Ergod. Th. & Dynam. Sys. (2011), **31**, 1269–1272 © *Cambridge University Press, 2010*
doi:10.1017/S0143385710000532

CORRIGENDUM

Avoiding early closing: 'Livšic theorems for non-commutative groups including diffeomorphism groups and results on the existence of conformal structures for Anosov systems' – CORRIGENDUM

RAFAEL DE LA LLAVE and ALISTAIR WINDSOR

doi:10.1017/S014338570900039X, Published by Cambridge University Press,
17 July 2009.

1. *Introduction*

This paper serves as a corrigendum to our paper [**dlLW09**]. In particular, in the proof of Theorem 6.3 we claimed the following.

CLAIM. *Let f be a topologically transitive Anosov diffeomorphism of a compact manifold M. For all $\epsilon > 0$ there exists $L \geq 1$ such that for every $n \in \mathbb{N}$ and $x \in M$ there exists a periodic point $p \in M$ satisfying:*

(1) *for all $0 \leq i \leq n$,*

$$d_M(f^i x, f^i p) < \epsilon;$$

and

(2) *p has minimal period $n + \ell$ with $0 \leq \ell \leq L$.*

Unfortunately, as B. Kalinin and V. Sadovskaya discovered, the proof sketched contained gaps. Using specification as was suggested in our paper leads to a weaker result than we claimed. In this paper we prove a uniform version of closing.

THEOREM. *Let f be a topologically transitive C^1 Anosov diffeomorphism of a compact connected manifold M. Given $\epsilon > 0$ there exists $D \geq 1$ and $N > 0$ such that for all $x \in M$ and $n \in \mathbb{N}$ there exists a periodic point $p \in M$ with minimal period $m \in \mathbb{N}$ and $d \in \mathbb{N}$ such that:*

(1) *for all $0 \leq i \leq n - 1$*

$$d_M(f^i x, f^i p) < \epsilon;$$

and

(2) *$n \leq d \cdot m \leq n + N$ and $1 \leq d \leq D$.*

This result is strong enough to complete the proof of Theorem 6.3.

2. *Results*

To prove our result for Anosov diffeomorphisms, we will first prove a similar statement for subshifts of finite type. Since every Anosov diffeomorphism is a factor of a subshift, this will allow us to establish the desired result.

Recall that a subshift of finite type can be described by a transition matrix A. Symbol j may follow symbol i in a word in Σ_A if $A_{i,j} = 1$. A finite sequence $(a_1, \ldots, a_n)$ is said to be admissible if $A_{a_i,a_{i+1}} = 1$ for $0 \le i \le n-1$. We say that a finite sequence $(a_1, \ldots, a_n)$ is periodic if it is admissible and $A_{a_n,a_1} = 1$ so that the sequence can be extended periodically to a point $a \in \Sigma_A$ of period n.

The following result is similar to one of Fine and Wilf in [**FW65**].

LEMMA 1. *Let (Σ_A, σ) be a subshift of finite type. Let $(a_1, \ldots, a_n)$ be a periodic sequence of period m_1. Let $(a_1, \ldots, a_n, \ldots, a_{n+L})$ be an extension of $(a_1, \ldots, a_n)$ that is periodic with period m_2.*

If $m_1 + m_2 \le n$, then $(a_1, \ldots, a_n)$ and $(a_1, \ldots, a_{n+L})$ are both periodic of period $\gcd(m_1, m_2)$.

Proof. Write $d := \gcd(m_1, m_2) = k_1 m_1 + k_2 m_2$ with $k_1, k_2 \in \mathbb{Z}$. Consider the following variation on the proof of Bézout's theorem that uses only numbers in the range $1, \ldots, n$. If $k_1 > 0$, then define $k_+ = k_1$, $m_+ = m_1$, $k_- = -k_2$ and $m_- = m_2$. If $k_2 > 0$, then define $k_+ = k_2$, $m_+ = m_2$, $k_- = -k_1$ and $m_- = m_1$.

Let $1 \le i \le n - d$ be arbitrary and initialize $k = i$.

(1) Add m_+ to k successively until either of the following holds:
  - (a) adding a further m_+ would give k above n; or
  - (b) all k_+ of the m_+ have been used.

(2) Subtract m_- from the new k successively until either of the following holds:
  - (a) subtracting a further m_- would give k below 1; or
  - (b) all k_- of the m_- have been used.

(3) If $k \neq i + d$, then return to step (1).

Notice that if $k + m_+ \ge n + 1$ and $k - m_- \le 0$, then $m_1 + m_2 \ge n + 1$, which is a contradiction. Thus the above procedure cannot terminate at an intermediate stage, and the algorithm must proceed to give $k = i + d$.

Since each of these steps involves one of the two periods and all of the numbers are among $1, \ldots, n$, this shows that $a_i = a_{i+d}$ for $1 \le i \le n - d$, i.e. the original sequence $(x_1, \ldots, x_n)$ is d-periodic. Since d divides m_2 and $m_2 < n$, we can conclude that the extended sequence $(a_1, \ldots, a_{n+L})$ is also d-periodic. □

Remark. The hypothesis that $m_1 + m_2 \le n$ is necessary. If we take the sequence

$$(0, 1, 0, 1, 0, 0, 1, 0, 1, 0)$$

of length 10 with period 5 and extend it by $(1, 0, 0, 1)$, we obtain the sequence

$$(0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1)$$

of length 14 that has period 7. Obviously, the original sequence is not constant even though 5 and 7 are relatively prime.

https://doi.org/10.1017/S0143385710000532 Published online by Cambridge University Press

THEOREM 1. *Let (Σ_A, σ) be a mixing subshift of finite type. Let L be such that $A^L > 2$. Let $n \geq L$. Let $(a_1, \ldots, a_n)$ be a periodic sequence. Then one of the following holds:*

(1) *$(a_1, \ldots, a_n)$ has minimal period n or $n/2$; or*

(2) *there exists an extension $(a_1, \ldots, a_{n+L})$ of $(a_1, \ldots, a_n)$ such that $(a_1, \ldots, a_{n+L})$ is periodic with minimal period $n + L$ or $(n + L)/2$.*

Proof. If $(a_1, \ldots, a_n)$ has minimal period m_1 being either n or $n/2$, then we are done; therefore we may suppose that $m_1 \leq n/3$. Let $(a_1, \ldots, a_{n+L})$ be an arbitrary periodic extension of $(a_1, \ldots, a_n)$. If $(a_1, \ldots, a_{n+L})$ has minimal period m_2 being either $n + L$ or $(n + L)/2$, then we are done; therefore we may suppose that $m_2 \leq (n + L)/3$.

In this case, since $L \leq n$ we must have $m_1 + m_2 \leq n$. Hence, by Lemma 1, we can show that the extended sequence $(a_1, \ldots, a_{n+L})$ has period m_1. There is a unique extension of $(a_1, \ldots, a_n)$ that makes $(a_1, \ldots, a_{n+L})$ have period m_1, but there are at least two ways of completing $(a_1, \ldots, a_{n+L})$. Using this other completion, we get that $(a_1, \ldots, a_{n+L})$ does not have period m_1. If we denote its minimal period by m_2, we see that we must have $m_1 + m_2 > n$. This means that m_2 must be at least $(n + L)/2$. □

Now we can state our main theorem.

THEOREM 2. *Let f be a topologically transitive C^1 Anosov diffeomorphism of a compact connected manifold M. Given $\epsilon > 0$, there exist $D \geq 1$ and $N > 0$ such that for all $x \in M$ and $n \in \mathbb{N}$ there exist a periodic point $p \in M$ with minimal period $m \in \mathbb{N}$ and $d \in \mathbb{N}$ such that:*

(1) *for all $0 \leq i \leq n - 1$,*

$$d_M(f^i x, f^i p) < \epsilon;$$

and

(2) *$n \leq d \cdot m \leq n + N$ and $1 \leq d \leq D$.*

Proof. Let $\epsilon > 0$ be arbitrary. There exists a Markov partition $\mathfrak{M}$ of M by 'rectangles' of diameter less than ϵ (see [**Bow70b**]). Let (Σ_A, σ) be the associated subshift of finite type with transition matrix A and alphabet $\mathscr{A}$. Every transitive Anosov diffeomorphism of a connected manifold is topologically mixing, so there exists $L \in \mathbb{N}$ such that A^L is a positive matrix. Immediately, we have $A^{2L} \geq 2$. By [**Bow70a**, Proposition 10], there exists $k \in \mathbb{N}$ such that the canonical projection $\pi : \Sigma_A \to M$ satisfies $\#\pi^{-1}(x) \leq k$ for all $x \in M$.

Consider one of the possible lifts of the point $x \in M$, $(\ldots, x_0, x_1, \ldots, x_{n-1}, \ldots)$. Consider the finite sequence $(x_0, \ldots, x_{n-1})$. We can extend this by $2L$ states to get a new finite sequence $(y_0, \ldots, y_{n-1+2L})$ that is periodic; we choose to extend by $2L$ rather than simply L so that $2L < n + 2L$. Now we can apply our symbolic extension lemma to obtain either a point q of period $n + 2L$ with minimal period at least $(n + 2L)/2$ or a point q of period $n + 4L$ with minimal period $(n + 4L)/2$. Let m be the minimal period of the point q. The orbit of the periodic point q consists of m distinct points. Projecting the

https://doi.org/10.1017/S0143385710000532 Published online by Cambridge University Press

orbit under π gives at least m/k distinct points. Hence the minimal period of the projected point $p = \pi(q)$ is at least m/k.

Taking $N = 4L$ and $D = 2k$, we see that we obtain a periodic point p of period $n \le n' \le n + N$ with minimal period at least n'/D.

Since we extended the original $(x_0, \ldots, x_{n-1})$, we have that p and x belong to the same rectangle for the first n iterations of f. This means that

$$d_M(f^i x, f^i p) < \epsilon$$

for $0 \le i \le n - 1$. □

3. *Completing the proof of Theorem 6.3*

Theorem 6.3 states that if the distortion of f along every periodic orbit is bounded, then the distortion of any iterate of f is uniformly bounded. The idea was that any orbit segment is close to a segment of a periodic orbit whose period is not very different from the length of the orbit segment. This led to the inequalities (54) in [**dlLW09**]:

$$\begin{aligned} K_{g,E^s}(f^n, p) &\le K_{g,E^s}(f^{n+\ell}, p) K_{g,E^s}(f^{-\ell}, p) \\ &\le C_{\text{per}} K_{g,E^s}(f^{-\ell}). \end{aligned} \tag{54}$$

Here we used $K_{g,E^s}(f^{n+\ell}, p) \le C_{\text{per}}$, since we were supposing that $n + \ell$ was the minimal period of the periodic point p. We are unable to show that this is the case; however, using our previous lemma, we can find a periodic point p with minimal period m such that for some $d \in \mathbb{N}$ with $1 \le d \le D$ we have $m \cdot d = n + \ell$ for $0 \le \ell \le N$. Now we have

$$\begin{aligned} K_{g,E^s}(f^{n+\ell}, p) &= K_{g,E^s}(f^{m\cdot d}, p) \\ &= K_{g,E^s}(f^m, p)^d \\ &\le K_{g,E^s}(f^m, p)^D \\ &\le C_{\text{per}}^D. \end{aligned}$$

This then leads immediately to the following replacement for (54):

$$\begin{aligned} K_{g,E^s}(f^n, p) &\le K_{g,E^s}(f^{n+\ell}, p) K_{g,E^s}(f^{-\ell}, p) \\ &\le C_{\text{per}}^D K_{g,E^s}(f^{-\ell}). \end{aligned} \tag{54$'$}$$

With estimate (54′), the remainder of the proof of [**dlLW09**, Theorem 6.3] carries through as stated. The only change is in the value of the constant obtained.

Acknowledgement. R. de la Llave was supported by NSF grant DMS-0901389.

References

[Bow70a] R. Bowen. Markov partitions and minimal sets for axiom A diffeomorphisms. *Amer. J. Math.* **92** (1970), 907–918.

[Bow70b] R. Bowen. Markov partitions for axiom A diffeomorphisms. *Amer. J. Math.* **92** (1970), 725–747.

[dlLW09] R. de la Llave and A. Windsor. Livšic theorems for non-commutative groups including diffeomorphism groups and results on the existence of conformal structures for Anosov systems. *Ergod. Th. & Dynam. Sys.* **30**(4) (2009), 1055–1100. Published online by Cambridge University Press 17 July 2009, doi: 10.1017/S014338570900039X.

[FW65] N. J. Fine and H. S. Wilf. Uniqueness theorems for periodic functions. *Proc. Amer. Math. Soc.* **16** (1965), 109–114.