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LETTERS TO THE EDITOR

THE EXPECTED VALUE OF THE ONE-SIDED CUSUM STOPPING TIME

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Abstract

We derive an asymptotic relation for the expected value of the stopping time $N_x = \inf\{k \geqq 0 \mid S_k - \min_{0 \leqq i \leqq k} S_i \geqq x\}$, $x > 0$, where S_k is a random walk with negative drift.

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Let $X_0 = 0$ and $X_k = \max(X_{k-1} + \xi_k, 0)$, where $\xi_1, \xi_2, \cdots$ are independent and identically distributed real-valued random variables with negative expected value. Let $S_0 = 0$, $S_k = \xi_1 + \cdots + \xi_k$ for $k \geqq 1$ and $N_x = \inf\{k \geqq 0 \mid X_k \geqq x\} = \inf\{k \geqq 0 \mid S_k - \min_{0 \leqq i \leqq k} S_i \geqq x\}$. This stopping time appears in the one-sided cusum procedure and is of interest in queueing theory, since $\{X_k, k \geqq 0\}$ can be interpreted as the waiting time process for a $GI/G/1$ system. In Stadje (1987) it is proved that

$$(1) \qquad (1 - a(x))/a(x) \leqq \boldsymbol{E}(N_x) \leqq 2\boldsymbol{E}(\tau)/a(x),$$

where $\tau = \inf\{k \geqq 1 \mid S_k \leqq 0\}$ and $a(x) = \boldsymbol{P}(\max_{0 \leqq k < \tau} S_k \geqq x)$. A result of Iglehart (1972) states that if ξ_1 is non-lattice and there exists a $\gamma > 0$ for which $\boldsymbol{E}(\exp(\gamma \xi_1)) = 1$ and $\boldsymbol{E}(\xi_1 \exp(\gamma \xi_1)) < \infty$, then

$$(2) \qquad a(x) \exp(\gamma x) \to [1 - \exp(\gamma S_\tau)]^2 / \gamma \boldsymbol{E}(\xi_1 \exp(\gamma \xi_1)) \boldsymbol{E}(\tau), \quad \text{as } x \to \infty.$$

The aim of this note is to supplement (1) by the asymptotic relation

$$(3) \qquad \lim_{x \to \infty} a(x) \boldsymbol{E}(N_x) = \boldsymbol{E}(\tau).$$

Under the above conditions on the moment-generating function of ξ_1, (2) and (3) together yield

$$(4) \qquad \lim_{x \to \infty} \exp(-\gamma x) \boldsymbol{E}(N_x) = \gamma \boldsymbol{E}(\xi_1 \exp(\gamma \xi_1)) (\boldsymbol{E}(\tau)/[1 - \exp(\gamma S_\tau)])^2.$$

Proof of (3). Let $A_x = \{\max_{0 \leqq k < \tau} X_k < x\}$. Then $\boldsymbol{P}(A_x) = 1 - a(x)$ and we can write

$$(5) \qquad \boldsymbol{E}(N_x) = \boldsymbol{E}((N_x - \tau) 1_{A_x}) + \boldsymbol{E}(\tau 1_{A_x}) + \boldsymbol{E}(N_x 1_{A_x^c}).$$

By the strong Markov property we obtain

$$\begin{aligned} \boldsymbol{E}((N_x - \tau) 1_{A_x}) &= \boldsymbol{P}(A_x) \boldsymbol{E}(N_x - \tau \mid A_x) \\ &= (1 - a(x)) \boldsymbol{E}(N_x) \end{aligned}$$

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so that by (5)

$$a(x)E(N_x) = E(\tau 1_{A_x}) + E(N_x 1_{A_x^c}). \tag{6}$$

Now obviously $P(A_x) \to 1$, as $x \to \infty$, because we assume $E(\xi_1) < 0$. Thus, $E(\tau 1_{A_x}) \to E(\tau)$. Further, on A_x^c we have $N_x \leqq \tau$; hence $E(N_x 1_{A_x^c}) \leqq E(\tau 1_{A_x^c}) \to 0$, since τ is integrable (again by the assumption $E(\xi_1) < 0$). Therefore, (6) implies that $a(x)E(N_x) \to E(\tau)$, as claimed.

Note that $a(x)E(N_x) \leqq E(\tau)$, so that the factor 2 on the right-hand side of (1) is unnecessary.

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