic to D , then S is said to be a *member defining set* of D .

Note that S is a defining set of D if and only if S is both a class and a member defining set of D .

Let $\{D_v\}$ be a family of Steiner t -($v, k, 1$) designs, where k and t are fixed. In Section 5 we show that, as $v \rightarrow \infty$, the fraction of blocks contained in a smallest member

defining set of D_v approaches zero. As a corollary to this, the sizes of smallest class defining sets and smallest defining sets are asymptotically equal. We discuss the implications of an analogue of this result for critical sets in Latin squares.

2. BACKGROUND

Let S_v denote the symmetric group of permutations on a v -set. Two designs, $D_1 = (V, \mathcal{B}_1)$ and $D_2 = (V, \mathcal{B}_2)$, are said to be *isomorphic* if there exists $\rho \in S_v$ such that $\rho\mathcal{B}_1 = \mathcal{B}_2$. If no such ρ exists, then D_1 and D_2 are *non-isomorphic*. If $\mathcal{B}_1 \neq \mathcal{B}_2$, then D_1 and D_2 are said to be *distinct*. If all distinct t - (v, k, λ) designs are isomorphic, we say that the design is *unique*.

Given a design D , if $\rho \in S_v$ is such that $\rho\mathcal{B} = \mathcal{B}$, then ρ is called an *automorphism* of D . The set of all automorphisms of D is a subgroup of S_v , and is denoted by $\text{aut}(D)$. If D is a t - (v, k, λ) design, then the class of designs isomorphic to D has size $|S_v|/|\text{aut}(D)| = v!/|\text{aut}(D)|$.

DEFINITION 2.1: A t - (v, k, λ) design whose blocks can be partitioned into sets in such a way that each set is a t - (v, k, μ_i) design, where $\sum_i \mu_i = \lambda$ and $0 < \mu_i < \lambda$, is called *decomposable*. If no such partition exists, the design is *indecomposable*.

For given t, v and k the number of indecomposable designs is finite (Engel [3]). This number is not known, in general. However, when all the indecomposable designs are known, all t - (v, k, λ) designs can be constructed (Grüttmüller [11]).

EXAMPLE 2.2. Let $V = \{1, 2, 3, 4, 5, 6, 7\}$, $D = (V, \mathcal{B})$ and $I = (V, \mathcal{I})$, where

$$\begin{aligned} \mathcal{B} &= \{124, 235, 346, 457, 561, 672, 713\} \cup \{124, 135, 346, 457, 562, 671, 723\} \\ &\quad \cup \{125, 134, 356, 457, 462, 671, 723\}, \\ \mathcal{I} &= \{123, 125, 127, 135, 136, 145, 146, 147, 167, 234, 236, 246, 247, 256, 257, \\ &\quad 345, 347, 357, 367, 456, 567\}. \end{aligned}$$

Then D is a 2- $(7, 3, 3)$ design whose blocks can be partitioned into Fano planes; thus D is decomposable. However, the 2- $(7, 3, 3)$ design I is indecomposable.

LEMMA 2.3. (K. Gray [6]) Let $D = D_1 \cup D_2$ be a t - $(v, k, \mu_1 + \mu_2)$ design, where D_1 and D_2 are t - (v, k, μ_1) and t - (v, k, μ_2) designs respectively. Let s, s_1 and s_2 denote the sizes of smallest defining sets of D, D_1 and D_2 respectively. Then $s \geq s_1 + s_2$.

NOTATION 2.4. If A is a collection of blocks, then

$$A^x = \{B \setminus \{x\} : x \in B \text{ and } B \in A\} \quad \text{and} \quad \overline{A^x} = \{B : x \notin B \text{ and } B \in A\}.$$

If A is a set of blocks and x is a new element not in any block of A , then we use the notation xA to stand for the set of blocks $\{\{x\} \cup B : B \in A\}$.

We say that x has *multiplicity* r_x in A if x is contained in r_x blocks of A ; that is, $r_x = |A^x|$. Suppose that $D = (V, \mathcal{B})$ is a t - (v, k, λ) design, and let x be any point in V .

The restriction of D on x is the $(t-1)-(v-1, k-1, \lambda)$ design $(V \setminus \{x\}, \mathcal{B}^x)$. Suppose that x is a new point not in V . Then it may be possible to extend D to a $(t+1)-(v+1, k+1, \lambda)$ design $(V \cup \{x\}, x\mathcal{B} \cup A)$, called an extension of D , for some set of blocks A . If A is the complement of $x\mathcal{B}$ (with respect to $V \cup \{x\}$), then the process of extension is called *extension by complementation*. The set of blocks of an extended design formed by complementation is necessarily self-complementary.

Let V be a v -set and T_1, T_2 be collections of m k -subsets of V . We say that T_1 and T_2 are t -balanced if each t -subset of V is contained in the same number of blocks of T_1 and of T_2 . If T_1 and T_2 are disjoint and t -balanced, then $T = T_1 - T_2$ is said to be a (v, k, t) trade of volume $m(T) = m$. The set of elements of V contained in T_1 is called the *foundation*, denoted by $F(T_1)$. Note that $F(T_1) = F(T_2)$, and so we define $F(T) = F(T_1)$. We write $f(T)$ for $|F(T)|$.

In $T = T_1 - T_2$, ‘ $-$ ’ does not represent the set-difference binary operation, which is always represented by ‘ \setminus ’. We think of the blocks of T_1 as being labelled ‘ $+$ ’ and those of T_2 as being labelled ‘ $-$ ’. If $T = T_1 - T_2$ is a (v, k, t) trade, we often refer to the single collection T_1 as a trade. If $D = (V, \mathcal{B})$ is a t - (v, k, λ) design with $T_1 \subseteq \mathcal{B}$, then the design is said to *contain* the trade. If the value of v is not of interest, we often speak of a (k, t) trade instead of a (v, k, t) trade. If $T_1 = T_2 = \emptyset$, then the trade is said to be *void*; we ignore void trades in what follows.

EXAMPLE 2.5. $T = T_1 - T_2 = +135 + 146 + 236 + 245 - 136 - 145 - 235 - 246$ is a $(6, 3, 2)$ trade, with $F(T) = \{1, 2, 3, 4, 5, 6\}$, $f(T) = 6$ and $m(T) = 4$.

Trades, which are also known as null t -designs, have many uses in the theory of designs. For example, they can be used to construct t -designs with different support sizes (Hedayat [12]) and are related to the design intersection problem (Billington [1]). Of relevance to us is the relationship between trades and defining sets of designs.

THEOREM 2.6. ([6]) *Suppose $D = (V, \mathcal{B})$ and $S \subseteq \mathcal{B}$. Then S is a defining set of D if and only if S intersects each trade in D .*

PROOF: Suppose S is a defining set of D . If T_1 is a trade in D , then $S \cap T_1 \neq \emptyset$, else S is also a subset of the design with blocks $(\mathcal{B} \setminus T_1) \cup T_2$, where $T_1 - T_2$ is a trade.

Conversely, suppose $S \subseteq \mathcal{B}$ intersects each trade in D . If S is not a defining set of D , then $S \subseteq D_2$ for some design D_2 with the same parameters as, but distinct from, D . Let T_1 comprise the blocks of D not in D_2 and T_2 comprise the blocks of D_2 not in D . Then $T_1 - T_2$ is a trade, with T_1 in D . Furthermore, S is disjoint from T_1 , which is a contradiction. \square

The final results we shall need give some simple properties of trades. Note that Lemma 2.8 follows from Lemma 2.7(1).

LEMMA 2.7. ([12], Hwang [13]) *Let $T = T_1 - T_2$ be a non-void (k, t) trade. Then:*

- (1) T is a (k, s) trade for all $0 < s < t$;
- (2) $m(T) \geq 2^t$;
- (3) $f(T) \geq k + t + 1$.

LEMMA 2.8. ([13]) *Suppose that $T = T_1 - T_2$ is a (k, t) trade. Then:*

- (1) $T^x = T_1^x - T_2^x$ is a $(k - 1, t - 1)$ trade of volume r_x ;
- (2) $\overline{T^x} = \overline{T_1^x} - \overline{T_2^x}$ is a $(k, t - 1)$ trade of volume $m(T) - r_x$;
- (3) $xT^x = xT_1^x - xT_2^x$ is a $(k, t - 1)$ trade of volume r_x .

3. VARYING t

Empirically (see [23, 24]), the higher the value of t , the smaller the fraction of the blocks needed to define a design. A classical example of this fact is the result by Curtis [2] that only eight out of the 759 blocks are needed to define the unique 5-(24, 8, 1) Mattheiu design.

Little is known in general about the effect on the sizes of defining sets in a t -design of varying t . K. Gray [6] showed that the size of a smallest defining set of the unique 3-(8, 4, 1) design is three, whereas when considered as one of the four 2-(8, 4, 3) designs the size of a smallest defining set is six. Greenhill and Street [9] showed that the size of a smallest defining set of the unique 3-(10, 4, 1) design is four, whereas the size of a smallest defining set is sixteen when this design is considered as a 2-(10, 4, 4) design (of which there are more than 1.7×10^6 , see Mathon and Rosa [19]).

LEMMA 3.1. *Let D be a t -(v, k, λ_t) design. For $0 < s < t$, D is also an s -(v, k, λ_s) design where λ_s and λ_t are related as in Lemma 1.3. If S is a defining set of D considered as an s -design, then S is also a defining set of D considered as a t -design.*

PROOF: By Lemma 2.7(1) any (k, t) trade in D is also a (k, s) trade in D and the result follows from Theorem 2.6. \square

Thus, if s_t and s_s are the sizes of smallest defining sets of D considered as a t -design and as an s -design respectively, then $s_t \leq s_s$. By considering the structure of trades and of defining sets, we now show that equality is not possible.

LEMMA 3.2. *If T is a non-void (k, t) trade, and $0 < s < t$, then the blocks of T can be partitioned into at least 2^{t-s} non-void (k, s) trades.*

PROOF: Clearly there exists $x \in F(T)$ such that $r_x \neq m(T)$. By Lemma 2.8, xT^x and $\overline{T^x}$ are two non-void $(k, t - 1)$ trades which partition the blocks of T . So T can be partitioned into two $(k, t - 1)$ trades. The result now follows by induction. \square

LEMMA 3.3. *Let M be a minimal defining set of a t -design D . Then there exists a trade T^M in D such that $|M \cap T^M| = 1$.*

PROOF: As M is a defining set of D , certainly $|M \cap T| \geq 1$ for every trade T in D . Suppose that $|M \cap T| \geq 2$ for every trade T in D . Then, if B is any block in M , $|(M \setminus \{B\}) \cap T| \geq 1$ for every trade T in D and $M \setminus \{B\}$ is also a defining set of D . This is a contradiction, since M is minimal. \square

THEOREM 3.4. *Let D be as in Lemma 3.1. Let s_t and s_s be the sizes of smallest defining sets of D considered as a t -design and as an s -design respectively. Then*

$$s_s \geq s_t + 2^{t-s} - 1.$$

PROOF: Let S be a smallest defining set of D considered as an s -design. Then, by Lemma 3.1, S is also a defining set of D considered as a t -design. Hence there exists $M \subseteq S$ such that M is a minimal defining set of D considered as a t -design. We show that

$$(*) \quad |S| \geq |M| + 2^{t-s} - 1.$$

The result then follows immediately, as $|S| = s_s$ and $|M| \geq s_t$.

To prove (*), we first note that if T is a (k, t) trade in D , then $|S \cap T| \geq 2^{t-s}$ by Lemma 3.2. But by Lemma 3.3, if $M \subseteq S$ is a minimal defining set of D considered as a t -design, then $|M \cap T^M| = 1$ for some (k, t) trade T^M in D . Thus M has at least $2^{t-s} - 1$ less blocks than S . \square

As the results quoted earlier for the 3-(8, 4, 1) and 3-(10, 4, 1) designs illustrate, the bound of Theorem 3.4 is certainly not tight. To conclude this section we show that, considered as 1-designs, Steiner designs need all but one of their blocks to define them. Thus the 3-(8, 4, 1) and 3-(10, 4, 1) designs, considered as 1-designs, have smallest defining sets of 13 and 29 blocks respectively.

LEMMA 3.5. *Let $D = (V, \mathcal{B})$ be a t -($v, k, 1$) design. Then, considered as a 1-design, D has a smallest defining set size of $b - 1$.*

PROOF: Since D is Steiner and $k \geq t + 1$, then any two blocks in \mathcal{B} intersect in at most $k - 2$ points. It is easy to see that any two blocks of size k that share at most $k - 2$ points must form one half of some $(k, 1)$ trade. The result now follows from Theorem 2.6. \square

4. VARYING λ

In this section we investigate the sizes of smallest defining sets of some families of designs in which λ varies. By combining some new upper bounds on the size of smallest defining sets with information on the designs' decomposability, we solve for the sizes of smallest defining sets of the 2-(6, 3, λ) designs and obtain good bounds on the sizes for the 2-(7, 3, λ) and 3-(8, 4, λ) designs.

LEMMA 4.1. *The size s of a smallest defining set of a t -($v, t + 1, \lambda_t$) design, $D = (V, \mathcal{B})$, with b blocks satisfies $s \leq b - r$.*

PROOF: Let $e \in V$ and let S be the collection of $b - r$ blocks of D which do not contain e . If W is a t -subset not containing e which does not occur λ_t times in the blocks of S , then S forces the block(s) eW . In this way, all the r blocks of D containing e are forced and S is a defining set of D . \square

LEMMA 4.2. *Let $D = (V, \mathcal{B})$ be a t - $(k + t + 1, k, \lambda_t)$ design. For $x \in V$, let D^x be the $(t - 1)$ - $(k + t, k - 1, \lambda_t)$ design $(V \setminus \{x\}, \mathcal{B}^x)$ (so D^x is the restriction of D on x). Suppose that s and s^x are the sizes of smallest defining sets of D and D^x respectively. Then $s \leq s^x \leq r$.*

PROOF: That $s^x \leq r$ is trivial. Let S^x be a smallest defining set of D^x , and set $S = xS^x$. We show that S is a defining set of D .

Certainly S forces the r blocks of D which contain the element x . The remaining $b - r$ blocks of D cannot contain a (k, t) trade as the foundation of such a trade would be strictly less than $k + t + 1$, contradicting Lemma 2.7(3). \square

EXAMPLE 4.3. The size of a smallest defining set of a 4-(11, 5, 1) design is five. It follows from the previous lemma that the size of a smallest defining set of a 5-(12, 6, 1) design is at most five. It is, in fact, straightforward to see that this size must equal five; see [9].

Suppose $k = t + 1$ and $v = k + t + 1$, so that $r = b/2$. Lemmas 4.1 and 4.2 each imply that $s \leq r$. However, we can improve this bound by combining the approaches of the two lemmas.

LEMMA 4.4. *The size s of a smallest defining set of a t - $(2t + 2, t + 1, \lambda_t)$ design, $D = (V, \mathcal{B})$, satisfies $s \leq r - \lambda_2$.*

PROOF: Let $a, b \in V$. Partition \mathcal{B} into sets X, Y and Z where X consists of the blocks of D that contain a but not b , Y consists of the blocks that contain a and b , and Z consists of the blocks that do not contain a .

Now given X, Y is forced since any t -subset that contains a but not b and which has not occurred λ_t times in X must form a block with b in Y . Thus X forces $X \cup Y$, which is a defining set of D as in the proof of Lemma 4.2. \square

THEOREM 4.5. *Let $\lambda \geq 2$ be even, J be a 2-(6, 3, λ) design, s_λ be the size of a smallest defining set of J , and μ_λ be the proportion of blocks in a smallest defining set of J . Then $s_\lambda = 3\lambda/2$ and $\mu_\lambda = 3/10$.*

PROOF: Let D be the unique 2-(6, 3, 2) design. K. Gray [7] has shown that the size of a smallest defining set of D equals three. The results quoted in Gronau [10] imply that J is necessarily decomposable into copies of D . By noting that $b = 10\lambda/2$, $r = 5\lambda/2$ and $\lambda_2 = \lambda$, the result now follows from Lemmas 2.3 and 4.4. \square

In general, the lower bound of Lemma 2.3 is not tight, even when all the decomposable designs must be decomposed into copies of a single design. For example, there is a

single 2-(9, 3, 1) design, with smallest defining set size of 4 [7]. There are 36 2-(9, 3, 2) designs. These designs, with their properties, are given by Morgan [21] and Mathon and Rosa [18], and the sizes of smallest defining sets are given by Khodkar [15]. From these we see that nine of the 2-(9, 3, 2) designs are decomposable, necessarily into copies of the unique 2-(9, 3, 1) design. However, only six of these nine have smallest defining set sizes of 8, the other three having sizes of 9.

We now consider the 2-(7, 3, λ) designs. Grüttmüller notes in [11] that Langdev [17] has shown that there are exactly two indecomposable 2-(7, 3, λ) designs. One of these designs is the 2-(7, 3, 1) design D introduced in Example 1.2 and the other is the 2-(7, 3, 3) design I introduced in Example 2.2. The number of blocks contained in smallest defining sets of D and I are 3 and 7 respectively; see [7].

Let μ be the fraction of blocks in a smallest defining set of a 2-(7, 3, λ) design, J say. Now J can be decomposed into isomorphic copies of I and D . Lemma 2.3 yields $\mu \geq \min(3/7, 7/21) = 1/3$. By Lemma 4.1, $\mu \leq 4/7$. We now prove a better upper bound, which shows that any 2-(7, 3, λ) design can be defined by less than half its blocks.

THEOREM 4.6. *If D is a 2-(7, 3, λ) design, then a smallest defining set has at most $16\lambda/5$ blocks.*

PROOF: Let $V = \{0, \dots, 6\}$, and suppose that $D = (V, \mathcal{B})$ is a 2-(7, 3, λ) design, with $b = 7\lambda$ and $r = 3\lambda$. Let $\mathcal{B} = X \cup Y$, where X is the set of blocks that do not contain the point 0. As in the proof of Lemma 4.1, X is a defining set of D , in $b - r = 4\lambda$ blocks. We show that X contains a set of at least $4\lambda/5$ blocks which can be deleted without destroying its unique completion property.

Consider the 3λ blocks in Y . These contain a total of 3λ pairs from $V \setminus \{0\}$. Since there are $\binom{6}{2} = 15$ possible pairs, there must be a pair, say ab , that occurs at most $3\lambda/15 = \lambda/5$ times in Y . Since each pair occurs λ times in D , there must be at least $4\lambda/5$ blocks in X that contain ab . Let the set of blocks in X that contain ab be Z .

Each element of $V \setminus \{0\}$ has multiplicity λ in Y , so X is a 1-(6, 3, 2λ) design. Since the blocks of Z all contain the pair ab , Z cannot contain any (3, 1) trades. So $X \setminus Z$ is a defining set for X . Hence $X \setminus Z$ is a defining set of D , and contains at most $4\lambda - 4\lambda/5 = 16\lambda/5$ blocks. \square

COROLLARY 4.7. *For any 2-(7, 3, λ) design J , $1/3 \leq \mu \leq 16/35$. If J can be decomposed into Fano planes, then $3/7 \leq \mu \leq 16/35$.*

The 3-(8, 4, λ) designs are self-complementary, and can all be obtained by extending (uniquely) the 2-(7, 3, λ) designs by complementation, as discussed by Khosrovshahi and Vatan [16]. It is easy to see that the extension of an indecomposable 2-(7, 3, λ) design is an indecomposable 3-(8, 4, λ) design, while any restriction of an indecomposable 3-(8, 4, λ) design is an indecomposable 2-(7, 3, λ) design. So there are precisely two indecomposable 3-(8, 4, λ) designs, one for $\lambda = 1$ and one for $\lambda = 3$. Let μ be the proportion of blocks

in a smallest defining set of a $3-(8, 4, \lambda)$ design. Since extension by complementation doubles the number of blocks, the following corollary of Lemma 4.2 and Theorem 4.6 is immediate.

COROLLARY 4.8. *For any $3-(8, 4, \lambda)$ design J , $1/6 \leq \mu \leq 8/35$. If J can be decomposed into $3-(8, 4, 1)$ designs, then $3/14 \leq \mu \leq 8/35$.*

Note that if $\lambda \leq 4$, then Corollaries 4.7 and 4.8 are sufficient to prove the sizes of smallest defining sets for those designs which can be decomposed into Fano planes or into $3-(8, 4, 1)$ designs.

5. VARYING v

Let S_d be the $2-(2^{d+1}-1, 3, 1)$ design obtained from the points and lines of $PG(d, 2)$ and let μ_d be the fraction of blocks in a smallest defining set of S_d . B. Gray [4] observed that $\{\mu_d\}_{d=2}^\infty$ is a non-decreasing sequence, and so $\mu_d \rightarrow l$ as $d \rightarrow \infty$, for some limiting value l . Moran [20] has shown that $\mu_3 = 16/35$ and so $l \geq 16/35$, but the exact value of l is unknown. This is an example of the problem of determining the limit (if it exists) of the size of a smallest defining set as v increases.

Although we make no progress on this particular problem, we do solve for the limiting value of the fraction of blocks in a smallest *member* defining set for Steiner designs. A simple corollary of this result shows that, asymptotically, the problem of determining the fraction of blocks in a smallest defining set of a Steiner design is equivalent to the problem of finding the fraction of blocks in a smallest class defining set.

LEMMA 5.1. *Let $\eta_v = (\log_2 v! + 1) \binom{k}{t} / \binom{v}{t}$, where k and t are fixed integers with $k > t > 1$. Then $\eta_v \rightarrow 0$ as $v \rightarrow \infty$.*

PROOF: Since k and t are fixed, it suffices to show that

$$\lim_{v \rightarrow \infty} \frac{\log_2 v!}{\binom{v}{t}} = 0.$$

Now $\log_2 v! \leq \log_2 v^v = v \log_2 v$, and $\binom{v}{t} \sim v^t/t!$. As $t \geq 2$, the result follows. □

THEOREM 5.2. *Define m_D to be the size of a smallest member defining set of a Steiner design D . Fix $k > t > 1$, let*

$$m_v = \max\{m_D : D \text{ is a } t-(v, k, 1) \text{ design}\},$$

and let b_v be the number of blocks in a $t-(v, k, 1)$ design. Then $m_v/b_v \rightarrow 0$ as $v \rightarrow \infty$.

PROOF: Let D be any $t-(v, k, 1)$ design, and let \mathcal{D} be the set of all distinct designs isomorphic to D . We present an algorithm to construct a member defining set M of

size at most $\log_2 v! + 1$ for some $D^* \in \mathcal{D}$. Now, using Lemma 5.1 and the fact that $b_v = \binom{v}{t} / \binom{k}{t}$, we obtain

$$\frac{m_v}{b_v} \leq \frac{|M|}{b_v} \leq \frac{\log_2 v! + 1}{b_v} \rightarrow 0 \text{ as } v \rightarrow \infty.$$

It remains to construct M . Consider the algorithm given in Figure 1. First note that the particular D^* for which a member defining set is constructed is irrelevant; in particular, it need not be specified initially. As $|\text{aut}(D)| \geq 1$, then $|\mathcal{D}| \leq v!$, and so at most $\log_2 v! + 1$ blocks are sufficient to select D^* , provided that each choice of block reduces the search-space by at least a half. Now if $|\mathcal{D}| \neq 1$ at STEP 1, then \mathcal{D} must contain at least two distinct designs. Thus the algorithm reaches STEP 5 after a finite number of passes through the loop represented by STEPS 2-4. Since the chosen t -subset T is contained in at least two distinct B_i , one B_i must occur in at most half of the designs in \mathcal{D} . Finally, since each t -subset is considered at most once the algorithm is guaranteed to terminate, and, when it does, M is a member defining set for D^* . □

REMARK 5.3. Unfortunately, the method of Theorem 5.2 appears unlikely to give a bound on the size of a smallest defining set of a design D . For example, Wilson [25] showed that the number of distinct $2-(v, 3, 1)$ designs is at least $(e^{-5}v)^{v^2/6}$, and $\log_2(e^{-5}v)^{v^2/6}$ dominates $b = v(v - 1)/6$.

Now, if S is a defining set of a design D then S is both a class and a member defining set of D . So, for Steiner designs, the problem of finding the asymptotic value for the fraction of blocks in a smallest defining set reduces to the problem of finding the fraction of blocks in a smallest class defining set. Rather than give the general result, we state how the result applies to the designs S_d derived from $PG(d, 2)$.

COROLLARY 5.4. *Let c_d and μ_d be the fraction of blocks in smallest class defining sets and smallest defining sets respectively of S_d . Then*

$$\lim_{d \rightarrow \infty} c_d = \lim_{d \rightarrow \infty} \mu_d = l, \text{ for some } l \geq \frac{16}{35}.$$

The method of Theorem 5.2 can be applied to other combinatorial structures. As an example, we consider the problem of finding critical sets of Latin squares.

DEFINITION 5.5: *A uniquely completable set in a Latin square L is a partial Latin square which has a unique completion to L . A uniquely completable set all of whose proper subsets complete to at least two distinct Latin squares is called a *critical set* of L .*

A uniquely completable (respectively, critical) set of a Latin square is conceptually equivalent to a defining (respectively, minimal defining) set of a design. A number of authors have studied the sizes of critical sets in Latin squares; see, for example, the survey paper by Keedwell [14]. It is conjectured that the number of entries in a smallest uniquely completable set of a Latin square of order n is at least $n^2/4$ (respectively, $(n^2 - 1)/4$) if n is even (respectively odd).

 FIGURE 1: Algorithm for member defining sets

INPUT: integers v, k and t , with $v > k > t > 1$;
 a t - $(v, k, 1)$ design D .
 OUTPUT: a design D^* isomorphic to D ;
 a member defining set M for D^* , with $|M| \leq \log_2 v! + 1$.
 SETUP: $\mathcal{D} \leftarrow$ set of designs isomorphic to D ;
 $M \leftarrow \emptyset$;
 $V_t \leftarrow$ set of t -subsets of v -set.
 STEP 1: if $|\mathcal{D}| = 1$ then
 output the unique $D^* \in \mathcal{D}$;
 output M ;
 terminate.
 STEP 2: pick a t -subset T from V_t .
 STEP 3: for each $D_i \in \mathcal{D}$ do
 $B_i \leftarrow$ block in D_i containing T .
 STEP 4: if all the B_i are equal then
 $V_t \leftarrow V_t \setminus$ set of t -subsets in any B_i ;
 goto step 2.
 STEP 5: $B \leftarrow$ a B_i that occurs least frequently.
 STEP 6: $M \leftarrow M \cup \{B\}$;
 $\mathcal{D} \leftarrow$ set of $D_i \in \mathcal{D}$ containing B ;
 $V_t \leftarrow V_t \setminus$ set of t -subsets of B ;
 goto step 1.

An *isotopism class* of a Latin square L of order n consists of all the Latin squares obtained from L by permuting rows, columns or entries of L . So an isotopism class contains at most $(n!)^3$ members. Define in the natural way a member uniquely completable set and a class uniquely completable set of a Latin square.

THEOREM 5.6. For a Latin square L of order n , let m_L be the size of a smallest member uniquely completable set. Let

$$m_n = \max\{m_L : L \text{ is a Latin square of order } n\}.$$

Then $m_n/n^2 \rightarrow 0$ as $n \rightarrow \infty$.

PROOF: Using a similar technique to that of Theorem 5.2, one obtains the inequality $m_n \leq \log_2((n!)^3 + 1)$. It is then simple to verify that the limit is as claimed. \square

Thus, asymptotically, the problem of determining the size of a smallest uniquely completable set is equivalent to determining the size of a smallest class uniquely completable set. It is hoped that this approach may shed some light on this unsolved problem.

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