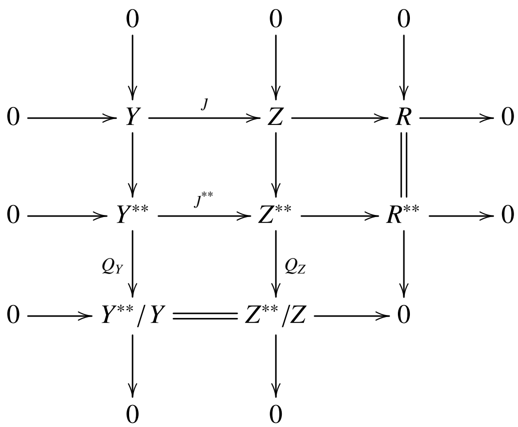
2

The Language of Homology

In this chapter we introduce the basic elements of the homological language and translate the statements about complemented and uncomplemented subspaces presented in Chapter 1 into this language. The homological language has a few advantages over the classical one:

- It allows us to present all available information about the problem in question at a glance. To give an example, assume we want to prove the following statement: *if Y is a subspace of Z such that Z/Y is reflexive and Y is complemented in Y^{**}, then Z is also complemented in Z^{**}*. Try to do it. Done? Good. Now, the homological way. All the information appears displayed in the diagram:



As for the proof (accept many terms here to be explained later), that Y is complemented in Y^{**} means that the left vertical sequence splits, so Q_Y admits a linear continuous section s. This obliges the middle vertical sequence to split since $\jmath^{**}s$ is a linear continuous section for Q_Z, thus Z

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is complemented in Z^{**}. Simple? As it should be. Why is it so simple? Answering this brings us to the main feature of the homological approach compared to the classical language:

- Diagrams encode a large amount of information in a simple way. Consequently, once the reader becomes familiar with the language, complicated things can be said in simple forms, usually simpler than in the classical language. Even at this early stage, an example can be given: unlike the classical language, the homological language treats subspaces and quotients symmetrically. For instance, saying 'each subspace of X is complemented' in classical terms with quotients requires some thought, while in the homological language, 'every exact sequence $0 \longrightarrow \cdot \longrightarrow X \longrightarrow \cdot \longrightarrow 0$ whose middle term is X splits' cannot be simpler or clearer.

Thus, the general strategy for tackling a problem the homological way is:

- **Draw diagrams** Formulating the problem using the homological language means stating the problem as the possibility of constructing a more or less complex diagram. Keep in mind the unspoken rule that diagrams must start and end in 0. No loose ends allowed. See Note 2.15.2 for details.
- **Simplify diagrams** Find a way to simplify diagrams. Simplifying means different things: usually, making the diagram 'split' into elementary diagrams. Techniques of homological algebra allow us to understand diagrams, show us how to manipulate them and determine when they split.
- **Interpret diagrams** Find out the meaning of the simplified diagrams inside Banach space theory.

Let us show off this strategy in action:

Claim Given a subspace Y of a Banach space Z, there is a natural isomorphism $(Z^{**}/Y^{**})/(Z/Y) \simeq (Z^{**}/Z)/(Y^{**}/Y)$.

Of course, there is a classical way to do that (please, be our guest!). To use homological language, one begins by observing that the first line in the data 'given a subspace Y of a Banach space Z' is pick the exact sequence $0 \longrightarrow Y \longrightarrow Z \longrightarrow Z/Y \longrightarrow 0$. Since the question involves biduals, observe also that biduals form an exact sequence $0 \longrightarrow Y^{**} \longrightarrow Z^{**} \longrightarrow (Z/Y)^{**} \longrightarrow 0$. This, and the very meaning of 'exact sequence', already yields $(Z/Y)^{**} \simeq Z^{**}/Y^{**}$. To get the isomorphism we are looking for, form a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & Z/Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y^{**} & \longrightarrow & Z^{**} & \longrightarrow & (Z/Y)^{**} & \longrightarrow & 0 \end{array}$$

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where the descending arrows are the natural inclusion maps, and complete the diagram so that no loose ends remain. One way, then, is to use brute force to check that the sequence of quotients is also exact. The other is to appeal to a far more general result known as the Snake lemma (Note 2.15.2). Whichever way, one arrives at the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & Z/Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y^{**} & \longrightarrow & Z^{**} & \longrightarrow & (Z/Y)^{**} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y^{**}/Y & \longrightarrow & Z^{**}/Z & \longrightarrow & \dfrac{(Z/Y)^{**}}{(Z/Y)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{2.1}$$

That this is a correct diagram we can be certain: it begins and ends with 0, as it should. Now, this diagram – more precisely, its right bottom corner – immediately provides the isomorphism claimed.

2.1 Exact Sequences of Quasi-Banach Spaces

The study of exact sequences of (quasi-) Banach spaces is the main theme of the book. The idea from now on is to consider the structure formed by a subspace and the correspondent quotient considered as a whole, and the same applies to a quotient operator and its kernel. Observe the asymmetry of these assertions: we spoke of the quotient by a subspace and the kernel of a quotient operator. To make everything symmetric, we should treat subspaces and quotients on equal terms by focusing on the operators rather than on the spaces themselves:

Definition 2.1.1 A short exact sequence of quasi-Banach spaces is a diagram composed by quasi-Banach spaces and operators

$$0 \longrightarrow Y \xrightarrow{\jmath} Z \xrightarrow{\rho} X \longrightarrow 0 \qquad (\mathsf{z})$$

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in which the kernel of each arrow coincides with the image of the preceding one. The middle space Z is usually called a *twisted sum* of Y and X.

The most natural (and, to some extent, the only) example of a short exact sequence is to start with an embedding $\jmath\colon Y \longrightarrow Z$ and form the diagram

$$0 \longrightarrow Y \xrightarrow{\ \jmath\ } Z \xrightarrow{\text{quotient}} Z/\jmath[Y] \longrightarrow 0$$

or to start with a quotient operator $\rho\colon Z \longrightarrow X$ and form the diagram

$$0 \longrightarrow \ker\rho \xrightarrow{\text{inclusion}} Z \xrightarrow{\ \rho\ } X \longrightarrow 0$$

In general, given a short exact sequence (z), exactness at Y means that $\jmath$ is injective; exactness at X means that ρ is onto and, by the open mapping theorem, open – i.e. a quotient map. The exactness at Z implies that $\jmath$ has closed range since $\jmath[Y] = \ker\rho$, and the open mapping theorem yields that $\jmath$ is an isomorphic embedding. Thus, any short exact sequence like (z) can be placed in a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\ \jmath\ } & Z & \xrightarrow{\ \rho\ } & X & \longrightarrow & 0\\
 & & \downarrow & & \| & & \uparrow & & \\
0 & \longrightarrow & \jmath[Y] & \longrightarrow & Z & \longrightarrow & Z/\jmath[Y] & \longrightarrow & 0
\end{array} \tag{2.2}$$

in which the vertical arrows are isomorphisms and the lower row is a ‘natural’ exact sequence. Summing up, the meaning of an exact sequence (z) is Y is (isomorphic to) a closed subspace of Z in such a way that the corresponding quotient is (isomorphic to) X. The sequence (z) is said to be *isometrically exact* if the embedding is an isometry and ρ is an isometric quotient in the sense that it maps the open unit ball of Z onto that of X.

Equality Notions for Short Exact Sequences

When must two short exact sequences be considered ‘equal’? Although this is a matter of perspective, it is plain that any reasonable notion of equality for exact sequences must involve the operators appearing in the sequence and not only the spaces.

The following definition is the classical one in homology, in which the subspace and quotient space are fixed:

Definition 2.1.2 Two exact sequences $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ and $0 \longrightarrow Y \longrightarrow Z' \longrightarrow X \longrightarrow 0$ are said to be equivalent if there exists an operator β making the following diagram commutative:

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$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\beta} & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & Z' & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

The following elementary lemma implies that the operator β must be an isomorphism, which somehow matches our expectations that equivalence of exact sequences is a true equivalence relation:

2.1.3 The 3-lemma *Given a commutative diagram of vector spaces and linear maps with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X' & \longrightarrow & 0 \end{array}$$

if α and γ are injective / surjective / bijective, then so is β.

The proof is just chasing diagrams since no topology is involved. And, speaking about trivial matters, the simplest exact sequence in sight is the direct product exact sequence $0 \longrightarrow Y \xrightarrow{\imath_1} Y \times X \xrightarrow{\pi_2} X \longrightarrow 0$, in which $\imath_1(y) = (y, 0)$ and $\pi_2(y, x) = x$.

Definition 2.1.4 An exact sequence is said to be trivial when it is equivalent to the direct product sequence.

We write $\mathtt{z} \sim \mathtt{z}'$ to mean that the sequences $\mathtt{z}$ and $\mathtt{z}'$ are equivalent, and we denote by $[\mathtt{z}]$ the *class* of all sequences that are equivalent to $\mathtt{z}$. The *set* of equivalence classes of exact sequences $0 \longrightarrow Y \longrightarrow \cdot \longrightarrow X \longrightarrow 0$, also called *extensions of X by Y*, will be denoted $\operatorname{Ext}(X, Y)$. The study of the assignment $X, Y \rightsquigarrow \operatorname{Ext}(X, Y)$ is the central topic of Chapter 4. There we will show, among other things, that $\operatorname{Ext}(\cdot, \cdot)$ is a functor and that $\operatorname{Ext}(X, Y)$ admits a natural vector space structure whose zero is the class of trivial sequences. Thus, $\operatorname{Ext}(X, Y) = 0$ means that all exact sequences $0 \longrightarrow Y \longrightarrow \cdot \longrightarrow X \longrightarrow 0$ are trivial. The trivial character of a sequence can be detected by looking at either the embedding or the quotient:

Lemma 2.1.5 *Given an exact sequence $0 \longrightarrow Y \xrightarrow{\jmath} Z \xrightarrow{\rho} X \longrightarrow 0$, the following are equivalent:*

(i) *The sequence is trivial.*

(ii) *The embedding $\jmath$ admits a left inverse in $\mathfrak{L}(Z, Y)$; i.e. there exists an operator $P \colon Z \longrightarrow Y$ such that $P\jmath = \mathbf{1}_Y$.*

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(iii) *The quotient map* ρ *admits a right inverse in* $\mathfrak{L}(X, Z)$*; i.e. there exists an operator* $s\colon X \longrightarrow Z$ *such that* $\rho s = \mathbf{1}_X$.

Proof Assume (i) and simply stare at the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\ \jmath\ } & Z & \xrightarrow{\ \rho\ } & X & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \beta} & & \| & & \\ 0 & \longrightarrow & Y & \underset{\imath_1}{\overset{\pi_1}{\rightleftarrows}} & Y \times X & \underset{\pi_2}{\overset{\imath_2}{\leftrightarrows}} & X & \longrightarrow & 0 \end{array}$$

until it becomes crystal clear that $\pi_1\beta$ is a left inverse of $\jmath$, while $\beta^{-1}\imath_2$ is a section of ρ. Hence, (i) implies (ii) and (iii). If (ii) holds and P is a left inverse of $\jmath$ then the restriction of ρ to $\ker P$ is an isomorphism onto X (it is injective since $\rho(x) = 0$ implies $x \in Y$, and since $x \in \ker P$, then $x = \rho(x) = 0$; it is surjective since $\rho(x - Px) = \rho(x)$), whose inverse is clearly a section of ρ. This shows (iii). That (iii) implies (ii) is by far the simplest implication: if s is a right inverse for ρ then $\mathbf{1}_Z - s\rho$ is a left inverse of $\jmath$. We finally prove that (iii) implies (i). If s is a section of ρ and $P = \mathbf{1}_Z - s\rho$ then the map $\beta\colon Z \longrightarrow Y \times X$ given by $\beta(x) = (P(x), \rho(x))$ is an isomorphism (its inverse is $(y, z) \mapsto \jmath(y) + s(z)$), making the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\ \jmath\ } & Z & \xrightarrow{\ \rho\ } & X & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \beta} & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & Y \times X & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

□

Left inverses are called *retractions* in the language of categories and *projections* in the language of Banach spaces. Right inverses are called *sections* in the language of categories and *liftings* in the language of Banach spaces. Retractions and sections appear in pairs: f is a retraction of $g \iff g$ is a section of f. Both sections and retractions are called *splitting morphisms* in algebraic jargon: if (ii) holds, then P splits Z as the direct sum $\jmath[Y] \oplus \ker P$; if (iii) holds, then $Z = s[X] \oplus \ker\rho$. For this reason, trivial sequences are said to *split*. The splitting of a sequence with embedding $\jmath\colon Y \longrightarrow Z$ means in classical terms that $\jmath[Y]$ is a complemented subspace of Z: if $P\jmath = \mathbf{1}_Y$, then $\jmath P$ is a projection of Z onto $\jmath[Y]$; we often say that P is a projection along $\jmath$. Conversely, if R is a projection of Z onto $\jmath[Y]$, then $\jmath^{-1}R$ is left inverse of $\jmath$.

It is important to realise that when an exact sequence splits, the middle space is isomorphic to the direct product of the other two, but the converse is not

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true: pick any non-trivial sequence $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$. 'Multiplying' by $Z \times X$ on the left one gets the sequence

$$0 \longrightarrow (Z \times X) \times Y \xrightarrow{\mathbf{1}_{Z\times X}\times \jmath} (Z \times X) \times Z \xrightarrow{0\oplus\rho} X \longrightarrow 0$$

which does not split (look at the quotient side). Proceeding analogously on the right side, we get the sequence $0 \longrightarrow Z \times X \times Y \longrightarrow Z \times X \times Z \times (Z \times Y) \longrightarrow X \times (Z \times Y) \longrightarrow 0$. Now, assuming that all the three spaces are isomorphic to their squares, this last sequence has the form $0 \longrightarrow X \times Y \times Z \longrightarrow (X \times Y \times Z)^2 \longrightarrow X \times Y \times Z \longrightarrow 0$ and is non-trivial. See Proposition 2.2.5 for less artificial examples.

This is a good place to discuss the norms of sections and projections in trivial isometric sequences of (quasi-) Banach spaces. It is clear that if $s \in \mathfrak{L}(X, Z)$ is a section for the quotient map then $P = \mathbf{1}_Z - s\rho$ is a projection onto $\jmath[Y]$, and we have $\|P\| \leq 1 + \|s\|$ when Z is a Banach space and $\|P\| \leq \Delta_Z(1 + \|s\|)$ in general. And, conversely, if $P \in \mathfrak{L}(Z)$ is a projection onto $\jmath[Y]$ then $\mathbf{1}_Z - P$ vanishes on $\ker\rho$ and induces an operator $s\colon X \longrightarrow Z$, which is a section of ρ. As before, $\|s\| \leq 1 + \|P\|$ in Banach spaces, and $\|s\| \leq \Delta_Z(1 + \|P\|)$ in general. The simplest sequence of Banach spaces $0 \longrightarrow \mathbb{K} \longrightarrow Z \longrightarrow X \longrightarrow 0$ admits a norm 1 projection, while it is not true, in general, that $0 \longrightarrow Y \longrightarrow Z \longrightarrow \mathbb{K} \longrightarrow 0$ admits a norm 1 section: that happens if and only if the norm 1 functional f used as a quotient map attains its norm on B_Z. And this happens for every $f \in Z^*$ if and only if Z is reflexive, by a famous theorem of James. Of course, norm $1+\varepsilon$ sections exist for all $\varepsilon > 0$, and thus hyperplanes are always $(2+\varepsilon)$-complemented. A precise calculus of the norm of projections onto hyperplanes of ℓ_1 and c_0 has been given in [46] with surprising results: every hyperplane of c_0 admits a projection with norm strictly less than 2, and there are hyperplanes in ℓ_1 for which a projection of norm 2 does not exist; an explicit example is the kernel of the functional given by $(1/2, 2/3, \ldots, n/(n+1), \ldots) \in \ell_\infty$ [202, p. 199]. In 2.14.9 we will encounter a special situation in which 1-sections exist.

Returning to the main topic, the standard notion of isomorphism of quasi-Banach spaces translates naturally to exact sequences as follows:

Definition 2.1.6 Two exact sequences $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ and $0 \longrightarrow Y' \longrightarrow Z' \longrightarrow X' \longrightarrow 0$ are said to be isomorphic if there exist isomorphisms α, β and γ making the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X' & \longrightarrow & 0 \end{array}$$

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Diagram (2.2) says that every short exact sequence is isomorphic to a natural one. The notion of isomorphism for sequences contains most of what it is expected to have: the overall meaning that the twisted sum spaces in isomorphic sequences are isomorphic via isomorphisms that somehow safeguard the positions of subspaces and quotients. It is clear that equivalent exact sequences are isomorphic. The converse is true for trivial sequences:

Lemma 2.1.7 *An exact sequence is isomorphic to the direct product sequence if and only if it splits.*

Proof As we said before, simply stare at the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & Y' & \underset{\imath_1}{\overset{\pi_1}{\leftrightarrows}} & Y' \times X' & \underset{\pi_2}{\overset{\imath_2}{\leftrightarrows}} & X' & \longrightarrow & 0 \end{array}$$

until it becomes crystal clear that $\alpha^{-1}\pi_1\beta$ is a retraction of $\jmath$ while $\beta^{-1}\imath_2\gamma$ is a section of ρ. □

The lemma might fuel a hope (or suspicion) that isomorphic and equivalent sequences coincide (when that is possible). But they do not: for an easy example of two isomorphic non-equivalent sequences, consider a quasi-Banach space Z and a subspace Y and form the corresponding sequence $0 \longrightarrow Y \longrightarrow Z \xrightarrow{\rho} Z/Y \longrightarrow 0$ that we may call z. Pick a scalar $c \neq 0, 1$ and consider the sequence

$$0 \longrightarrow Y \xrightarrow{c} Z \xrightarrow{\rho} Z/Y \longrightarrow 0, \qquad (\mathsf{c})$$

where c denotes multiplication by c. The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\text{inclusion}} & Z & \xrightarrow{\rho} & Z/Y & \longrightarrow & 0 \quad (\mathsf{z}) \\ & & \uparrow{\scriptstyle c\mathbf{1}_Y} & & \| & & \| & & \\ 0 & \longrightarrow & Y & \xrightarrow{c} & Z & \xrightarrow{\rho} & Z/Y & \longrightarrow & 0 \quad (\mathsf{c}) \end{array}$$

shows that the two sequences are isomorphic. They are, however, equivalent if and only if z is trivial: indeed, assume that $u \in \mathfrak{L}(Z)$ makes the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\text{inclusion}} & Z & \xrightarrow{\rho} & Z/Y & \longrightarrow & 0 \quad (\mathsf{z}) \\ & & \| & & \uparrow{\scriptstyle u} & & \| & & \\ 0 & \longrightarrow & Y & \xrightarrow{c} & Z & \xrightarrow{\rho} & Z/Y & \longrightarrow & 0 \quad (\mathsf{c}) \end{array}$$

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The operator $\mathbf{1}_Z - cu$ vanishes on Y, so it induces an operator $S : Z/Y \longrightarrow Z$, defined by $S(z + Y) = z - cu(z)$. Composing with ρ, we obtain $\rho(S(z + Y)) = \rho(z - cu(z)) = \rho(z) - c\rho(z) = (1 - c)(z + Y)$, which shows that $(1 - c)^{-1}S$ is a section of ρ. When the operators (α, β, γ) appearing in the definition of isomorphic sequences are isometries, the sequences are said to be *isometric*. When $Y' = Y$ and $X' = X$, and both α and γ are scalar multiples of the identity, the exact sequences are said to be *projectively equivalent*. The preceding example shows that this notion is strictly weaker than usual equivalence. It is clear that isometric or projectively equivalent sequences are isomorphic. There is a topologised version of the 3-lemma:

Lemma 2.1.8 *Assume one has a commutative diagram of quasi-Banach spaces and operators with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\jmath'} & Z' & \xrightarrow{\rho'} & X' & \longrightarrow & 0 \end{array}$$

(a) *If α and γ have dense range then β has dense range.*
(b) *If α and γ are isomorphic embeddings then so is β.*

Proof We may assume that Z' carries a p-norm for some $0 < p < 1$, that Y' is a subspace of Z' and that $X' = Z'/Y'$ carries the quotient quasinorm. (a) Pick $z' \in Z'$ and $\varepsilon > 0$. Set $x' = \rho'(z')$ and take $x \in X$ such that $\|x' - \gamma(x)\| < \varepsilon$. Now choose $z \in Z$ such that $x = \rho(z)$. As $\|\rho'(\beta(z) - z')\| < \varepsilon$, there is $y' \in Y'$ such that $\|y' + \beta(z) - z'\| < \varepsilon$, and since α has dense range, we may assume that $y' = \alpha(y)$ for some $y \in Y$. Clearly, $\|\beta(\jmath(y) + z) - z'\| < \varepsilon$. To prove (b), it suffices to check that if (z_n) is a sequence in Z and $\beta(z_n) \longrightarrow 0$ in Z', then $z_n \longrightarrow 0$ in Z. We have $\rho'(\beta(z_n)) = \gamma(\rho(z_n)) \longrightarrow 0$, hence $\rho(z_n) \longrightarrow 0$ in X, and we can write $z_n = \jmath(y_n) + \tilde{z}_n$, where $y_n \in Y$ and $\tilde{z}_n$ converges to zero in Z. Since $\beta(z_n) = \beta(\jmath(y_n) + \tilde{z}_n) = \jmath'(\alpha(y_n)) + \beta(\tilde{z}_n)$ and $\beta(\tilde{z}_n) \longrightarrow 0$, $\jmath'(\alpha(y_n)) \longrightarrow 0$, and thus y_n and z_n converge to zero. □

2.2 Basic Examples of Exact Sequences

In general, an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ cannot split when B enjoys a certain property preserved by isomorphisms (say, the DPP or 'being an $\mathscr{L}_1$-space') that passes to complemented subspaces while either A or C does not have that property. A variation of the same argument is that if B has some hereditary property and C contains a subspace failing to have it, then the sequence cannot split.

Folklore on Exact Sequences Involving $C(K)$-Spaces

Let S and T be compact spaces. Every continuous mapping $\varphi\colon S \longrightarrow T$ induces an operator (well, a homomorphism of Banach algebras) $\varphi^\circ\colon C(T) \longrightarrow C(S)$. Clearly, $\|\varphi^\circ\| = 1$. There are two cases in which φ° can appear in a short exact sequence:

- φ is injective (hence a homeomorphism onto its range) $\iff$ φ° is surjective $\iff$ φ° is an isometric quotient.

All this is Tiezte's extension theorem. Assuming that S is a closed subset of T, we can interpret φ° as plain restriction, call it r and obtain the sequence

$$0 \longrightarrow \ker r \longrightarrow C(T) \xrightarrow{\ r\ } C(S) \longrightarrow 0 \tag{2.3}$$

in which the nature of the subspace cannot be clearer: it is the ideal of those functions on T vanishing on S, and thus it is naturally isometric to $C_0(T\backslash S)$. When $S = T'$ is the subset of accumulation points of T, we have $C_0(T\backslash S) = c_0(I)$, where I is the (discrete) set of isolated points of T. Bounded linear sections of r are called extension operators. They always exist if S is metrisable, and equivalently, if $C(S)$ is separable: this is the content of the Borsuk–Dugundji theorem, which will be generalised in Theorem 2.14.5.

- φ is surjective $\iff$ φ° is injective $\iff$ φ° is an isometry.

This is obvious and yields the exact sequence

$$0 \longrightarrow C(T) \xrightarrow{\ \varphi^\circ\ } C(S) \longrightarrow \cdot \longrightarrow 0 \tag{2.4}$$

A projection along φ° is called an averaging operator: the reason is that $C(T)$ sits in $C(S)$ as the subspace of those functions that remain constant on the fibers of φ – the sets $\varphi^{-1}(t)$ for $t \in T$. Thus, if P is a projection of $C(S)$ onto $\varphi^\circ[C(T)]$, then $P(f)$ must be a kind of average of f. No general criterion is known for the splitting of (2.4). The nature of the quotient space is also unclear, and many natural questions about it remain unanswered: perhaps the most glaring one is whether it is isomorphic to a $\mathscr{C}$-space, or even to a Lindenstrauss space. Even so, one can easily compute norms in the quotient space thanks to [377, Corollary 9.10]:

Lemma 2.2.1 *Let $\varphi\colon S \longrightarrow T$ be a surjection between compact spaces. Then the norm in $C(S)/\varphi^\circ[C(T)]$ is given by (real case)*

$$\left\|f + \varphi^\circ[C(T)]\right\| = \sup_{\varphi(s)=\varphi(s')} \frac{|f(s) - f(s')|}{2}.$$

A generalised form for this estimate [42, Proposition 1.18] will be useful later. Recall that the oscillation of a function $f\colon K \longrightarrow \mathbb{R}$ at a point $s \in K$ is defined by $\operatorname{osc} f(s) = \inf_V \sup_{r,t\in V} (f(r) - f(t))$, where V runs over the neighbourhoods of s in K. The oscillation of f on K is $\operatorname{osc}_K(f) = \sup_{s\in K} \operatorname{osc} f(s)$.

Lemma 2.2.2 *Let K be a compact space and $f\colon K \longrightarrow \mathbb{R}$ a bounded function. Then*

$$\operatorname{dist}(f, C(K)) = \|f + C(K)\| = \tfrac{1}{2}\operatorname{osc}_K(f).$$

Proof Only one of the inequalities needs a proof. Define

$$f_{\rm lsc}(s) = \sup_V \inf_{t\in V} f(t) = \max\big(f(s), \liminf_{t\to s} f(t)\big),$$
$$f^{\rm usc}(s) = \inf_V \sup_{t\in V} f(t) = \min\big(f(s), \limsup_{t\to s} f(t)\big),$$

where V runs over the neighbourhoods of s. Clearly, $f^{\rm usc}$ is upper semicontinuous, $f_{\rm lsc}$ is lower semicontinuous and $f^{\rm usc} \le f \le f_{\rm lsc}$. If $\delta = \frac{1}{2}\operatorname{osc}_K f$, it is clear that $f_{\rm lsc} - \delta \le f^{\rm usc} + \delta$. The Hahn–Tong separation theorem [430, 6.4.4. theorem] gives us a continuous function h satisfying $f_{\rm lsc} - \delta \le h \le f^{\rm usc} + \delta$. Hence,

$$f - \delta \le f_{\rm lsc} - \delta \le h \le f^{\rm usc} + \delta \le f + \delta \quad \Longrightarrow \quad \|f - h\| \le \delta. \qquad \square$$

It is important to realise that many Banach algebras are $\mathscr{C}$-spaces in disguise: a commutative, unital Banach algebra A is isometrically isomorphic to the algebra of all continuous functions on some compact space if and only if it is a C^*-algebra (complex case) or for every f, $g \in A$ we have $2\|fg\| \le \|f^2 + g^2\|$ (real case). The complex case is the celebrated Gelfand–Naimark theorem; its real companion is due to Albiac and Kalton [5, Theorem 4.2.1]. In both cases we recover the underlying compact space as the set of unital homomorphisms $A \longrightarrow \mathbb{K}$ with the relative weak* topology. The result applies, for instance, to the spaces $L_\infty(\mu)$ and $\ell_\infty(I)$ and their unital (and self-adjoint in the complex case) subalgebras, to the ultraproducts of families of $\mathscr{C}$-spaces and to many others that we will meet along the way. This implies that if $u\colon A \longrightarrow B$ is a unital homomorphism and A and B satisfy the corresponding condition, then there are compact spaces S and T isometric isomorphisms $\alpha\colon A \longrightarrow C(T)$,

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$\beta\colon B \longrightarrow C(S)$ and continuous mapping $\varphi\colon S \longrightarrow T$ forming a commutative diagram which one should keep in mind for subsequent examples:

$$\begin{array}{ccc} A & \xrightarrow{\quad u \quad} & B \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} \\ C(T) & \xrightarrow{\quad \varphi^\circ \quad} & C(S) \end{array}$$

The Foiaş–Singer Sequence and Its Variations

This construction appears in [176, Theorem 6]. Let $\Delta = \{0, 1\}^{\mathbb{N}}$ be the Cantor set equipped with the product topology and the lexicographic order. We denote by Δ_0 the countable and dense subset of those $t \in \Delta$ having finitely many 1s. Let $D = D(\Delta; \Delta_0)$ be the space of all functions $\Delta \longrightarrow \mathbb{R}$ that are continuous at every $t \notin \Delta_0$ and left continuous with right limits at every $t \in \Delta_0$. It is *really* easy to prove that the sup norm makes $D(\Delta)$ into a Banach space containing $C(\Delta)$ and that the quotient $D/C(\Delta)$ is isometric to c_0. Indeed, if $J\colon D \longrightarrow \ell_\infty(\Delta_0)$ denotes the 'jump' function $J(f)(q) = \frac{1}{2}(f(q^+) - f(q))$, where $f(q^+)$ is the right limit of f at q, then J maps D onto $c_0(\Delta_0)$ and $\operatorname{dist}(f, C(\Delta)) = \|Jf\|_\infty$ for every $f \in D$, as an obvious application of Lemma 2.2.2. Thus one has an exact sequence

$$0 \longrightarrow C(\Delta) \xrightarrow{\ \imath\ } D \xrightarrow{\ J\ } c_0 \longrightarrow 0 \tag{2.5}$$

Besides, the space D is a unital subalgebra of $\ell_\infty(\Delta)$, and since $\imath$ is a homomorphism, it follows that D is a $\mathscr{C}$-space and that the Foiaş–Singer sequence (2.5) has the form (2.4).

Lemma 2.2.3 *Let (f_q) be any sequence in D such that $J(f_q) = e_q$ for every $q \in \Delta_0$. Then, given $\lambda^1, \ldots, \lambda^n \in \mathbb{R}; q^1, \ldots, q^n \in \Delta_0$ and $\varepsilon > 0$, there exist $q \in \Delta_0 \backslash \{q^1, \ldots, q^n\}$ and $\lambda = \pm 1$ such that*

$$\left\| \lambda f_q + \sum_{k=1}^{n} \lambda^k f_{q^k} \right\|_D \geq 1 + \left\| \sum_{k=1}^{n} \lambda^k f_{q^k} \right\|_D - \varepsilon.$$

Proof With no serious loss of generality we may assume that there is q in $\Delta_0 \backslash \{q^1, \ldots, q^n\}$ such that

$$\sum_{k=1}^{n} \lambda^k f_{q^k}(q) > \left\| \sum_{k=1}^{n} \lambda^k f_{q^k} \right\|_D - \varepsilon.$$

But $\sum_{k=1}^n \lambda^k f_{q^k}$ is continuous at q, and so $\sum_{k=1}^n \lambda^k f_{q^k}(q^+) = \sum_{k=1}^n \lambda^k f_{q^k}(q)$. Now, since $Jf_q = e_q$, if $f_q(q) \geq -1$, then $f_q(q^+) \geq 1$, hence

$$\left\| f_q + \sum_{k=1}^n \lambda^k f_{q^k} \right\|_D \geq \left(f_q + \sum_{k=1}^n \lambda^k f_{q^k} \right)(q^+) > 1 + \left\| \sum_{k=1}^n \lambda^k f_{q^k} \right\|_D - \varepsilon,$$

as required. And if $f_q(q) < -1$, then

$$\left\| -f_q + \sum_{k=1}^n \lambda^k f_{q^k} \right\|_D \geq \left(-f_q(q) + \sum_{k=1}^n \lambda^k f_{q^k}(q) \right) > 1 + \left\| \sum_{k=1}^n \lambda^k f_{q^k} \right\|_D - \varepsilon.$$

This is enough to make the sequence (2.5) non-trivial. □

Other arguments can be given [1, Remark (ii); 42, Example 1.20, p. 24; or 73, Lemma 2.2]. A variation of this construction, working now in $[0, 1]$, was presented by Aharoni and Lindenstrauss in [1]. Fix a countable subset $N \subset [0, 1]$ and form the space $D([0, 1]; N)$ of all real, bounded functions on $[0, 1]$ which are continuous except at points of N, where they are left-continuous and have right limits. Again, $C[0, 1]$ is a closed subspace of $D([0, 1]; N)$ and $D([0, 1]; N)/C[0, 1]$ is isometric to $c_0(N)$ via $Jf = \frac{1}{2}(f(q_n^+) - f(q_n))$. All this gives an exact sequence $0 \longrightarrow C[0, 1] \longrightarrow D([0, 1]; N) \longrightarrow c_0 \longrightarrow 0$ whose splitting depends on the location of N inside $[0, 1]$. Precisely:

Lemma 2.2.4 *If N is dense in* $[0, 1]$ *then no lifting of* (e_n) *is weakly Cauchy.*

Proof The assumption $J(f_n) = e_n$ means that $f_n(q_n^+) - f_n(q_n) = 2$ for all n. Let us assume (f_n) is weakly Cauchy and hence bounded. We first note that if I is any non-empty open interval in $(0, 1)$, $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}$, there exists $n > m$ and a non-empty open interval J with $\overline{J} \subset I$ such that for some β with $|\beta - \alpha| \geq 1$, we have $|f_n(t) - \beta| \leq \frac{1}{4}$ for $t \in J$. Indeed, we just pick $n > m$ such that $q_m \in I$ and then let β be either $f_n(q_n)$ or $f_n(q_n^+)$. The interval J can then be chosen using the left- or right-hand limit condition. Now we can use this inductively to create a subsequence (f_{n_k}) of (f_n), a sequence of non-empty intervals (I_k) with $\overline{I}_{k+1} \subset I_k$ and a sequence of reals (α_k) with $|\alpha_{k+1} - \alpha_k| \geq 1$ such that $|f_{n_k}(t) - \alpha_k| \leq \frac{1}{4}$ for $t \in I_k$. If we pick $t_0 \in \bigcap_{k=1}^{\infty} I_k$ (which is non-empty by compactness), it is clear that $|f_{n_k}(t_0) - f_{n_{k+1}}(t_0)| \geq 1/2$ for all k, and this yields a contradiction. □

We now identify the isometry type of the spaces $D(\Delta; \Delta_0)$ and $D([0, 1]; N)$.

Proposition 2.2.5 *$D(\Delta; \Delta_0)$ is isometric to $C(\Delta)$ and so is $D([0, 1]; N)$ when N is a countable dense subset of* $[0, 1]$.

Proof We give the proof for $D([0, 1]; N)$. The case of $D(\Delta; \Delta_0)$ is easier. We know that the Banach algebra $A = D([0, 1]; N)$ is isometrically isomorphic to

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a $C(K)$ for some compact space K. Three obvious properties of A force this K to be homeomorphic to Δ.

- K is totally disconnected $\iff$ A is generated by its idempotents, which is clear since if $s, t \in N$, $s < t$, then $1_{(s,t]} \in A$.
- K does not have isolated points $\iff$ A does not have minimal idempotents, which is clear since each idempotent in A is the sum of two idempotents.
- K is metrisable $\iff$ A is separable, which is obvious. □

The Foiaş–Singer and Aharoni–Lindenstrauss constructions produce uncomplemented copies of $C(\Delta)$ and of $C[0,1]$ inside $C(\Delta)$ with quotient c_0. A variation of Amir and Lindenstrauss [10] considers uncountable sets of jumps, such as the set $\mathbb{I} \subset [0,1]$ of irrationals, thus obtaining a non-trivial sequence $0 \longrightarrow C[0,1] \longrightarrow D([0,1]; \mathbb{I}) \longrightarrow c_0(\mathbb{I}) \longrightarrow 0$.

We now consider countable compacta. To produce a 'countable' version of the Foiaş–Singer sequence, we need the following 'reordering':

Lemma 2.2.6 *Any countable compact space K can be embedded in $\mathbb{R}$ in such a way that for each $n \in \mathbb{N}$ and each x in the nth derived set $K^{(n)}$ and each $\varepsilon > 0$, both the left neighbourhood $(x - \varepsilon, x)$ and the right neighbourhood $(x, x + \varepsilon)$ contain points of $K^{(n-1)}$.*

Proof By 1.6.1, we can think of K as a countable ordinal. The proof proceeds easily by transfinite induction: the case $\alpha = 1$ is clear, as well as the inductive step for $\alpha + 1$ assuming it is true for α. We thus prove the inductive step for $\alpha = \lim \alpha_n$. The induction hypothesis is that each interval $(\alpha_n, \alpha_{n+1}]$ admits an embedding into $\mathbb{R}$ with the required properties, so embed it into $(-1)^n(\frac{1}{n+1}, \frac{1}{n})$ and send α to the origin. □

Let K be a countable compact which we assume is embedded in the line as in the lemma. The simple plan now is to define $D(K; K')$ as the space of all functions $K \longrightarrow \mathbb{R}$ that are left continuous and possess right limits at every point of K'. The space $D(K; K')/C(K)$ is isometric to $c_0(K')$; via the jump function $J(f) = \frac{1}{2}(f(t^+) - f(t))_{t \in K'}$. As before, we have an exact sequence

$$0 \longrightarrow C(K) \xrightarrow{\text{inclusion}} D(K; K') \xrightarrow{J} c_0(K') \longrightarrow 0 \tag{2.6}$$

We denote the unit basis in $c_0(K')$ by $(e_t)_{t \in K'}$.

Lemma 2.2.7 *Let $\{f_x \colon x \in K'\}$ be any collection of functions in $D(K; K')$ for which $Jf_x = e_x$. Fix $\delta > 0$ and $n \in \mathbb{N}$ and choose a point $x_n \in K^{(n)}$. Then there exist $t \in K$ and distinct points $x_i \in K^{(i)}$ for $1 \le i < n$ and signs $\varepsilon_i = \pm 1$ for $1 \le i \le n$ such that $\varepsilon_i f_{x_i}(t) > 1 - \delta$ for $1 \le i \le n$.*

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Proof We will write f_j instead of f_{x_j}. We are told that $f_n(x_n^+) - f_n(x_n) = 2$, which obviously implies that either $f_n(x_n^+) \geq 1$ or $f_n(x_n) \leq -1$. In the former case, put $\varepsilon_n = 1$, in the latter, $\varepsilon_n = -1$. In either case, x_n has a one-sided neighbourhood $O_n \subset K$ on which $\varepsilon_n f_n > 1 - \delta$. Thanks to our embedding of K into $\mathbb{R}$, we may choose a point $x_{n-1} \in K^{(n)} \cap O_n$. The same argument then gives some $\varepsilon_{n-1} = \pm 1$ and a one-sided neighbourhood O_{n-1} of x_{n-1} contained in O_n on which $\varepsilon_{n-1} f_{n-1} > 1 - \delta$. Repeat the process until exhaustion. □

Corollary 2.2.8 *If K is a countable compact such that $K^{(n)} \neq \emptyset$, then any linear section of J in* (2.6) *has norm at least n.*

Proof If $L \colon c_0(K') \longrightarrow D(K; K')$ is a section of J, then the preceding lemma applies to the family $f_x = L(e_x)$. Taking $x_n \in K^{(n)}$ and $s = \sum_{1 \leq i \leq n} \varepsilon_i e_{x_i}$, it is clear that $\|s\| = 1$, while the norm of $L(s) = \sum_{1 \leq i \leq n} \varepsilon_i f_{x_i}$ is at least $(1 - \delta) n$. □

The reader is invited to ponder the choices $K = \omega^N$ for $1 \leq N < \omega$.

Exact Sequences Involving $c_0(I)$

We present three essentially different non-trivial exact sequences of the type $0 \longrightarrow c_0(I) \longrightarrow \cdot \longrightarrow c_0(J) \longrightarrow 0$:

- The Nakamura–Kakutani sequences 2.2.10, mutated into the Johnson–Lindenstrauss sequences of Diagram (2.38) in Section 2.12.
- The Ciesielski–Pol sequence of 2.2.11.
- The Bell–Marciszewski construction in Proposition 2.2.15.

The first two types provide non-WCG (hence non-trivial) twisted sums of two $c_0(I)$ spaces, which is already weird; the last one provides non-trivial WCG twisted sums of two $c_0(I)$ spaces, which is weirder still.

Definition 2.2.9 A family $\mathcal{M}$ of infinite subsets of $\mathbb{N}$ is called almost disjoint if the intersection of any two elements of $\mathcal{M}$ is finite.

The existence of such families of cardinal $\mathfrak{c}$ was first observed by Sierpiński [433]. A good example could be enumerating the nodes of the dyadic tree: $\mathcal{M}$ will be the set of branches such that each α will be the set of naturals assigned to the nodes in the branch α. Or else, pick an enumeration of all rationals and identify each irrational α with the set of natural numbers corresponding to a sequence of rationals converging to it. Nakamura and Kakutani [369] observe that the images of the characteristic functions of an almost disjoint family generate an isometric copy of $c_0(\mathcal{M})$ in ℓ_∞ / c_0.

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2.2.10 The Nakamura–Kakutani sequences *An uncountable almost disjoint family $\mathcal{M}$ generates a non-trivial exact sequence*

$$0 \longrightarrow c_0 \longrightarrow C_0(\wedge_{\mathcal{M}}) \xrightarrow{\rho} c_0(\mathcal{M}) \longrightarrow 0$$

where $C_0(\wedge_{\mathcal{M}}) = \overline{[\{1_n : n \in \mathbb{N}\} \cup \{1_\alpha : \alpha \in \mathcal{M}\}]} \subset \ell_\infty$ *is not WCG.*

The sequence cannot split because the points of ℓ_∞, and so those of its subspace $C_0(\wedge_{\mathcal{M}})$, can be separated by a countable family of functionals; since those of $c_0(\mathcal{M})$ cannot, the conclusion is that no injective linear map from $c_0(\mathcal{M})$ to ℓ_∞ exists, let alone a bounded linear section of the quotient ρ. This also shows that $C_0(\wedge_{\mathcal{M}})$ cannot be WGC; see Proposition 1.7.7. To understand the nature of the space $C_0(\wedge_{\mathcal{M}})$ and justify the seemingly eccentric notation, just observe that $C_0(\wedge_{\mathcal{M}})$ is a subring of ℓ_∞ but does not have a unit. It follows that it can be represented as the ring of all continuous functions vanishing at infinity on some locally compact space $\wedge_{\mathcal{M}}$, which we now describe. The space $\wedge_{\mathcal{M}}$ has two classes of points: those corresponding to points of $\mathbb{N}$, which we declare isolated, and those corresponding to the elements of $\mathcal{M}$; a typical neighbourhood of A must contain A together with almost all the 'elements' of A. This space is locally compact, but not compact. The one point compactification of $\wedge_{\mathcal{M}}$ will be denoted by $\triangle_{\mathcal{M}}$. It is a fairly run-of-the-mill scattered compactum of height 3. The space $C(\triangle_{\mathcal{M}})$ can be viewed as the unitisation of $C_0(\wedge_{\mathcal{M}})$ in ℓ_∞ and can be placed in the obvious exact sequence $0 \longrightarrow c_0 \longrightarrow C(\triangle_{\mathcal{M}}) \longrightarrow c(\mathcal{M}) \longrightarrow 0$, which is of type (2.3). When $\mathcal{M}$ is the family of branches of the dyadic tree, in which case $|\mathcal{M}| = \mathfrak{c}$, the space $C(\triangle_{\mathcal{M}})$ could well be called the Johnson–Lindenstrauss space for reasons that will become clear in Section 2.12. Analogous constructions can be carried out for larger cardinals using the fact that given an infinite set I, there exists a family $\mathcal{M}$ of infinite subsets of I such that $|A \cap B| < |I|$ for each $A, B \in \mathcal{M}$ and with $|\mathcal{M}| > |I|$; see [314].

The Ciesielski–Pol Space

Ciesielski and Pol obtained the thereafter so-called *Ciesielski–Pol* compacta, namely height 3 compact spaces K such that (a) both $K \setminus K'$ and K' are uncountable and (b) for every $\alpha \in K' \setminus \{\infty\}$ there is an infinite countable set $C_\alpha \subset K \setminus K'$ such that (b.1) $C_\alpha \cup \{\alpha\}$ is clopen, and (b.2) every uncountable subset of $K \setminus K'$ contains some C_α. If CP is a Ciesielski–Pol compact, the space $C(\mathsf{CP})$ is far from being WCG, as 2.2.11 shows. The intricate construction of Ciesielski–Pol compacta together with the proof of 2.2.11 probably cannot be better described than in [148, VI, Example 8.8], so we do not even try:

2.2.11 Ciesielski–Pol sequence *Let* CP *be a Ciesielski–Pol compact. There is a non-trivial exact sequence* $0 \longrightarrow c_0(\mathfrak{c}) \longrightarrow C(\mathsf{CP}) \longrightarrow c_0(\mathfrak{c}) \longrightarrow 0$ *such that no injective operator* $C(\mathsf{CP}) \longrightarrow c_0(I)$ *exists for any set* I.

A WCG Non-trivial Twisted Sum of $c_0(\Gamma)$

The discussion leading to Definition 1.7.11 suggests Yost's question: must every copy of $c_0(I)$ in a WCG space be complemented? The previous examples also lead to, does a non-trivial WCG twisted sum of two spaces $c_0(I)$ exist? The answer to this second question is yes, and thus the answer to the first question is no. We will deal with the two preceding questions together, adding a cardinal delicacy: that $C(\aleph_\omega)$ is not K-Sobczyk for no $K > 0$, which moreover shows that Proposition 1.7.14 is optimal.

Given a set S, the closed subset $\sigma_n(2^S) = \{a \in 2^S : |a| \leq n\}$ of 2^S is an Eberlein compact and of height $n+1$ when S is infinite. It is an Eberlein compact because $\sigma_n(2^S)$ is a weakly compact subset of $c_0(S)$ under the identification $a \leftrightarrow 1_a$, and it is of finite height because the set of its isolated points is $\{a \in 2^S : |a| = n\}$, thus its derived set is $\sigma_n(2^S)' = \sigma_{n-1}(2^S)$. Obviously the compact $\sigma_n(2^S)$ is scattered. A cornerstone result here is from Godefroy, Kalton and Lancien [190, Theorem 4.8, plus comment on p. 800]:

2.2.12 *For a compact space K of weight strictly lesser than $\aleph_\omega$, we have $C(K) \simeq c_0(I)$ if and only if K is an Eberlein compact of finite height.*

It therefore follows that $C(\sigma_n(2^S)) \simeq c_0(I)$ for some I. One can rummage around to see what else is in this pocket: the natural exact sequence $0 \longrightarrow c_0(I) \longrightarrow C(\sigma_n(2^S)) \longrightarrow C(\sigma_{n-1}(2^S)) \longrightarrow 0$ splits since Granero [199] showed that every copy of $c_0(I)$ inside of $c_0(J)$ is complemented. Marciszewski [353, Proposition 3.1] provides a nice improvement:

Lemma 2.2.13 *Let $K \subset \sigma_n(2^S)$ be compact and let $r\colon C(\sigma_n(2^S)) \longrightarrow C(K)$ be the restriction operator. The following sequence splits:*

$$0 \longrightarrow \ker r \longrightarrow C(\sigma_n(2^S)) \longrightarrow C(K) \longrightarrow 0$$

Thus, if K is a compact that can be embedded in some $\sigma_n(2^S)$ for some S and some $n \in \mathbb{N}$, then $C(K) \simeq c_0(I)$ for some I. The converse also holds [353, Theorem 1.1]:

Lemma 2.2.14 *A compact K can be embedded in some $\sigma_n(2^S)$ for some S and some $n \in \mathbb{N}$ if and only if $C(K) \simeq c_0(I)$ for some I.*

Proof If $C(K) \simeq c_0(I)$ then K is an Eberlein compact that, by 2.2.12, must be of finite height, hence scattered. Some combinatorial work [353, Lemma 2.2], see also [36, Lemma 1.1] yields the existence of a family $\mathcal{F}$ of clopen subsets of K such that (i) given two points of K, there is some element of $\mathcal{F}$ containing

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exactly one of them, and (ii) there is $n \in \mathbb{N}$ such that intersection of any n elements of $\mathcal{F}$ is empty. With this family $\mathcal{F} = \{U(s) : s \in S\}$ in hand, we can form the embedding $e : K \longrightarrow \sigma_n(2^S)$ given by $e(k)(s) = 1_{U(s)}(k)$. □

Proposition 2.2.15 *There is an Eberlein compact* BM *of weight* $\aleph_\omega$ *and height* 3 *such that* $C(\mathsf{BM})$ *contains an uncomplemented copy of* $c_0(\aleph_\omega)$ *for which there is a non-trivial exact sequence* $0 \longrightarrow c_0(\aleph_\omega) \longrightarrow C(\mathsf{BM}) \longrightarrow c_0(\aleph_\omega) \longrightarrow 0$.

Proof Let $S = \bigcup_n \mathcal{P}_n(\omega_n)$. Consider the set

$$\mathcal{A}_n = \{\emptyset\} \cup \{\{b\} : b \in \mathcal{P}_n(\omega_n) \cup \{A \subset \mathcal{P}_n(\omega_n) : |A| = n+1 \ \wedge \ |\cup_{a\in A} a| = n+1\}$$

and $\mathcal{A} = \bigcup_n \mathcal{A}_n$. Identifying $a \in \mathcal{A}$ with 1_a in 2^S, let $\mathsf{BM} = \{1_a : a \in \mathcal{A}\}$. This is a compact space of weight $\aleph_\omega$ and height 3 since

$$\mathsf{BM} = \{\emptyset\} \cup \bigcup_{n=1}^{\infty} \Big(\underbrace{\{1_{\{a\}} : a \in \mathcal{P}_n(\omega_n)\}}_{\mathsf{BM}'} \cup \underbrace{\{1_A : A \subset \mathcal{P}_n(\omega_n) : |A| = |\cup A| = n+1\}}_{\text{isolated points}} \Big)$$

with $\{\emptyset\} = \mathsf{BM}''$. If we set $K_n = \{1_a : a \in \mathcal{A}_n\}$ then for each point $a \in \mathcal{P}_n(\omega_n)$, it turns out that $1_{\{a\}}$ is the unique accumulation point of $\{1_A \in K_n : a \in A\}$, and that is why BM' corresponds to $\bigcup_n \mathcal{P}_n(\omega_n)$. Each K_n is an Eberlein compact, and thus, BM is an Eberlein compact. Since the isolated points of BM form a discrete subset I and $C(\mathsf{BM}') \simeq c_0(J)$ because of its height 3, the natural exact sequence $0 \longrightarrow C_0(I) \longrightarrow C(\mathsf{BM}) \longrightarrow C(\mathsf{BM}') \longrightarrow 0$ becomes

$$0 \longrightarrow c_0(I) \longrightarrow C(\mathsf{BM}) \longrightarrow c_0(J) \longrightarrow 0$$

It remains to show that the sequence does not split or, equivalently, that $C(\mathsf{BM})$ is not isomorphic to some $c_0(I)$. It is then enough to show that BM cannot be embedded into any $\sigma_n(2^T)$. To do that, we show that K_{2^n} cannot be embedded into any $\sigma_n(2^T)$. The combinatorial core of the argument is two lemmata: the almost obvious but dismaying

Lemma 2.2.16 *Let* B *be a set and* $n \in \mathbb{N}$*. There is a family* $\{V_a : a \in \sigma_n(2^B)\}$ *of open subsets such that* $a \in V_a$ *for every* a *and the intersection of any* $2^n + 1$ *elements of the family is empty.*

Proof Pick for each a the clopen neighbourhood $V_a = \{b : a \subset b\}$ of a. □

plus a disguised form of the diagonal argument that no surjective map from a set into its power set exists, no less dismaying nevertheless:

Lemma 2.2.17 *Let* $n \in \mathbb{N} \cup \{0\}$ *and let* $\varphi : \mathcal{P}_n(\omega_n) \longrightarrow \mathrm{fin}(\omega_n)$ *be a map. There is* $f \in \mathcal{P}_{n+1}(\omega_n)$ *such that* $a \notin \varphi(f \setminus \{a\})$ *for every* $a \in f$.

Proof The proof proceeds by induction:

(0) If $n = 0$ then $\{a \in 2^\omega : |a| = 0\} = \{\emptyset\}$. Pick $k \in \omega \setminus \varphi(\emptyset)$ and set $b = \{k\}$.

(n) Assume that the lemma holds for n and let $\varphi: \mathcal{P}_{n+1}(\omega_{n+1}) \longrightarrow \mathrm{fin}(\omega_{n+1})$. Since $\left|\bigcup_{b\in\mathcal{P}_{n+1}(\omega_n)} \{\, \varphi(b) \,\}\right| \le \omega_n$, there is $\beta \in \omega_{n+1} \setminus \left(\bigcup_{b\in\mathcal{P}_{n+1}(\omega_n)} \cup\, \omega_n\right)$, and we can define an auxiliary map $\psi: \mathcal{P}_n(\omega_n) \longrightarrow \mathrm{fin}(\omega_n)$ by $\psi(b) = \varphi(b \cap \{\beta\} \cap \omega_n$. The induction hypothesis provides some $c \in \mathcal{P}_{n+1}(\omega_{n+1})$ such that $a \notin \psi(c \setminus \{a\})$ for every $a \in c$. Then $c \cup \{\beta\}$ is the desired element f that works for φ. □

Let's go for the proof of Proposition 2.2.15. If K_{2^n} embeds into some $\sigma_n(2^T)$ then the first lemma provides a family of open neighbourhoods of its points such that every $2^n + 1$ elements have empty intersection. So, it is enough to prove that every family $\{V_a : a \in K_n\}$ of open neighbourhoods in K_n has $n + 1$ elements whose intersection is non-empty. Given $b \in \mathcal{P}_n(\omega_n)$, pick the clopen neighbourhood of $1_{\{b\}}$ given by $U_b = \{1_A \in K_n : b \in A\}$. Since $1_{\{b\}}$ is the only accumulation point, every neighbourhood of $1_{\{b\}}$ contains a set of the form $W_b = U_b \setminus F_b$, where F_b is a finite subset of $U_b \setminus \{1_{\{b\}}\}$. It is then enough to show that there exist $n + 1$ elements in $\{W_b : b \in \mathcal{P}_n(\omega_n)\}$ whose intersection is non-empty.

Claim For every $1_A \in U_b\setminus\{1_{\{b\}}\}$, we have $A = \mathcal{P}_n(b\cup\{\alpha\})$ for some $\alpha \in \omega_n\setminus b$.

This is in the definition: if $A \subset \mathcal{P}_n(\omega_n)$ satisfies the conditions $|A| = n + 1$ and $|\cup A| = n + 1$ then it has the form $A = \mathcal{P}_n(B)$ for some $B \in \mathcal{P}_{n+1}(\omega_n)$ of the form $B = b \cup \{\alpha\}$. We can thus define the function $\varphi: \mathcal{P}_n(\omega_n) \longrightarrow \mathrm{fin}(\omega_n)$ given by $\varphi(b) = \{\alpha \in \omega_n : 1_{\mathcal{P}_n(b\cup\{\alpha\})} \in F_b\}$, to which the second lemma applies to yield $f \in \mathcal{P}_{n+1}(\omega_n)$ such that for every $\alpha \in f$, we have $\alpha \notin \varphi(f \setminus \{\alpha\})$. This means that if $f = \{\alpha_1, \dots, \alpha_n + 1\}$ then

$$1_{\mathcal{P}_n(f)} \in \bigcap_{j=1}^{n+1} W_{f\setminus\{\alpha_j\}}.$$

□

A less glittering example appears in [16]:

2.2.18 *Let* $\alpha = \lim \alpha_n$ *with* $\alpha_0 = \aleph_1$, $\alpha_{n+1} = 2^{\alpha_n}$. *The one-point compactification* ACGJM *of* $\cup_n \sigma_n(2^{\alpha_n})$ *is an Eberlein compact, and* $C(\mathsf{ACGJM})$ *contains an uncomplemented copy of some* $c_0(I)$

The argument is by showing that $\|P\| \ge 1 + \frac{n}{2}$ for any projection P in the natural sequence $0 \longrightarrow \ker r_n \longrightarrow C(\sigma_n(2^{\alpha_n})) \longrightarrow C(\sigma_{n-1}(2^{\alpha_n})) \longrightarrow 0$. Such projections exist by Lemma 2.2.13 or, under GCH, by Proposition 1.7.13 since, in this case, the compact $\sigma_n(2^{\alpha_n})$ has weight $\alpha_n = \aleph_n$. The space ACGJM is the one-point compactification of $\bigcup_n \sigma_n(2^{\alpha_n})$ so that the c_0-sum

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$$0 \longrightarrow c_0(\mathbb{N}, \ker r_n) \longrightarrow c_0(\mathbb{N}, C(\sigma_n(2^{\alpha_n})) \longrightarrow c_0(\mathbb{N}, C(\sigma_{n-1}(2^{\alpha_n})) \longrightarrow 0$$

becomes $0 \longrightarrow c_0(I) \longrightarrow C(\mathsf{ACGJM}) \longrightarrow C(\mathsf{ACGJM}) \longrightarrow 0$. The space $C(\mathsf{ACGJM})$ cannot be isomorphic to any $c_0(J)$.

Ultraproduct Sequences

Given an ultrafilter $\mathcal{U}$ on a set I, the ultrapower $X_\mathcal{U}$ of a space X is the quotient space in the exact sequence $0 \longrightarrow c_0^\mathcal{U}(I, X) \longrightarrow \ell_\infty(I, X) \longrightarrow X_\mathcal{U} \longrightarrow 0$ (see Section 1.4). From now on the quotient map in this ultraproduct sequence will be denoted $[\cdot]_\mathcal{U}$ – when it is necessary to make $\mathcal{U}$ explicit – and $[\cdot]$ otherwise. A standard argument, see [102], shows that given $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$, the natural sequence of ultrapowers $0 \longrightarrow Y_\mathcal{U} \longrightarrow Z_\mathcal{U} \longrightarrow X_\mathcal{U} \longrightarrow 0$ is exact. No criterion is known to decide when the ultrapower sequence splits.

The Pełczyński–Lusky Sequence

Let X be a quasi-Banach space, and let (X_n) be an increasing sequence of subspaces whose union is dense in X. Let $c(\mathbb{N}, X_n)$ denote the space of those converging sequences (x_n) such that $x_n \in X_n$ for every $n \in \mathbb{N}$, equipped with the sup quasinorm. There is an obvious exact sequence

$$0 \longrightarrow c_0(\mathbb{N}, X_n) \longrightarrow c(\mathbb{N}, X_n) \xrightarrow{\lim} X \longrightarrow 0. \tag{2.7}$$

Proposition 2.2.19 *Let X be a separable p-Banach space, and let (X_n) be an increasing sequence of finite-dimensional subspaces whose union is dense in X. The Pełczyński–Lusky sequence* (2.7) *splits if and only if X has the BAP.*

Proof The space $c(\mathbb{N}, X_n)$ has the BAP since the sequence of finite-rank operators $T_k((x_n)) = (x_1, x_2, \ldots, x_{k-1}x_k, x_k, x_k, \ldots)$ converges pointwise to the identity. Thus, if the sequence splits, X, as a complemented subspace of $c(\mathbb{N}, X_n)$, would have the BAP. Now assume that X has the λ-AP and that there is an increasing sequence of integers $n(k)$ and a sequence of finite-rank operators (T_k) with $T_0 = 0$ and such that $T_k[X] \subset X_{n(k)}$ with $\|T_k\| \leq \lambda^+$ and $T_{k+1}(x) = x$ for $x \in X_{n(k)}$. A linear continuous section for the limit map is provided by the map $s\colon X \longrightarrow c(\mathbb{N}, X_n)$ given by $s(x)(n) = T_{k-1}(x)$ for $n(k) \leq n < n(k+1)$. □

The following key application is a formal adaptation of [384, Lemma 1.2]:

Lemma 2.2.20 *Every separable quasi-Banach space with the BAP is isomorphic to a complemented subspace of a space with a* 1*-FDD.*

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Proof Assume X is a separable p-Banach space with the λ-AP. Then there is a sequence of finite-rank operators $(f_n)_{n\geq 1}$ converging pointwise to $\mathbf{1}_X$, with $\|f_n\| \leq \lambda$ for all n. Assuming $f_1 = 0$, we define $a_n = f_{n+1} - f_n$ so that $\|a_n\| \leq 2^{1/p}\lambda$ and $x = \sum_{n=1}^{\infty} a_n(x)$ for all $x \in X$; i.e. the sequence (a_n) is a finite-dimensional expansion of the identity in X. Set $Y_n = a_n[X]$, and consider the vector space

$$\Sigma(Y_n) = \left\{(y_n) \in \prod_n Y_n : \sum_{n\geq 1} y_n \text{ converges in } X\right\}$$

p-normed by $\|(y_n)\| = \sup_k \|\sum_{n\leq k} y_n\|$. The following facts are nearly trivial:

- $\Sigma(Y_n)$ has a 1-FDD.
- The sum operator $s\colon \Sigma(Y_n) \longrightarrow X$ given by $s((y_n)_n) = \sum_{n\geq 1} y_n$ is contractive.
- The operator $a\colon X \longrightarrow \Sigma(Y_n)$ defined by $a(x) = (a_n x)_n$ is a right inverse of s, with $\|a\| \leq \lambda$. □

If $(Y_n)_{n\geq 1}$ is a chain of subspaces of X with dense union and $Y_1 = 0$ then $\Sigma(Y_n)$ is isomorphic to $c(\mathbb{N}, Y_n)$ via telescoping the series. Thus, Lemma 2.2.20 is roughly equivalent to Proposition 2.2.19.

The Bourgain ℓ_1-Sequence

We now revisit Bourgain's embedding $\ell_1 \longrightarrow \ell_1$ mentioned in Proposition 1.3.1. In proving [48, Theorem 7], Bourgain shows that there is some constant $C > 0$ such that for every $\varepsilon > 0$ and every sufficiently large $n \in \mathbb{N}$, there is $N(n)$ and an n-dimensional subspace E_n of $\ell_1^{N(n)}$ which is C-isomorphic to ℓ_1^n and such that every projection $P\colon \ell_1^{N(n)} \longrightarrow E_n$ has $\|P\| \geq C^{-1}(\log\log n)^{1-\varepsilon}$. Form the sequences $0 \longrightarrow E_n \longrightarrow \ell_1^{N(n)} \longrightarrow \ell_1^{N(n)}/E_n \longrightarrow 0$ and then their adjoints $0 \longrightarrow E_n^{\perp} \longrightarrow \ell_\infty^{N(n)} \longrightarrow E_n^* \longrightarrow 0$ and observe that each E_n^* is C-isomorphic to ℓ_∞^n. Amalgamating, we obtain the exact sequence

$$0 \longrightarrow c_0(\mathbb{N}, E_n^{\perp}) \longrightarrow c_0(\mathbb{N}, \ell_\infty^{N(n)}) \longrightarrow c_0(\mathbb{N}, E_n^*) \longrightarrow 0, \quad (2.8)$$

and its adjoint is

$$0 \longrightarrow \ell_1(\mathbb{N}, E_n) \longrightarrow \ell_1(\mathbb{N}, \ell_1^{N(n)}) \longrightarrow \ell_1(\mathbb{N}, \ell_1^{N(n)}/E_n) \longrightarrow 0.$$

Neither of these splits because if $P\colon \ell_1(\mathbb{N}, \ell_1^{N(n)}) \longrightarrow \ell_1(\mathbb{N}, E_n)$ is a projection, the restriction to the nth coordinate yields a projection $P_n\colon \ell_1^{N(n)} \longrightarrow E_n$ with $\|P_n\| \leq \|P\|$. Thus, $\ell_1(\mathbb{N}, E_n)$ provides an uncomplemented copy of ℓ_1 inside ℓ_1.

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If we call $\mathcal{B} = c_0(\mathbb{N}, E_n^{\perp})$ then, since $c_0(\mathbb{N}, \ell_\infty^{N(n)}) = c_0$ and $c_0(\mathbb{N}, E_n^*) \simeq c_0$, the sequence (2.8) becomes

$$0 \longrightarrow \mathcal{B} \longrightarrow c_0 \longrightarrow c_0 \longrightarrow 0 \tag{2.9}$$

and its adjoint

$$0 \longrightarrow \ell_1 \longrightarrow \ell_1 \longrightarrow \mathcal{B}^* \longrightarrow 0. \tag{2.10}$$

The space $\mathcal{B}$ cannot be an $\mathscr{L}_\infty$-space since $\mathscr{L}_\infty$-subspaces of c_0 are complemented by 1.6.3 (b). The space $\mathcal{B}^*$ cannot therefore be an $\mathscr{L}_1$-space.

2.3 Topologically Exact Sequences

In categories where no open mapping theorem exists (say, normed spaces), the exactness of a sequence

$$0 \longrightarrow Y \xrightarrow{\ \jmath\ } Z \xrightarrow{\ \rho\ } X \longrightarrow 0 \tag{2.11}$$

no longer means that $\jmath$ embeds Y as a subspace of Z or that ρ is a quotient map: consider the (very) short sequences where either $\jmath$ or ρ is the formal identity $\ell_p^0 \longrightarrow \ell_q^0$ with $0 < p < q \leq \infty$ and draw your own conclusions. Here ℓ_r^0 is the quasinormed space of finitely supported sequences with the restriction of the quasinorm of ℓ_r. Thus, what we used to get for free must now be courteously requested. A map $f\colon A \longrightarrow B$ acting between topological spaces is said to be relatively open if, whenever $U \subset A$ is an open set, $f[U]$ is open in (the relative topology of) $f[A]$.

Definition 2.3.1 An exact sequence of topological vector spaces is topologically exact if its arrows are relatively open operators.

This notion retains the meaning that Y is (embedded by $\jmath$) a closed subspace of Z in such a way that the corresponding quotient is (isomorphic to) X. In categories where operators with closed range are relatively open, such as quasi-Banach spaces, exactness implies topological exactness. A 3-space problem has the form: given a topologically exact sequence (2.11) in which Y, X have a certain property, does Z have it? Any 3-space problem implicitly carries a category, or at least a class of spaces, where the action takes place, which until further notice will be the category of topological vector spaces and operators. The first thing to know in the 3-space business is:

2.3.2 Roelcke's lemma *Let Z be a linear space and $Y \subset Z$ a linear subspace. Comparable linear topologies on Z that induce the same topologies on Y and X/Y agree.*

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Proof The comparability assumption is necessary even if the linear topologies on a vector space form a lattice. Let $\mathcal{T} \leq \mathcal{T}'$ be linear topologies on Z that induce the same topologies on Y and X/Y. We denote by $\mathcal{O}$ and $\mathcal{O}'$ the respective filters of neighbourhoods at the origin of Z. Pick $U \in \mathcal{O}$ and then $U_1 \in \mathcal{O}$ such that $U_1 \pm U_1 \subset U$. As $U_1 \cap Y$ is a neighbourhood of zero in Y for the restriction of $\mathcal{T}'$, we can pick $V \in \mathcal{O}'$ such that $(V \pm V) \cap Y \subset U_1 \cap Y$. Let $\pi \colon Z \longrightarrow Z/Y$ be the natural quotient map, and take $W \in \mathcal{O}'$ such that $W \subset V$ and $\pi[W] \subset \pi[U_1 \cap V]$, that is, $W + Y \subset (U_1 \cap V) + Y$. Hence,

$$W \subset (U_1 + Y) \cap (V + Y) \subset (U_1 \cap V) + ((V - V) \cap Y) \subset U_1 + U_1 \subset U. \quad \square$$

The reader is invited to freely interpret Roelcke's lemma and uncover some of its many consequences. The following result gathers together some interesting 3-space properties:

Lemma 2.3.3 *The Hausdorff character, metrisability, local boundedness and completeness are* 3*-space properties.*

Proof Let $0 \longrightarrow Y \longrightarrow Z \xrightarrow{\pi} X \longrightarrow 0$ be a topologically exact sequence and assume $Y = \ker \pi$. In what follows, $\mathcal{O}_Y, \mathcal{O}_Z$ and $\mathcal{O}_X$ denote the filters of neighbourhoods of 0 in those spaces.

Hausdorff character Pick any non-zero $z \in Z$. If $\pi(z) \neq 0$ then there is $V \in \mathcal{O}_X$ such that $\pi(z) \notin V$ and every $U \in \mathcal{O}_Z$ such that $\pi[U] \subset V$ separates z from the origin in Z. If $\pi(z) = 0$ then z belongs to Y, and since Y is Hausdorff, there is $U \in \mathcal{O}_Z$ such that $z \notin U \cap Y$, which is enough.

Metrisability We use the Birkhoff–Kakutani theorem: a topological vector space is metrisable (by a translation-invariant metric) if and only if it is Hausdorff and there is a countable base of neighbourhoods of 0; see the argument leading to the corollary in [283, p. 5] for an elegant proof. Take countably many sets $U_n \in \mathcal{O}_Z$ such that $\{U_n \cap Y\}$ and $\{\pi[U_n]\}$ are bases for $\mathcal{O}_Y$ and $\mathcal{O}_X$, respectively. If $\mathcal{T}$ is the least linear topology on X containing the sets U_n, then $\mathcal{T}$ is metrisable, and the sequence $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ remains topologically exact when Z carries $\mathcal{T}$. By Roelcke's lemma, $\mathcal{T}$ must be the original topology of Z.

Completeness Let $\mathcal{F}$ be a Cauchy filter on Z. Then $\pi[\mathcal{F}]$ is a Cauchy filter and converges in X. Applying a translation, if necessary, we may and do assume that $\pi[\mathcal{F}]$ converges to zero. Put $\mathcal{G} = \{F + U \colon F \in \mathcal{F}, U \in \mathcal{O}_Z\}$. This is another Cauchy filter on Z. Moreover, for each $U \in \mathcal{O}_X$, there is $F \in \mathcal{F}$ such that $\pi[F] \subset \pi[U]$, whence $F \subset Y + U$. Consequently, $\mathcal{G} \cap Y$ is a filter, hence a Cauchy filter on Y, and converges, say, to $y \in Y$. Thus, y is adherent to $\mathcal{G}$, which implies that $\mathcal{G}$ and $\mathcal{F}$ converge to y in Z.

Local boundedness Take a balanced $U \in \mathcal{O}_Z$ such that $(U + U) \cap Y$ and $\pi[U]$ are bounded. Let us verify that U is bounded. Given a balanced $W \in \mathcal{O}_Z$, there are $m, k \in \mathbb{N}$ such that $(U + U) \cap Y \subset mW$ and $\pi[U] \subset k\pi[U \cap W]$. Hence,

$$U \subset k(U + U) \cap Y + kW \subset mkW + kW \subset mk(W + W). \qquad \square$$

Since quasi-Banach = Hausdorff + locally bounded + complete, we get:

Proposition 2.3.4 *To be a quasi-Banach space is a* 3*-space property.*

However, to be a Banach space is not a 3-space property, as shown by Ribe's counterexample in Section 3.2. A careful study of the 3-space problem for local convexity can be found in Section 3.4. Let us close this section with a result that will be needed in due course. Let $\alpha\colon A \longrightarrow B$ be an operator acting between quasinormed spaces. If $\widehat{A}$ and $\widehat{B}$ are completions that we will consider to contain the corresponding spaces, it is obvious that there exists a unique operator $\widehat{\alpha}$ making the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{\ \alpha\ } & B \\ {\scriptstyle \text{inclusion}}\downarrow & & \downarrow{\scriptstyle \text{inclusion}} \\ \widehat{A} & \xrightarrow{\ \widehat{\alpha}\ } & \widehat{B} \end{array}$$

Now, if $0 \longrightarrow Y \xrightarrow{\ \jmath\ } Z \xrightarrow{\ \rho\ } X \longrightarrow 0$ is an exact sequence of quasinormed spaces and operators, we have a commutative diagram (the vertical arrows are the corresponding inclusions)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\ \jmath\ } & Z & \xrightarrow{\ \rho\ } & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widehat{Y} & \xrightarrow{\ \widehat{\jmath}\ } & \widehat{Z} & \xrightarrow{\ \widehat{\rho}\ } & \widehat{X} & \longrightarrow & 0 \end{array}$$

2.3.5 Completion of an exact sequence *If the upper row in the preceding diagram is topologically exact then the lower row is exact.*

Proof Three things have to be proved: that $\widehat{\jmath}$ is injective, that $\ker\widehat{\rho} = \widehat{\jmath}[\widehat{Y}]$ and that $\widehat{\rho}$ is surjective. We may assume that $\widehat{Z}$ is a p-Banach space containing Z as a dense subspace; that Y and X carry the induced p-norms and that the p-norms of $\widehat{Y}$ and $\widehat{X}$ extend those of Y and X, respectively. Assume $\widehat{\jmath}(y) = 0$ for some $y \in \widehat{Y}$. Pick a sequence $y_n \longrightarrow y$, with $y_n \in Y$. Then $\jmath(y_n) \longrightarrow 0$ in X and so $y_n \longrightarrow 0$ in Y since $\jmath$ is an embedding. Let $\widehat{\rho}(z) = 0$, and pick a sequence $z_n \longrightarrow z$ with $z_n \in Z$. Then $\rho(z_n) \longrightarrow 0$ in X, and since ρ is open, we can pick a sequence (y_n) in Y for which $z_n - y_n \longrightarrow 0$ in Z. Since (z_n) is Cauchy, (y_n) is Cauchy in Y as well and thus convergent to some point of $\widehat{Y}$ which agrees with z. This shows that $\ker\widehat{\rho} \subset \widehat{\jmath}[\widehat{Y}]$ and that the reverse containment is trivial. To check that $\widehat{\rho}$ is surjective, pick $x \in \widehat{X}$ and write $x = \sum_{n=1}^{\infty} x_n$ with $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$ and $x_n \in X$. Then choose $z_n \in Z$ such that $\rho(z_n) = x_n$ and $\|z_n\| \leq C\|x_n\|$ for some C independent on n. Then $\sum_n z_n$ converges to some $z \in \widehat{Z}$ whose image under $\widehat{\rho}$ is x. □

2.4 Categorical Constructions for Absolute Beginners

Anyone reading or even just flipping through this book will surely know what categories and functors are. And for those who do not, there are much better places than this book to learn such things, say [350; 27; 402]. So, rather than annoying anyone with definitions, let us present the rather short list of categories appearing onstage, in a speaking part, in this book.

A domestic atlas of categories

Name	Objects	Arrows (Morphisms)
A	Boolean algebras	Boolean homomorphisms
B	Banach spaces	operators
$\mathbf{B}_1$	Banach spaces	contractive operators
K	compact spaces	continuous maps
$\mathbf{K}_0$	Stone compacta	continuous maps
$p\mathbf{B}$	p-Banach spaces	operators
$p\mathbf{B}_1$	p-Banach spaces	contractive operators
Q	quasi-Banach spaces	operators
$\mathbf{Q}_1$	quasi-Banach spaces	contractive operators
sQ	semi-quasi-Banach spaces	operators
$\mathbf{s}(p\mathbf{B})$	semi-p-Banach spaces	operators
S	sets	mappings
V	vector spaces	linear maps

Kernel and Cokernel

A widespread slogan in category theory is ‘its the arrows that really matter’. Accordingly, one should define everything by means of arrows. For instance, kernel. In its categorical definition, the kernel of an arrow f is an arrow k such that $fk = 0$ and with the universal property that whenever $fg = 0$, the arrow g factorises through k. Thus, the kernel of an operator $f : A \longrightarrow B$ is the inclusion of the subspace $\ker f = \{x \in A : f(x) = 0\}$ into A. Composing on the

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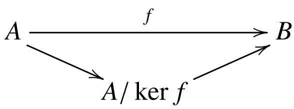
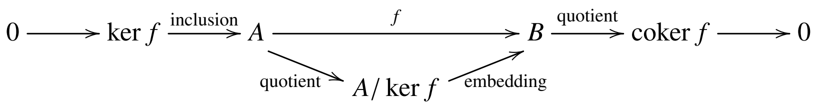
left, one obtains the categorical definition of the cokernel of f: an arrow c such that $cf = 0$ and with the universal property that whenever $gf = 0$, the arrow g factorises through c. To identify the cokernel of an operator $f: A \longrightarrow B$, just observe that if $g: B \longrightarrow C$ is an operator in $\mathbf{Q}$ such that $gf = 0$ then g vanishes on $f[A]$, hence on $\overline{f[A]}$, and therefore it factors through the natural quotient map $\pi: B \longrightarrow B/\overline{f[A]}$. Since $\pi f = 0$, it is clear that the cokernel of 'coker f' in $\mathbf{Q}$ is precisely $\pi: B \longrightarrow B/\overline{f[A]}$. We relapse into bad habits and write 'coker f' for the space $B/\overline{f[A]}$. The notion of a cokernel is most useful for relatively open operators (equivalently, with closed range); these include embeddings, whose cokernel is the corresponding quotient, and quotient maps, whose cokernel is zero. Every operator $f: A \longrightarrow B$ factors as



The left descending arrow is always a quotient map. The right ascending one is always injective and an embedding if and only if f has closed range, in which case, we can 'expand' it to a complete diagram with exact horizontal row



A non-closed range operator and its cokernel have a problematic relationship.

Product and Direct Sum

Elementary products of quasi-Banach spaces have already appeared in Chapter 1. Anyway, just to fix ideas, the product $A \times B$ of two quasi-Banach spaces A, B is the vector space product endowed with the $\|\cdot\|_\infty$ norm – or any equivalent quasinorm if one is prone to ignoring estimates. The product has the universal property that if $\pi_A: A \times B \longrightarrow A$ and $\pi_B: A \times B \longrightarrow B$ are the canonical projections then for every pair of operators $a: \diamond \longrightarrow A$ and $b: \diamond \longrightarrow B$, there exists a unique operator $c: \diamond \longrightarrow A \times B$ such that $a = \pi_A c$ and $b = \pi_B c$, and, moreover, such that $\|c\| \leq \max\{\|a\|, \|b\|\}$ if we rightly set $\|\cdot\|_\infty$. The direct sum $A \oplus B$ of two quasi-Banach spaces A, B is (again) the vector space $A \times B$ (endowed with any equivalent quasinorm if estimates are to be ignored) and enjoys the universal property that if $\imath_A: A \longrightarrow A \times B$ and $\imath_B: B \longrightarrow A \times B$ are the canonical injections then for every pair of operators $a: A \longrightarrow \diamond$ and $b: B \longrightarrow \diamond$, there exists a unique operator $c: A \times B \longrightarrow \diamond$ such that

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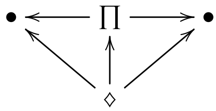
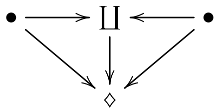
$a = c\iota_A$ and $b = c\iota_B$. If A, B are p-Banach spaces and $A \times B$ is given the p-norm $\|\cdot\|_p$, which we denote $A \oplus_p B$, then the estimate $\|c\| \leq \max\{\|a\|, \|b\|\}$ holds. However, no quasinorm yields that same estimate for arbitrary quasi-Banach spaces. There is a categorical way to say all this, including the caveat about estimates. The product of two objects in a category is an object $\prod$ of the category and two arrows $\bullet \longleftarrow \prod \longrightarrow \bullet$ with the universal property with respect to this diagram: for any other object $\diamond$ yielding a similar diagram $\bullet \longleftarrow \diamond \longrightarrow \bullet$, there is a unique arrow $\diamond \longrightarrow \prod$ making a commutative diagram



The (categorically speaking) dual notion of coproduct $\coprod$ or direct sum is defined by the diagram



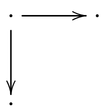
Summing up, this is how things are: given two objects A, B,

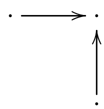
(a) the product in $\mathbf{Q}, p\mathbf{B}$ and in $\mathbf{B}_1, \mathbf{Q}_1$ and $p\mathbf{B}_1$ is $A \times B$,
(b) the direct sum in $\mathbf{Q}$ and $p\mathbf{B}$ is $A \oplus B$,
(c) the direct sum in $p\mathbf{B}_1$ is $A \oplus_p B$,
(d) no direct sum exists in $\mathbf{Q}_1$.

Moving beyond, a *universal construction* is a compressed way of speaking about a correspondence that assigns to a certain family of (quasi-) Banach spaces another (quasi-) Banach space in a canonical way (not a mere witticism). The categorical term for universal construction is *limit*, with its prefix *colimit* to isolate the corresponding construction obtained reversing arrows. Note 2.15.1 contains a We are Groot presentation of categorical limits.

2.5 Pullback and Pushout

Let us now consider the diagrams





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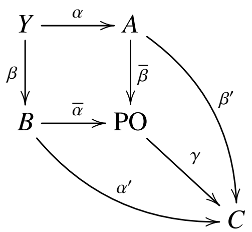
The pushout is the limit of the first (two arrows with the same domain), which means that given a diagram in the category of (quasi-) Banach spaces

$$\begin{CD} Y @>{\alpha}>> A \\ @V{\beta}VV \\ B \end{CD}$$

the pushout is a (quasi-) Banach space PO and two operators $\overline{\alpha}\colon B \longrightarrow \mathrm{PO}$ and $\overline{\beta}\colon A \longrightarrow \mathrm{PO}$ making the diagram

$$\begin{CD} Y @>{\alpha}>> A \\ @V{\beta}VV @VV{\overline{\beta}}V \\ B @>{\overline{\alpha}}>> \mathrm{PO} \end{CD} \tag{2.12}$$

commute and with the universal property that given any other (quasi-) Banach space C and operators $\beta' : A \longrightarrow C$ and $\alpha' : B \longrightarrow C$ such that $\beta'\alpha = \alpha'\beta$, there is a unique operator $\gamma\colon \mathrm{PO} \longrightarrow C$ such that $\alpha' = \gamma\overline{\alpha}$ and $\beta' = \gamma\overline{\beta}$, i.e. making the following diagram commute:



(2.13)

Pushouts exist in (quasi-) Banach spaces: if $\Delta = \{(\alpha y, -\beta y) : y \in Y\}$ then

$$\mathrm{PO} = \mathrm{PO}(\alpha,\beta) = (A \oplus B)/\overline{\Delta}.$$

The map $\overline{\alpha}$ is the composition of the inclusion of B into $A \oplus B$ and the natural quotient map $A\oplus B \longrightarrow (A\oplus B)/\overline{\Delta}$, such that $\overline{\alpha}(b) = (0, b) + \overline{\Delta}$ and, analogously, $\overline{\beta}(a) = (a, 0) + \overline{\Delta}$. All this makes a commutative diagram: $\overline{\beta}\alpha = \overline{\alpha}\beta$. Moreover, if $\beta' : A \longrightarrow C$ and $\alpha' : B \longrightarrow C$ are operators such that $\beta'\alpha = \alpha'\beta$ then there is a unique operator $\gamma\colon \mathrm{PO} \longrightarrow C$ given by $\gamma((a, b) + \overline{\Delta}) = \beta'(a) + \alpha'(b)$ such that $\alpha' = \gamma\overline{\alpha}$ and $\beta' = \gamma\overline{\beta}$. Pushouts are unique, up to isomorphisms in the ambient category. It is simple to check that

2.5.1 *If α is an embedding, then Δ is closed and $\overline{\alpha}$ is an embedding.*

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Indeed, the operator $(\alpha, -\beta)\colon Y \longrightarrow A \oplus B$ is an embedding and its range $(\alpha, -\beta)[Y] = \Delta$ is closed. The second part is trivial. Assume $A \oplus B$ carries the sum quasinorm. Then since Δ is closed, letting $c = \min\left((\|\beta\|\,\|\alpha^{-1}\|)^{-1}, 1\right)$,

$$\begin{aligned}\|\overline{\alpha}(b)\| &= \|(0,b)+\Delta\| = \inf_{y\in Y}\|(0,b)+(\alpha(y),-\beta(y))\| = \inf_{y\in Y}\|(\alpha(y), b-\beta(y))\|\\ &= \inf_{y\in Y}\|\alpha(y)\| + \|b-\beta(y)\| \geq c\inf_{y\in Y}\|\beta(y)\| + \|b-\beta(y)\| \geq \frac{c}{\Delta_B}\|b\|, \quad (2.14)\end{aligned}$$

where Δ_B is the modulus of concavity of B. If we now move to the category of p-Banach spaces and perform the pushout via the right direct sum $A \oplus_p B$ then the maps in Diagram (2.12) enjoy additional metric properties:

Lemma 2.5.2

(a) $\|\gamma\| \leq \max\left(\|\alpha'\|, \|\beta'\|\right)$.
(b) $\max\left(\|\overline{\alpha}\|, \|\overline{\beta}\|\right) \leq 1$.
(c) *If α is an isometry and $\|\beta\| \leq 1$ then $\overline{\alpha}$ is an isometry.*
(d) *If $\|\beta\| \leq 1$ and α is an isomorphism then $\overline{\alpha}$ is an isomorphism and*

$$\|(\overline{\alpha})^{-1}\| \leq \max\{1, \|\alpha^{-1}\|\}.$$

Proof (a) is a direct consequence of the p-Banach structure involved:

$$\|\gamma((a,b)+\overline{\Delta})\|^p = \|\beta' a + \alpha' b\|^p \leq \|\beta' a\|^p + \|\alpha' b\|^p \leq \max\left(\|\beta'\|, \|\alpha'\|\right)^p \|(a,b)\|_p^p.$$

(b) is clear. To prove (c), keep in mind that Δ is closed. If $\|\beta\| \leq 1$ then

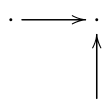
$$\|\overline{\alpha}(b)\|^p = \|(0,b)+\Delta\|^p = \inf_{y\in Y}\|\alpha y\|^p + \|b-\beta y\|^p \geq \inf_{y\in Y}\|\beta y\|^p + \|b-\beta y\|^p \geq \|b\|^p.$$

To prove (d), we first check that $\overline{\alpha}$ is onto. Pick $(a,b) \in A \oplus_p B$. Take $y \in Y$ such that $a = \alpha(y)$ and then set $b' = b + \beta(y)$. Clearly, $\overline{\alpha}(b') = (0, b') + \Delta = (a,b) + \Delta$ since $(a,b) - (0,b') = (\alpha(y), -\beta(y))$. To get a lower bound for $\|\overline{\alpha}(b)\|$, we can use the string of inequalities (2.14) and the p-subadditivity of the quasinorms to obtain

$$\|\overline{\alpha}(b)\| \geq \min\left(1, \frac{1}{\|\beta\|\|\alpha^{-1}\|}\right)\|b\| \geq \min\left(1, \frac{1}{\|\alpha^{-1}\|}\right)\|b\|,$$

which is exactly what the estimate in (d) says. □

The pullback is the colimit of the other diagram at the beginning of this section



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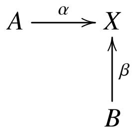
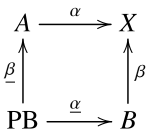
(two arrows with the same codomain), which means that the pullback of a diagram of (quasi-) Banach spaces

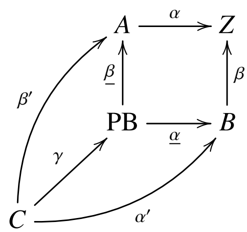


is a (quasi-) Banach space PB and operators $\underline{\alpha}$: PB $\longrightarrow B$ and $\underline{\beta}$: PB $\longrightarrow A$, making a commutative diagram



(2.15)

and with the universal property that given any other (quasi-) Banach space C and operators $\alpha' : C \longrightarrow B$ and $\beta' : C \longrightarrow A$ such that $\alpha\beta' = \alpha'\beta$, there is a unique operator $\gamma : C \longrightarrow$ PB such that $\beta' = \underline{\beta}\gamma$ and $\alpha' = \underline{\alpha}\gamma$; i.e. making a commutative diagram



Pullbacks exist in (quasi-) Banach spaces:

$$\text{PB} = \text{PB}(\alpha, \beta) = \{(a, b) \in A \oplus_\infty B : \alpha(a) = \beta(b)\},$$

with operators $\underline{\alpha}(a, b) = b$ and $\underline{\beta}(a, b) = a$. All properties are immediate, $\gamma(c) = (\beta'(c), \alpha'(c))$, and we have the additional estimate $\|\gamma\| \leq \max(\|\alpha'\|, \|\beta'\|)$ for free. It is simple to check that when α is onto, $\underline{\alpha}$ is onto. Do it.

2.6 Pushout and Exact Sequences

Suppose we are given an exact sequence $0 \longrightarrow Y \xrightarrow{\jmath} Z \xrightarrow{\rho} X \longrightarrow 0$ and an operator $\tau : Y \longrightarrow B$. Consider the pushout of the pair $(\jmath, \tau)$ and draw the corresponding arrows:

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$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \overline{\tau}} & & & & \\
 & & B & \xrightarrow{\overline{\jmath}} & \mathrm{PO} & & & &
\end{array}$$

By Lemma 2.5.2(a), $\overline{\jmath}$ is an embedding. Now, the quotient operator ρ and the null operator $0\colon B \longrightarrow X$ satisfy $\rho\jmath = 0\tau = 0$, and thus the universal property of the pushout gives a unique operator $\overline{\rho}\colon \mathrm{PO} \longrightarrow X$, making a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \overline{\tau}} & & \Vert & & \\
0 & \longrightarrow & B & \xrightarrow{\overline{\jmath}} & \mathrm{PO} & \xrightarrow{\overline{\rho}} & X & \longrightarrow & 0
\end{array} \qquad (2.16)$$

To make it explicit, $\overline{\rho}((x, b) + \Delta) = \rho(x)$. It is easy to check that the lower sequence in the preceding diagram is exact since $\ker \overline{\rho} = \{(x, b) + \Delta\colon \rho(x) = 0\} = \{(y, b) + \Delta\colon y \in Y\} = \{(0, b) + \Delta\colon b \in B\} = \overline{\jmath}\,[B]$. As $\jmath$ is an embedding, the operator $\overline{\jmath}$ is injective, and $\overline{\rho}$ is surjective since $\overline{\rho\tau} = \rho$, so the lower row in 2.16 is a short exact sequence, from now on referred to as the *pushout sequence*. Actually, the universal property of the pushout makes Diagram (2.16) work as the definition of pushout:

2.6.1 *Given a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\
 & & {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle T} & & \Vert & & \\
0 & \longrightarrow & B & \xrightarrow{\iota} & Z' & \xrightarrow{\pi} & X & \longrightarrow & 0
\end{array} \qquad (2.17)$$

the lower sequence is equivalent to the pushout sequence (2.16).

Proof Indeed, the universal property of the pushout implies that since $\iota\tau = T\jmath$, there must be an operator $\gamma\colon \mathrm{PO} \longrightarrow Z$ such that $\iota = \gamma\overline{\jmath}$ and $T = \gamma\overline{\tau}$; this makes the left bottom square commutative in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \overline{\tau}} & & \Vert & & \\
0 & \longrightarrow & B & \xrightarrow{\overline{\jmath}} & \mathrm{PO} & \xrightarrow{\overline{\rho}} & X & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle \gamma} & & \Vert & & \\
0 & \longrightarrow & B & \xrightarrow{\iota} & Z' & \xrightarrow{\pi} & X & \longrightarrow & 0
\end{array}$$

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To check the commutativity on the right, observe that $\pi\gamma\overline{\tau} = \pi T = \rho$ and $\pi\gamma\overline{\jmath} = 0$. Since the arrow $\overline{\rho}$ appearing in the definition of *pushout* is unique, $\pi\gamma = \overline{\rho}$. □

For this reason, we usually refer to a diagram like (2.17) as a pushout diagram. It is implicit in the preceding argument that

2.6.2 *Making pushout preserves the equivalence of extensions. Precisely, if* z *and* z' *are exact sequences,* τ *is an operator and* $\tau\,\mathsf{z}$ *denotes the pushout sequence, then* $[\mathsf{z}] = [\mathsf{z}'] \implies [\tau\,\mathsf{z}] = [\tau\,\mathsf{z}']$.

The following result is the key piece that connects the pushout construction with operator extension properties:

Lemma 2.6.3 *The lower sequence in a pushout diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\ & & {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\overline{\tau}} & & \| & & \\ 0 & \longrightarrow & B & \xrightarrow{\overline{\jmath}} & Z' & \xrightarrow{\overline{\rho}} & X & \longrightarrow & 0 \end{array} \tag{2.18}$$

splits if and only if there is an operator $T\colon Z \longrightarrow B$ *such that* $T\jmath = \tau$.

Proof One implication is trivial: if the lower sequence splits, composing $\overline{\tau}$ with any left inverse of $\overline{\jmath}$, one obtains the required 'extension' of τ. We provide two (two?) proofs for the converse. First, assume that the left square is the straight pushout diagram and $Z' = \mathrm{PO}$: if $T\colon Z \longrightarrow B$ satisfies $\tau = T\jmath$ then by applying the universal property of PO to the operators $T, \mathbf{1}_B$, we obtain $\gamma\colon Z' \longrightarrow B$ such that $\gamma\overline{\jmath} = \mathbf{1}_B$, so the lower row splits. The general case follows from this in view of 2.6.1. But even ignoring these facts, the proof is easy: if an extension operator T exists, there is an operator $s\colon X \longrightarrow Z'$ such that $\overline{\tau} - \overline{\jmath}T = s\rho$ since $(\overline{\tau} - \overline{\jmath}\,T)\jmath = 0$. This s is a linear continuous selection for $\overline{\rho}$ since $\overline{\rho}s\rho = \overline{\rho\tau} - \overline{\rho\jmath}\,T = \rho$, hence $\overline{\rho}s = \mathbf{1}_X$. □

Diagram (2.16) describes the pushout space PO as an enlargement of B whose main characteristic is that PO enlarges B in the same way that X enlarges Y, i.e. $\mathrm{PO}/B = Z/Y$. From the point of view of the operators, PO provides an enlargement of B that enables us to extend $\tau\colon Y \longrightarrow B$ to the whole of Z. The next question to ponder is then: where does one encounter pushouts? Or worse: are there pushout sequences at all? Yes, everywhere! Indeed, all exact sequences 'are' pushout sequences. That is the content of the next section.

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2.7 Projective Presentations: the Universal Property of ℓ_p

The spaces $\ell_p(I)$ enjoy, for $0 < p \leq 1$, two very special properties among all p-Banach spaces. One of them is that they are *projective*, which means that whenever $\pi\colon Z \longrightarrow X$ is a quotient map between two p-Banach spaces, every operator $\tau\colon \ell_p(I) \longrightarrow X$ can be lifted to Z, that is, there exists an operator $T\colon \ell_p(I) \longrightarrow Z$ such that $\pi T = \tau$. In other words, there is a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\ \pi\ } & X \\ & {\nwarrow}^{T} & \uparrow{\scriptstyle \tau} \\ & & \ell_p(I) \end{array}$$

To obtain T, it is enough to pick a bounded family of elements z_i such that $\pi(z_i) = \tau(e_i)$ and define $T(e_i) = z_i$. This property admits the following reformulation: every exact sequence of p-Banach spaces

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow \ell_p(I) \longrightarrow 0$$

splits; indeed, every quotient map $Z \longrightarrow \ell_p(I)$ admits a linear continuous section, namely any lifting of the identity. By Proposition 1.2.3, one has:

2.7.1 *$\ell_p(I)$ are the only projective p-Banach spaces,* $0 < p \leq 1$.

The other is that every p-Banach space X is a quotient of some $\ell_p(\alpha)$. A quotient map $\pi\colon \ell_p(\alpha) \longrightarrow X$ can be defined explicitly as follows: take $(x_i)_{i\in\alpha}$ a set of size α that is dense in the unit ball of X and set $\pi((\lambda_i)_{i\in\alpha}) = \sum_i \lambda_i x_i$. It is very easy to see that π is an operator onto X; the associated exact sequence $0 \longrightarrow \ker\pi \longrightarrow \ell_p(\alpha) \stackrel{\pi}{\longrightarrow} X \longrightarrow$ is called a projective presentation of X in the category of p-Banach spaces. Whether a space is projective depends upon the category we consider it in: the space ℓ_1 is projective in the category of Banach spaces but not in the category of p-Banach spaces $0 < p < 1$ because there is a quotient map $\ell_p \longrightarrow \ell_1$ for which no linear continuous section is possible since ℓ_1 is not a subspace of ℓ_p. In general, an exact sequence $0 \longrightarrow \kappa \longrightarrow \mathcal{P} \longrightarrow X \longrightarrow 0$ in a category in which the object $\mathcal{P}$ is projective is called a *projective presentation* of X. When every object of a category admits a projective presentation, we sometimes say that the category has *enough projectives*. Therefore, the category of p-Banach spaces has enough projectives.

2.7.2 *Every exact sequence of p-Banach spaces* $0 \longrightarrow Y \stackrel{J}{\longrightarrow} Z \stackrel{\rho}{\longrightarrow} X \longrightarrow 0$ *is (equivalent to) a pushout of any projective presentation of X.*

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Indeed, if $\pi\colon \ell_p(I) \longrightarrow X$ is a projective presentation then π can be lifted to Z, so let $T\colon \ell_p(I) \longrightarrow Z$ be an operator such that $\rho T = \pi$. The restriction of T to $\ker\pi$ takes values in $\jmath[Y]$ since $\pi T(k) = \pi(k) = 0$ for $k \in \ker\pi$. Hence, if $\tau\colon \ker\pi \longrightarrow Y$ is given by $\tau(k) = \jmath^{-1}T(k)$, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\pi & \longrightarrow & \ell_p(I) & \xrightarrow{\pi} & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \end{array} \tag{2.19}$$

which, according to 2.6.1, is a pushout diagram. Therefore, exact sequences $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ and operators $\ker\pi \longrightarrow Y$ are, roughly speaking, equivalent objects. Simple examples show that projective presentations are not unique, not even isomorphic: if X is a separable p-Banach space and J is uncountable, the two sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\pi & \longrightarrow & \ell_p & \xrightarrow{\pi} & X & \longrightarrow & 0 \\ & & & & & & \| & & \\ 0 & \longrightarrow & \ell_p(J)\times\ker\pi & \longrightarrow & \ell_p(J)\times\ell_p & \xrightarrow{0\oplus\pi} & X & \longrightarrow & 0 \end{array}$$

define non-isomorphic projective presentations of X. Classical Banach space theory already noticed the phenomenon that all projective presentations of a separable Banach space are ‘essentially the same’ and provided the following ad hoc explanation: if X is a separable Banach space not isomorphic to ℓ_1, the kernels of any two quotient maps from ℓ_1 to X are isomorphic. But that is just a part of the picture:

Proposition 2.7.3 *Let $\pi\colon \ell_p(I) \longrightarrow X$ and $\pi'\colon \ell_p(J) \longrightarrow X$ be two quotient maps. Then there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\pi\times\ell_p(J) & \longrightarrow & \ell_p(I)\times\ell_p(J) & \xrightarrow{\pi\oplus 0} & X & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \| & & \\ 0 & \longrightarrow & \ell_p(I)\times\ker\pi' & \longrightarrow & \ell_p(I)\times\ell_p(J) & \xrightarrow{0\oplus\pi'} & X & \longrightarrow & 0 \end{array}$$

in which α and β are isomorphisms. In particular, $\ker\pi\times\ell_p(J) \simeq \ker\pi'\times\ell_p(I)$, *and the rows are isomorphic sequences.*

Proof Consider the quotient operator $Q\colon \ell_p(I) \times \ell_p(J) \longrightarrow X$ given by $Q(x,y) = \pi x - \pi' y$ whose kernel is $\{(x,y)\colon \pi x = \pi' y\}$. The map $\ker Q \longrightarrow \ell_p(J)$ given by $(x,y) \longmapsto y$ is surjective, and thus it admits a linear bounded section $\ell_p(J) \longrightarrow \ker Q$ given by $y \longmapsto (sy, y)$. We define an isomorphism $u\colon \ker\pi \times \ell_p(J) \longrightarrow \ker Q$ as $u(x,y) = (x + sy, y)$. It is well defined since $\pi(x + sy) = \pi sy = \pi' y$. It is obviously injective since $(x + sy, y) = (0,0)$ implies

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$(x, y) = (0, 0)$. And it is surjective since $(x, y) = u(x-sy, y)$, and if $(x, y) \in \ker Q$ then $\pi x = \pi' y$ and thus $x - sy \in \ker\pi$ since $\pi(x - sy) = \pi x - \pi' y = 0$. Analogously, there is an isomorphism $v : \ell_p(I) \times \ker\pi' \longrightarrow \ker Q$ given by $(a, b) \longmapsto (a, s'a + b)$, where $x \longmapsto (x, s'x)$ is a linear continuous section for the map $\ker Q \longrightarrow \ell_p(J)$ given by $(x, y) \longmapsto x$. Then, define $\alpha = v^{-1}u$. The map $\beta(x, y) = \big(x + sy, y - s'(x + sy)\big)$ is an automorphism of $\ell_p(I) \times \ell_p(J)$ whose inverse is $(a, b) \longmapsto (a - s(s'a + b), s'a + b)$. □

What you have just seen is an example of the use of the diagonal principles we will present in Section 2.11. From Proposition 2.7.3 we derive the result from Banach space folklore mentioned earlier. For this reason, we will use the notation $\kappa_p(X)$ to denote the kernel of any quotient map $\ell_p(I) \longrightarrow X$.

Corollary 2.7.4 *Let X be a separable p-Banach space, and let π, π' be two quotient maps $\ell_p \longrightarrow X$. Then $\ker\pi \times \ell_p$ and $\ker\pi' \times \ell_p$ are isomorphic. If $p = 1$ and X is not isomorphic to ℓ_1, then $\ker\pi$ and $\ker\pi'$ are isomorphic.*

Proof The first part is contained in the preceding proposition. Infinite-dimensional subspaces of ℓ_1 contain complemented copies of ℓ_1, thus we have $\ker\pi \simeq \ell_1 \times A \simeq \ell_1 \times \ell_1 \times A \simeq \ell_1 \times \ker\pi$; analogously, $\ker\pi' \simeq \ell_1 \times \ker\pi'$. □

This raises the apparently open question of whether subspaces of ℓ_p contain complemented copies of ℓ_p (the different issue of whether a subspace of ℓ_p contains a copy of ℓ_p complemented *in* ℓ_p is treated in [443; 444; 397; 21; 261]). We conclude the section by connecting projective presentations of the subspace and the quotient space in an exact sequence.

Lemma 2.7.5 *Given an exact sequence $0 \longrightarrow Y \stackrel{\jmath}{\longrightarrow} Z \stackrel{\rho}{\longrightarrow} X \longrightarrow 0$ of p-Banach spaces and projective presentations of Y and X, there exists a projective presentation of Z forming a commutative diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker\pi_Y & \longrightarrow & \ker\pi_Z & \longrightarrow & \ker\pi_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{P}_Y & \longrightarrow & \mathcal{P}_Z & \longrightarrow & \mathcal{P}_X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi_Z} & & \downarrow{\scriptstyle \pi_X} & & \\
0 & \longrightarrow & Y & \stackrel{\jmath}{\longrightarrow} & Z & \stackrel{\rho}{\longrightarrow} & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proof Let $Q\colon \mathcal{P}_X \longrightarrow Z$ be a lifting of π_X, set $\mathcal{P}_Z = \mathcal{P}_Y \times \mathcal{P}_X$ and define $\pi_Z\colon \mathcal{P}_Z \longrightarrow Z$ by $\pi_Z(a,b) = \jmath\pi_Y(a) + Q(b)$. □

2.8 Pullbacks and Exact Sequences

Consider an exact sequence $0 \longrightarrow Y \xrightarrow{\jmath} Z \xrightarrow{\rho} X \longrightarrow 0$ and an operator $\tau\colon A \longrightarrow X$. Let us form the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\ &&&& \uparrow{\scriptstyle \underline{\tau}} && \uparrow{\scriptstyle \tau} && \\ &&&& \mathrm{PB} & \xrightarrow{\underline{\rho}} & A && \end{array}$$

Recalling that $\underline{\rho}$ is onto and setting $\underline{\jmath}(y) = (0, \jmath(y))$, it is easily seen that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\ && \| && \uparrow{\scriptstyle \underline{\tau}} && \uparrow{\scriptstyle \tau} && \\ 0 & \longrightarrow & Y & \xrightarrow{\underline{\jmath}} & \mathrm{PB} & \xrightarrow{\underline{\rho}} & A & \longrightarrow & 0 \end{array} \tag{2.20}$$

The lower sequence is exact and shall be referred to as the *pullback sequence*. Actually, the universal property of the pullback makes this diagram work as the definition of the pullback, which is why we will usually refer to the diagram in 2.8.1 as a pullback diagram:

2.8.1 *Given a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\ && \| && \uparrow && \uparrow && \\ 0 & \longrightarrow & Y & \longrightarrow & Z' & \longrightarrow & A & \longrightarrow & 0 \end{array} \tag{2.21}$$

the lower exact sequence is equivalent to the pullback sequence (2.20).

The proof is entirely dual of that of 2.6.1 and, thus, it is implicit that

2.8.2 *Taking pullbacks preserves the equivalence of extensions. Precisely, if* z *and* z' *are exact sequences,* τ *is an operator and* $\mathsf{z}\,\tau$ *denotes the pullback sequence, then* $[\mathsf{z}] = [\mathsf{z}'] \Longrightarrow [\mathsf{z}\,\tau] = [\mathsf{z}'\,\tau]$.

The following result is the key piece that connects pullback properties with operator lifting properties.

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Lemma 2.8.3 *The lower sequence in the pullback diagram* (2.20) *splits if and only if there is an operator* $T: A \longrightarrow Z$ *such that* $\rho T = \tau$.

Proof If $s: A \longrightarrow \text{PB}$ is a section of $\overline{\rho}$, then $T = \underline{\tau} s$ is a lifting of τ. And if $T: A \longrightarrow Z$ is a lifting of τ, then $s(a) = (T(a), a)$ is a section of $\underline{\rho}$. □

Where does one encounter pullback diagrams? Are there pullback sequences at all? Yes: everywhere! Indeed, all exact sequences *are* pullback sequences, and that is the content of the next section.

2.9 Injective Presentations: the Universal Property of ℓ_∞

This section is dual, almost word-by-word, to Section 2.7, except for 2.9.1. Apart from that, if one goes to Section 2.7, fixes $p = 1$, changes ‘quotient’ to ‘embedding’, reverses arrows in diagrams, etc.... one gets this section. We stress the correspondence by reproducing the presentation as closely as possible. The spaces $\ell_\infty(I)$ enjoy two very special properties among all Banach spaces. First, they are *injective*: a p-Banach space X is said to be injective if all operators $A \longrightarrow X$ can be extended to any p-Banach superspace. If norm one operators can be extended to norm λ operators then X is called λ-injective.

2.9.1 *If* $0 < p < 1$, *the only injective space in* $p\mathbf{B}$ *is* 0.

Wow, that’s a surprise, right? The proof follows from the corollary in Note 1.8.3: injective spaces Y in $p\mathbf{B}$ must obviously be ultrasummands, but if some non-zero $y \in Y$ exists then the operator $c \in \mathbb{K} \longmapsto cy \in Y$ cannot extend to L_p. Thus, this section is about Banach spaces only.

The space $\ell_\infty(I)$ is 1-injective among Banach spaces, but there are many others. Most of what is known about injective Banach spaces has been collected in [22]. Injectivity admits the formulation: every exact sequence of Banach spaces $0 \longrightarrow \ell_\infty(I) \longrightarrow \cdot \longrightarrow \cdot \longrightarrow 0$ splits; think of an extension of the identity of $\ell_\infty(I)$. Moreover, if I is a dense set of a Banach space Y then Y is isometric to a closed subspace of $\ell_\infty(I)$. An embedding $\jmath: Y \longrightarrow \ell_\infty(I)$ can be defined explicitly as follows: take for each $i \in I$ a norm one functional i^* such that $\langle i^*, i\rangle = \|i\|$ and set $\jmath(y) = \langle i^*, y\rangle_{i\in I}$. The associated exact sequence

$$0 \longrightarrow Y \longrightarrow \ell_\infty(I) \longrightarrow \ell_\infty(I)/Y \longrightarrow 0$$

or, more generally, any exact sequence $0 \longrightarrow Y \longrightarrow \mathcal{I} \longrightarrow \mathcal{I}/Y \longrightarrow 0$ in which $\mathcal{I}$ is an injective Banach space, is called an *injective presentation* of Y. The cokernel space $\mathcal{I}/Y$ is sometimes denoted $c\kappa(Y)$. A category in which every object admits an injective presentation is said to have *enough injectives*. Since the spaces $\ell_\infty(I)$ are injective, the category of Banach spaces admits enough injectives. A category may have injective objects, but not enough: the category of separable Banach spaces has injective objects (c_0, thanks to Sobczyk's theorem), but they are not enough by Zippin's theorem (c_0 is the only separable separably injective space). Exact sequences $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ and injective presentations of Y can be connected to form commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & \ell_\infty(I) & \longrightarrow & \ell_\infty(I)/Y & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \end{array} \tag{2.22}$$

In this way, exact sequences $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ and operators $X \longrightarrow \ell_\infty(I)/Y$ are, roughly speaking, equivalent objects. There are many non-(isomorphically) equivalent injective presentations, but the result dual to Proposition 2.7.3 works and can be proved cleanly either with the Diagonal principle 2.11.7 or through an ad hoc dual rewriting of the proof of Proposition 2.7.3. However we go, we get

Proposition 2.9.2 *Let $\imath\colon Y \longrightarrow \ell_\infty(I)$ and $\jmath\colon Y \longrightarrow \ell_\infty(J)$ be two emdeddings. Then there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & \ell_\infty(I)\times\ell_\infty(J) & \longrightarrow & (\ell_\infty(I)/Y)\times\ell_\infty(J) & \longrightarrow & 0 \\ & & \| & & \beta\uparrow & & \gamma\uparrow & & \\ 0 & \longrightarrow & Y & \longrightarrow & \ell_\infty(I)\times\ell_\infty(J) & \longrightarrow & \ell_\infty(I)\times(\ell_\infty(J)/Y) & \longrightarrow & 0 \end{array}$$

in which β and γ are isomorphisms. In particular, $(\ell_\infty(I)/Y)\times\ell_\infty(J) \simeq \ell_\infty(I)\times(\ell_\infty(J)/Y)$, and the rows are isomorphic sequences.

Corollary 2.9.3 *If $\imath$ and $\jmath$ are embeddings of a separable space Y into ℓ_∞, then $\ell_\infty/\imath[Y]$ and $\ell_\infty/\jmath[Y]$ are isomorphic.*

Proof The quotient $\ell_\infty/\imath[Y]$ must contain ℓ_∞ by Rosenthal's property (V) and therefore $\ell_\infty/\imath[Y] \simeq \ell_\infty/\imath[Y]\times\ell_\infty$. Analogously, $\ell_\infty/\jmath[Y] \simeq \ell_\infty/\jmath[Y]\times\ell_\infty$. □

The result applies to all subspaces Y of ℓ_∞ such that ℓ_∞/Y contains ℓ_∞. See 7.2.2 for further developments. Injective presentations of the subspace and the quotient of an exact sequence can be connected:

Lemma 2.9.4 *Given an exact sequence $0 \longrightarrow Y \stackrel{\jmath}{\longrightarrow} Z \stackrel{\rho}{\longrightarrow} X \longrightarrow 0$ of Banach spaces and injective presentations of Y and X, there exists an injective presentation of Z forming a commutative diagram*

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$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota_Y} & & \downarrow{\scriptstyle \iota_Z} & & \downarrow{\scriptstyle \iota_X} & & \\
0 & \longrightarrow & \mathcal{J}_Y & \longrightarrow & \mathcal{J}_Z & \longrightarrow & \mathcal{J}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{J}_Y/Y & \longrightarrow & \mathcal{J}_Z/Z & \longrightarrow & \mathcal{J}_X/X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Proof Indeed, let $I\colon Z \longrightarrow \mathcal{J}_Y$ be an extension of ι_Y, set $\mathcal{J}_Z = \mathcal{J}_Y \times \mathcal{J}_X$ and define $\iota_Z\colon Z \longrightarrow \mathcal{J}_Z$ as $\iota_Z(x) = I(x) + \iota_X(\rho x)$. □

2.10 All about That Pullback/Pushout Diagram

Once we are aware of their existence, there are a few essential things to know about pullback/pushouts.

First thing: how to recognise them. Imagine that reading and writing diagrams is reading and writing Japanese. The two basic ideograms to learn are *pullback* and *pushout*: we must learn to recognise that

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & 0
\end{array}
$$

is *always* a pullback diagram (Section 2.8), while

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & 0
\end{array}
$$

is *always* a pushout diagram (Section 2.6). This pictorial recognition of pullback/pushouts makes evident that taking the pushout along α and *then* along α' is the same as taking pushout along $\alpha'\alpha$ and that taking pullback along γ and *then* along γ' is the same as taking pullback along $\gamma\gamma'$.

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And taking a pushout along α and then a pullback along γ? Keep reading.

Second thing: they commute. Namely, given an exact sequence $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ and two operators $\alpha\colon Y \longrightarrow Y'$ and $\gamma\colon X' \longrightarrow X$, first taking pushout along α and then the pullback along γ,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\jmath_o} & \mathrm{PO} & \xrightarrow{\rho_o} & X & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\underline{\gamma}} & & \uparrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\jmath_{bo}} & \mathrm{PB(PO)} & \xrightarrow{\rho_{bo}} & X' & \longrightarrow & 0 \end{array} \tag{2.23}$$

or first taking the pullback along γ and then the pushout along α,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & Y & \xrightarrow{\jmath_b} & \mathrm{PB} & \xrightarrow{\rho_b} & X' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\jmath_{ob}} & \mathrm{PO(PB)} & \xrightarrow{\rho_{ob}} & X' & \longrightarrow & 0 \end{array} \tag{2.24}$$

produces equivalent sequences. Indeed, that the final resulting sequences are equivalent means that there is an operator $T: \mathrm{PO(PB)} \longrightarrow \mathrm{PB(PO)}$ making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y' & \xrightarrow{\jmath_{ob}} & \mathrm{PO(PB)} & \xrightarrow{\rho_{ob}} & X' & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\jmath_{bo}} & \mathrm{PB(PO)} & \xrightarrow{\rho_{bo}} & X' & \longrightarrow & 0 \end{array} \tag{2.25}$$

There are two ways to get such a map: one relying on the fact that PO(PB) is a pushout and the other relying on the fact that PB(PO) is a pullback. We show the second case. Thus, consider the pullback square

$$\begin{array}{ccc} \mathrm{PO} & \xrightarrow{\rho_o} & X \\ \uparrow{\scriptstyle\underline{\gamma}} & & \uparrow{\scriptstyle\gamma} \\ \mathrm{PB(PO)} & \xrightarrow{\rho_{bo}} & X' \end{array} \tag{2.26}$$

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Let us form another commutative square;

$$\begin{array}{ccc} \mathrm{PO} & \xrightarrow{\rho_{\mathrm{o}}} & X \\ {\scriptstyle\delta}\uparrow & & \uparrow{\scriptstyle\gamma} \\ \mathrm{PO(PB)} & \xrightarrow{\rho_{\mathrm{bo}}} & X' \end{array} \qquad (2.27)$$

in which the arrow δ is obtained from the universal property of the pushout square

$$\begin{array}{ccc} Y & \xrightarrow{\jmath_{\mathrm{b}}} & \mathrm{PB} \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\underline{\alpha}} \\ Y' & \xrightarrow{\jmath_{\mathrm{ob}}} & \mathrm{PO(PB)} \end{array}$$

in combination with the fact that the square obtained by juxtaposition of the upper left squares of Diagrams 2.24 and 2.25, namely

$$\begin{array}{ccc} Y & \xrightarrow{\jmath_{\mathrm{b}}} & \mathrm{PB} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\overline{\alpha}\,\overline{\gamma}} \\ Y' & \xrightarrow{\jmath_{\mathrm{o}}} & \mathrm{PO} \end{array}$$

is also commutative. Thus, there is a unique operator $\delta\colon \mathrm{PO(PB)} \longrightarrow \mathrm{PO}$ such that $\delta\underline{\alpha} = \overline{\alpha}\,\overline{\gamma}$ and $\jmath_{\mathrm{ob}} = \jmath_{\mathrm{o}}$.

Finally, the commutativity of Diagram (2.27) and the universal property of (2.26) immediately yield the existence of an operator $T\colon \mathrm{PO(PB)} \longrightarrow \mathrm{PB(PO)}$ such that $\rho_{\mathrm{bo}}T = \rho_{\mathrm{ob}}$ and $\underline{\gamma}T = \delta$. The first of those equalities is the commutativity of the right square in Diagram (2.25). Let us prove the commutativity of the left square, i.e. $T\jmath_{\mathrm{ob}} = \jmath_{\mathrm{bo}}$: since $\rho_{\mathrm{bo}}T = \rho_{\mathrm{bo}}$, it is clear that $\rho_{\mathrm{bo}}T\jmath_{\mathrm{ob}} = \rho_{\mathrm{ob}}\jmath_{\mathrm{ob}} = 0$, and therefore some operator $u\colon Y' \longrightarrow Y'$ must exist such that $T\jmath_{\mathrm{ob}} = \jmath_{\mathrm{bo}}u$. But since $\jmath_{\mathrm{o}}u = \underline{\gamma}\jmath_{\mathrm{bo}}u = \underline{\gamma}T\jmath_{\mathrm{ob}} = \delta\jmath_{\mathrm{ob}} = \jmath_{\mathrm{o}}$, it follows that u is the identity on Y', and this concludes the proof.

Third thing: they mix. They mix in a single diagram – this one:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\jmath'} & Z' & \xrightarrow{\rho'} & X' & \longrightarrow & 0 \end{array} \qquad (2.28)$$

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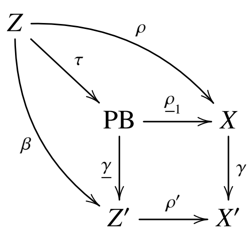
since it can be artfully decomposed as

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\overline{\alpha}} & & \| & & \\
0 & \longrightarrow & Y' & \xrightarrow{\overline{\jmath}} & \mathrm{PO} & \xrightarrow{\overline{\rho}} & X & \longrightarrow & 0 \\
 & & \| & & & & \| & & \\
0 & \longrightarrow & Y' & \xrightarrow{\underline{\jmath}'} & \mathrm{PB} & \xrightarrow{\underline{\rho}'} & X & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\underline{\gamma}} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & Y' & \xrightarrow{\jmath'} & Z' & \xrightarrow{\rho'} & X' & \longrightarrow & 0
\end{array} \tag{2.29}$$

where the two middle sequences are equivalent, as we show now (the argument is quite similar to the one used earlier). The commutative diagram

$$\begin{array}{ccc}
Z & \xrightarrow{\rho} & X \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\
Z' & \xrightarrow{\rho'} & X'
\end{array}$$

yields an operator $\tau\colon Z \longrightarrow \mathrm{PB}$ forming a commutative diagram



which therefore means that it also makes a commutative square

$$\begin{array}{ccc}
Y & \xrightarrow{\jmath} & Z \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\tau} \\
Y' & \xrightarrow{\underline{\jmath}'} & \mathrm{PB}
\end{array}$$

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and in turn there is an operator $T\colon \mathrm{PO} \longrightarrow \mathrm{PB}$ forming a commutative diagram

$$\begin{array}{ccccc}
Y & \xrightarrow{\ \jmath\ } & Z & & \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \overline{\alpha}} & \searrow{\scriptstyle \tau} & \\
Y' & \xrightarrow{\ \overline{\jmath}\ } & \mathrm{PO} & & \\
& \searrow{\scriptstyle \underline{\jmath}'} & & \searrow{\scriptstyle T} & \\
& & & & \mathrm{PB}
\end{array}$$

which, inserted correctly in (2.29), yields the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\ \jmath\ } & Z & \xrightarrow{\ \rho\ } & X & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \underline{\gamma}\, T\, \overline{\alpha}} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & Y' & \xrightarrow{\ \jmath'\ } & Z' & \xrightarrow{\ \rho'\ } & X' & \longrightarrow & 0
\end{array}$$

Only one task remains (for the sceptics): check that $\beta = \underline{\gamma}\, T\, \overline{\alpha}$. Really? Yes, $\underline{\gamma}\, T\, \overline{\alpha} = \underline{\gamma}\, \tau = \beta$.

Perhaps the best example one can give of Diagram (2.29) comes from considering an exact sequence of Banach spaces and combining it with projective and injective presentations of the quotient and subspace, respectively, namely

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\rho & \longrightarrow & \mathcal{P} & \longrightarrow & X & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & \nearrow & \downarrow & & \\
& & & & Z & & & & \\
& & \downarrow & \nearrow & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{I}/Y & \longrightarrow & 0
\end{array} \tag{2.30}$$

in which it is clear that the middle diagonal sequence is both a pushout of the upper sequence and a pullback of the lower one.

Fourth thing: they can be completed. It is time to show the value of completing diagrams. The natural context in which diagrams can be completed is when the involved operators are either quotient maps or embeddings, which

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are the two main cases of closed-range operators. Start with a pullback Diagram (2.20) whose upwards operator is a quotient map to obtain

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Y & \longrightarrow & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle \pi} & & \\
0 & \longrightarrow & Y & \longrightarrow & \mathrm{PB}(\rho, \pi) & \longrightarrow & Z' & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \ker \underline{\pi} & = & \ker \pi & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\tag{2.31}
$$

When the operator is an embedding, the completed diagram is

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \operatorname{coker} \underline{\imath} & = & \operatorname{coker} \imath & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Y & \longrightarrow & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle \underline{\imath}} & & \uparrow{\scriptstyle \imath} & & \\
0 & \longrightarrow & Y & \longrightarrow & \mathrm{PB}(\rho, \imath) & \longrightarrow & Y' & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\tag{2.32}
$$

in which $\mathrm{PB}(\rho, \imath)$ is naturally isomorphic to $\rho^{-1}[\imath[Y']]$. The completion of a pushout Diagram (2.16) when the left downwards operator is a quotient map yields

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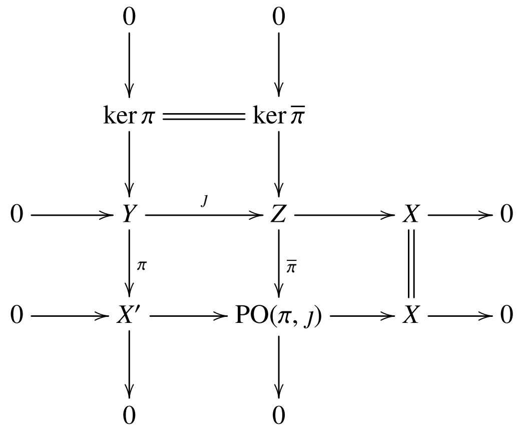
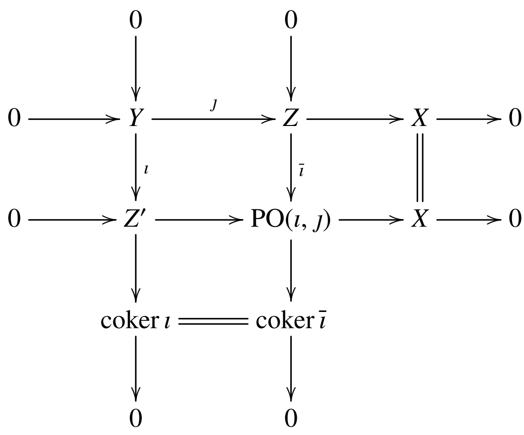


(2.33)

and when it is an embedding, we get



(2.34)

This makes a total of $2 + 2 = 3$ (!) diagrams, since (2.32) and (2.33) are exactly the same … when rotated. *This* diagram could then be written as

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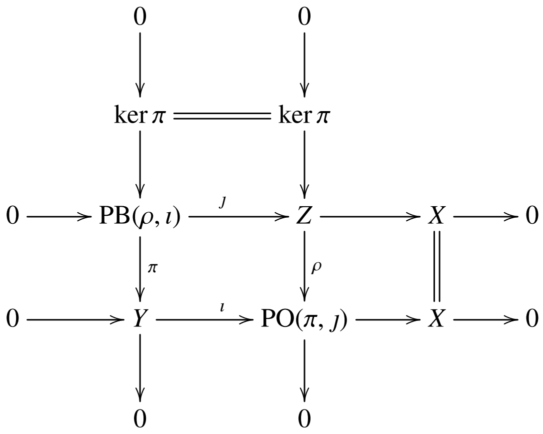
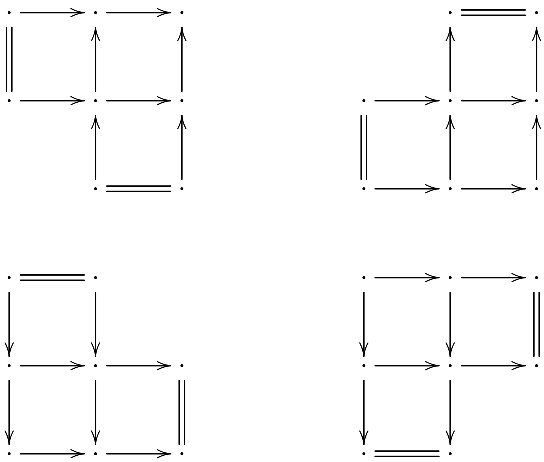


(2.35)

Summing the situation up, the four fundamental diagrams



are actually three, since those in the positions $(0, 0)$ and $(1, 1)$ are the same.

Diagonals

Concealed in the diagrams are other 'diagonal' exact sequences. While at first glance they may seem to be the oompa loompas of homology, they are in fact essential both for the understanding of pairs of exact sequences and for the construction of counterexamples. Since the reader should start to think in terms of equivalence classes, we will from now on write $[\mathtt{z}]$ instead of $\mathtt{z}$ whenever $\mathtt{z}$

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can be replaced by any other equivalent sequence to obtain equivalent results. Thus, given a pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle \underline{\tau}} & & \uparrow{\scriptstyle \tau} & & \\ 0 & \longrightarrow & Y & \xrightarrow{\imath} & \mathrm{PB} & \xrightarrow{\underline{\rho}} & X' & \longrightarrow & 0 \end{array} \qquad [\mathsf{z}]$$

the very definition of pullback space generates a *diagonal pullback sequence*:

$$0 \longrightarrow \mathrm{PB} \longrightarrow Z \times X' \xrightarrow{\rho \oplus (-\tau)} X \longrightarrow 0$$

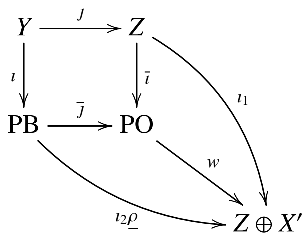
(the unnamed arrow is plain inclusion).

2.10.1 *The diagonal pullback sequence is the pushout sequence* [$\imath$ z].

The truth of the assertion is witnessed by the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \imath} & & \downarrow{\scriptstyle \bar\imath} & & \| & & \\ 0 & \longrightarrow & \mathrm{PB} & \xrightarrow{\bar\jmath} & \mathrm{PO} & \longrightarrow & X & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle w} & & \| & & \\ 0 & \longrightarrow & \mathrm{PB} & \longrightarrow & Z \times X' & \xrightarrow{\rho \oplus (-\tau)} & X & \longrightarrow & 0 \end{array} \qquad \begin{array}{c} [\mathsf{z}] \\ \\ [\imath\,\mathsf{z}] \\ \\ \\ \end{array}$$

in which the arrow w has been obtained from the universal property of the pushout



Given a pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \bar\tau} & & \| & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\bar\jmath} & \mathrm{PO} & \xrightarrow{\bar\rho} & X & \longrightarrow & 0 \end{array}$$

the very nature of the pushout space yields a *diagonal pushout sequence*,

$$0 \longrightarrow Y \xrightarrow{(\tau, \jmath)} Y' \oplus Z \xrightarrow{\bar\jmath \oplus (-\bar\tau)} \mathrm{PO} \longrightarrow 0$$

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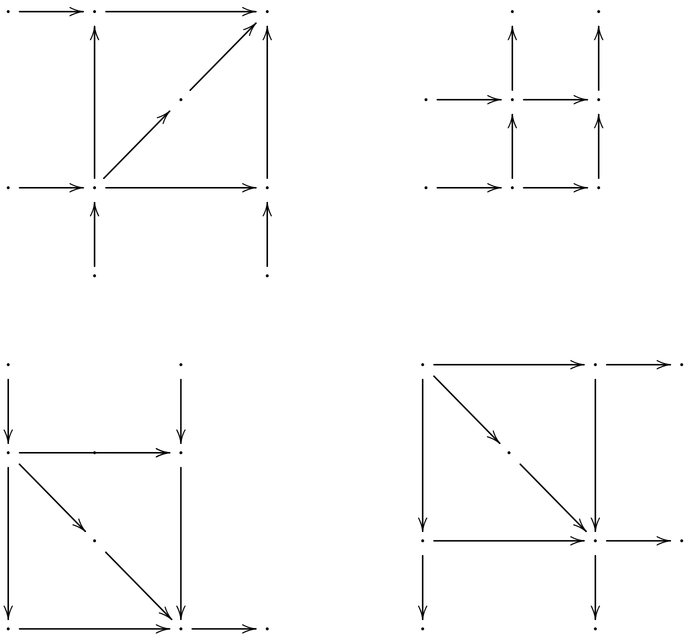
2.10.2 *The diagonal pushout sequence is the pullback sequence* $[\mathsf{z}\,\overline{\rho}]$.

That can be seen dualising what was done in the pullback case. We hope the reader will have fun with it. Summing up again, the four (three) pullback/pushout diagrams contain in them the diagonals



When do these diagonal sequences split? Good question. Observe that no matter where they are placed, those diagonals are plain pullbacks/pushouts: indeed, the pullback diagonal $0 \longrightarrow \mathrm{PB} \longrightarrow Z \times X' \longrightarrow X \longrightarrow 0$ is the pushout sequence $[\imath\,\mathsf{z}]$ (see 2.10.1) and the diagonal pushout sequence is the lower sequence in the pullback diagram (see your own diagram)

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 & & [\mathsf{z}] \\ & & \| & & \uparrow & & \uparrow{\scriptstyle \overline{\rho}} & & & & \\ 0 & \longrightarrow & Y & \longrightarrow & Y' \times Z & \longrightarrow & \mathrm{PO} & \longrightarrow & 0 & & [\mathsf{z}\,\overline{\rho}] \end{array}$$

2.10.3 *The diagonal pullback sequence splits if and only if* $\imath\colon Y \longrightarrow \mathrm{PB}$ *can be extended to* Z. *The diagonal pushout sequence splits if and only if* $\overline{\rho}\colon \mathrm{PO} \longrightarrow X$ *can be lifted to* Z.

Ok, this is, admittedly, better than nothing, but not terribly informative for the reader hungry for something more substantial. But wait: the splitting of the diagonal pullback sequence obviously implies that $\mathrm{PB} \times X \simeq Z \times X'$, while the splitting of the diagonal pushout sequence implies $Y \times \mathrm{PO} \simeq Y' \times Z$. And a little bit of *this* (when pullback/pushout sequences split) in combination with some additional little bit of *that* (the pullback/pushout spaces are isomorphic to some product) leads to the main course served up next: diagonal principles.

2.11 Diagonal and Parallel Principles

We have, in fact, already encountered diagonal and parallel principles earlier. The sceptical reader is invited to look again to Propositions 2.7.3 and 2.9.2. Thus our aim here is to give (homological) shape to a real-life phenomenon and thus bring understanding. The *diagonal* and *parallel* principles we present now govern the behaviour of pairs of exact sequences and work in categories where pullback, pushout, finite products and exact sequences exist:

- The parallel principles are concerned with the following problem: assume we have an exact sequence $[\mathtt{z}]$ that is pullback (resp. pushout) of another $[\mathtt{z}']$; when can we conclude that $[\mathtt{z}']$ is also a pullback (resp. pushout) of $[\mathtt{z}]$?
- The diagonal principles are concerned with what occurs if we have success in the previous situation: if one of the exact sequences $[\mathtt{z}]$ and $[\mathtt{z}']$ is a pullback (resp. pushout) of the other then ... what?

The starting point is to get a criterion to detect when an exact sequence $[\mathtt{z}']$ is a pushout (or pullback) of another sequence $[\mathtt{z}]$.

Proposition 2.11.1 *Given two exact sequences*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 & \qquad & (\mathtt{z}) \\ & & & & & & \| & & & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\jmath'} & Z' & \xrightarrow{\rho'} & X & \longrightarrow & 0 & \qquad & (\mathtt{z}') \end{array}$$

$[\mathtt{z}']$ *is a pushout of* $[\mathtt{z}]$ *if and only if* $[\mathtt{z}'\rho] = 0$.

Proof The necessity is clear: since $\mathbf{1}_Z$ is a lifting of $\rho\colon Z \longrightarrow X$, we have $[\mathtt{z}\rho] = 0$ and, therefore, $[\mathtt{z}'] = [\tau\,\mathtt{z}] \implies [\mathtt{z}'\rho] = [(\tau\mathtt{z})\rho)] = [\tau(\mathtt{z}\rho)] = 0$, by the commutativity of pullbacks and pushouts. The sufficiency is clear too:

$[\mathsf{z}'\rho] = 0$ means that ρ can be lifted to a map $L \in \mathfrak{L}(Z, Z')$, which yields a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle L} & & \| & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\jmath'} & Z' & \xrightarrow{\rho'} & X & \longrightarrow & 0 \end{array}$$

because $\rho' L\jmath = \rho\jmath = 0$ means that $L\jmath$ takes values in $\jmath'[Y'] = \ker\rho'$. □

Definition 2.11.2 Two exact sequences

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 & \qquad & (\mathsf{z}) \\ & & & & & & \| & & & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\jmath'} & Z' & \xrightarrow{\rho'} & X & \longrightarrow & 0 & \qquad & (\mathsf{z}') \end{array}$$

will be called semi-equivalent (*on the quotient end*, if one needs to specify) if each is a pushout of the other.

It is clear now that z and z' are semi-equivalent if and only if $[\mathsf{z}\rho'] = 0$ and $[\mathsf{z}'\rho] = 0$. The pullback version is analogous:

Proposition 2.11.3 *Given two exact sequences*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 & \qquad & (\mathsf{z}) \\ & & \| & & & & & & & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\jmath'} & Z' & \xrightarrow{\rho'} & X' & \longrightarrow & 0 & \qquad & (\mathsf{z}') \end{array}$$

$[\mathsf{z}']$ *is a pullback of* $[\mathsf{z}]$ *if and only if* $[\jmath\,\mathsf{z}'] = 0$.

Proof The necessity is clear: since $\mathbf{1}_Z$ is an extension of $\jmath$, we have $[\jmath\,\mathsf{z}] = 0$, and thus $[\mathsf{z}'] = [\mathsf{z}\,\tau] \Longrightarrow [\jmath\,\mathsf{z}'] = [(\jmath\,\mathsf{z})\tau] = 0$. The sufficiency is clear too: $[\jmath\,\mathsf{z}'] = 0$ means that $\jmath$ can be extended to an operator $J \colon Z' \longrightarrow Z$, which yields a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\ & & \| & & {\scriptstyle J}\uparrow & & \uparrow & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\jmath'} & Z' & \xrightarrow{\rho'} & X' & \longrightarrow & 0 \end{array}$$

because $\rho J\jmath' = \rho\jmath = 0$ forces ρJ to factorise through ρ'. □

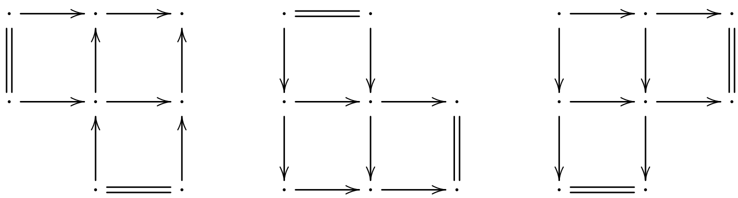
Definition 2.11.4 Two exact sequences

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 & \qquad & (\mathsf{z}) \\ & & \| & & & & & & & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\jmath'} & Z' & \xrightarrow{\rho'} & X' & \longrightarrow & 0 & \qquad & (\mathsf{z}') \end{array}$$

will be called semi-equivalent (*on the subspace end*, if necessary) if one is a pullback of the other, and vice versa.

The sequences $(\mathtt{z})$ and $(\mathtt{z}')$ are thus semi-equivalent if and only if $[\jmath\,\mathtt{z}'] = 0$ and $[\jmath'\,\mathtt{z}] = 0$. In what follows, we will call two exact sequences semi-equivalent if they are semi-equivalent on the appropriate end. Only when necessary will we specify the end. With these tools in hand, we get

2.11.5 Parallel lines principle *In any of the three diagrams*



the two vertical sequences are semi-equivalent if and only if the two horizontal sequences are semi-equivalent.

Proof In all cases, the announced semi-equivalence corresponds to the splitting of the diagonal sequence. □

Of course, one can give standard proofs for each of these results, though each is over-long, gnarled and different to the others. That's why we prefer the homological approach. We now present the diagonal principles, furthering our understanding of semi-equivalence. Some more names will be of great help here. Given an exact sequence

$$0 \longrightarrow Y \xrightarrow{\ \jmath\ } Z \xrightarrow{\ \rho\ } X \longrightarrow 0 \qquad (\mathtt{z})$$

and a quasi-Banach space E, $E \times \mathtt{z}$ will denote the sequence obtained by multiplying on the left by E:

$$0 \longrightarrow E \times Y \xrightarrow{\mathbf{1}_E \times \jmath} E \times Z \xrightarrow{0 \oplus \rho} X \longrightarrow 0$$

The sequence obtained multiplying by E on the right will be denoted $\mathtt{z} \times E$:

$$0 \longrightarrow Y \xrightarrow{(\jmath, 0)} Z \times E \xrightarrow{\rho \times \mathbf{1}_E} X \times E \longrightarrow 0$$

2.11.6 Diagonal principle: projective case *If the sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\ \jmath\ } & Z & \xrightarrow{\ \rho\ } & X & \longrightarrow & 0 \qquad (\mathtt{z}) \\ & & & & & & \| & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\ \jmath'\ } & Z' & \xrightarrow{\ \rho'\ } & X & \longrightarrow & 0 \qquad (\mathtt{z}') \end{array}$$

are semi-equivalent then the sequences $Z' \times \mathtt{z}$ and $Z \times \mathtt{z}'$ are isomorphic.

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Proof The hypothesis means that there are operators α,β such that $[\alpha\,\mathsf{z}] = [\mathsf{z}']$ and $[\beta\,\mathsf{z}'] = [\mathsf{z}]$, and therefore the diagonal pushout sequence

$$0 \longrightarrow Y \xrightarrow{(\jmath,\alpha)} Z \times Y' \longrightarrow Z' \longrightarrow 0$$

splits, which yields an isomorphism $\phi\colon Z\times Y' \longrightarrow Z'\times Y$ such that $\phi(\jmath,\alpha)(y) = (0,y)$. Back in the fast lane, we notice that since $[\alpha\,\mathsf{z}] = [\mathsf{z}']$ and $[\jmath\,\mathsf{z}] = 0$, $[Z\times\mathsf{z}'] = [(\jmath,\alpha)\,\mathsf{z}]$ according with the agreed names, and therefore

$$[\phi\,(Z\times\mathsf{z}')] = [\phi\,(\jmath,\alpha)\,\mathsf{z}] = [Z'\times\mathsf{z}]$$

thus there is an operator $\overline{\phi}$ making the following diagram commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z\times Y' & \longrightarrow & Z\times Z' & \longrightarrow & X & \longrightarrow & 0\\
 & & {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\overline{\phi}} & & \| & & \\
0 & \longrightarrow & Z'\times Y & \longrightarrow & Z'\times Z & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

The operator $\overline{\phi}$ is an isomorphism since ϕ is an isomorphism. □

Proposition 2.7.3 is an easy victim of this principle. There is a dual (pullback, injective, left end) version which anyone can perform by simple dualisation of this one. Let us present a classically knitted proof:

2.11.7 Diagonal principle: injective case *If the sequences*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 & \qquad & (\mathsf{z})\\
 & & \| & & & & & & & & \\
0 & \longrightarrow & Y & \xrightarrow{\jmath'} & Z' & \xrightarrow{\rho'} & X' & \longrightarrow & 0 & \qquad & (\mathsf{z}')
\end{array}$$

are semi-equivalent then $(\mathsf{z}\times Z')$ *and* $(\mathsf{z}'\times Z)$ *are isomorphic sequences.*

Proof The hypothesis yields operators $I\colon Z' \longrightarrow Z$ and $J\colon Z \longrightarrow Z'$ such that $I\jmath = \jmath'$ and $J\jmath' = \jmath$. The maps $\tau,\tau' \in \mathfrak{L}(Z\times Z')$ given by

$$\tau(z,z') = (z, z' + J(z)),$$
$$\tau'(z,z') = (z - I(z'), z')$$

are both isomorphisms. So, $T = \tau'\tau$ is the isomorphism we are looking for since $T(z,z') = \tau'\tau(z,z') = \tau'(z,z'+J(z)) = (z - I(z'+J(z)), z'+J(z))$ and thus $T(\jmath'(y),0) = (\jmath'(y) - I(0 + J(\jmath'(y)), 0 + J(\jmath'(y))) = (0,\jmath'(y))$. □

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2.12 Homological Constructions Appearing in Nature

We have seen that all exact sequences of p-Banach spaces are pushouts of a projective presentation of the quotient space and that all exact sequences of Banach spaces are pullbacks of an injective presentation of the subspace. We now record a few more entries in the directory of natural situations in which one encounters pushouts and pullbacks and other homological or categorical constructions.

The Natural Embedding of X into $C(B_X^*)$

Here is an everyday example of functor and natural transformation (notions to be defined in Chapter 4). Recall that B_X^* denotes the unit ball of the dual of X with the weak* topology, which is a compact space by the Banach–Alaoglu theorem. There is a natural isometry $\delta_X\colon X \longrightarrow C(B_X^*)$ given by $\delta_X(x)(x^*) = \langle x^*, x\rangle$. We have

Lemma 2.12.1 *Every $\mathscr{C}$-valued operator defined on a Banach space X admits a 1-extension through the embedding $\delta_X\colon X \longrightarrow C(B_X^*)$.*

Proof Assume without loss of generality that $\tau\colon X \longrightarrow C(K)$ is a contractive operator. Let $\delta_K\colon K \longrightarrow C(K)^*$ be the canonical embedding, in which $\delta_K(k) = \delta_k$ is the evaluation functional at k. It is clear that this map is continuous when $C(K)^*$ carries the weak* topology. The sought-after extension $T\colon C(B_X^*) \longrightarrow C(K)$ is $T(f)(k) = f(\tau^*(\delta_k))$. The operator T is well defined since $T(f)$, being the composition of three continuous maps, is a continuous function. The linearity of T and the bound $\|T\| \le 1$ are clear. That T extends τ through δ_X is clear as well:

$$(T\delta_X(x))(k) = \delta_X(x)\,(\tau^*(\delta_k)) = \langle \tau^*\delta_k, x\rangle = \langle \delta_k, \tau(x)\rangle = \tau(x)(k). \qquad \square$$

Regarding complementation, δ_X is the 'best' embedding that X can have into a $\mathscr{C}$-space:

2.12.2 *A Banach space X is complemented in a $\mathscr{C}$-space if and only if it is complemented in $C(B_X^*)$ through the natural embedding.*

Proof Let K be a compact space, and let $\jmath\colon X \longrightarrow C(K)$ and $P\colon C(K) \longrightarrow X$ be operators with $P\jmath = \mathbf{1}_X$. Let $J\colon C(B_X^*) \longrightarrow C(K)$ be an extension of $\jmath$ through δ_X, with $\|J\| = \|\jmath\|$. Then $Q = \delta_X PJ$ is a projection of $C(B_X^*)$ onto $\delta_X[X]$ and, clearly, $\|Q\| \le \|\jmath\|\,\|P\|$. The other implication is obvious. $\square$

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Interpolation Theory

We will have to wait until 10.8 for a fine-brush painting of the interlacing connections between complex interpolation and twisted sums. The purpose of that section is to derive the construction of the fundamental Kalton–Peck Z_p spaces from complex interpolation theory. However, we can take a broad-brush approach now. The reader can consider this and the next section as trailers for forthcoming films, including some spoilers.

As far as we currently know, most interpolation methods for pairs of Banach spaces follow the following schema. One starts with a pair (X_0, X_1) of Banach spaces that one assumes are linear and continuously embedded into some Banach space Σ. Then, there is a Banach space $\mathcal{H}$ and an operator $\Phi\colon \mathcal{H} \longrightarrow \Sigma$, which we will call an *interpolator on* $\mathcal{H}$, such that, for every linear operator $t\colon \Sigma \longrightarrow \Sigma$ acting continuously sending $X_0 \longrightarrow X_0$ and $X_1 \longrightarrow X_1$, there is an operator $T\colon \mathcal{H} \longrightarrow \mathcal{H}$ such that $t \circ \Phi = \Phi \circ T$. We denote by X_Φ the space $\Phi(\mathcal{H})$ endowed with the quotient norm $\|x\|_\Phi = \inf\{\|f\|_{\mathcal{H}} \colon f \in \mathcal{H}, \Phi f = x\}$, which is a Banach space. Given two interpolators Ψ, Φ on $\mathcal{H}$, consider the map $(\Psi, \Phi)\colon \mathcal{H} \longrightarrow \Sigma \times \Sigma$, and let $X_{\Psi,\Phi}$ denote the space $(\Psi, \Phi)[\mathcal{H}]$ endowed with the quotient norm. One thus has the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \ker\Psi \cap \ker\Phi & = & \ker(\Psi,\Phi) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \ker\Phi & \longrightarrow & \mathcal{H} & \xrightarrow{\Phi} & X_\Phi & \longrightarrow & 0 \\
 & & {\scriptstyle\Psi}\downarrow & & {\scriptstyle(\Psi,\Phi)}\downarrow & & \| & & \\
0 & \longrightarrow & \Psi[\ker\Phi] & \xrightarrow{\imath} & X_{\Psi,\Phi} & \xrightarrow{\rho} & X_\Phi & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{2.36}$$

where $\Psi[\ker\Phi]$ is endowed with the obvious quotient norm. The maps $\imath, \rho$ are defined by $\imath(\Psi g) = (\Psi g, 0)$ and $\rho(\Psi f, \Phi f) = \Phi f$.

The complex interpolation method, as described in Section 10.8, as well as the K and J real methods [82] and, in general, the unifying method of Cwikel, Kalton, Milman, and Rochberg [139], can be fit into this schema.

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The 3-Space Problem for Dual Spaces

In [452], Vogt posed a quite natural problem: must an exact sequence

$$0 \longrightarrow A^* \longrightarrow X \longrightarrow B^* \longrightarrow 0$$

be the dual sequence of another exact sequence? Must X be a dual space? The answer is no. We present now a no-frills counterexample, while Section 10.5 displays a much more natural and elaborated example.

Proposition 2.12.3 *Let X be a Banach space such that X^{**}/X is an ultrasummand with the RNP. Then X is a complemented subspace of a twisted sum of two dual spaces.*

Proof Let $\delta\colon X \longrightarrow X^{**}$ be the canonical inclusion, and let $\pi\colon \ell_1(I) \longrightarrow X^{**}$ be a quotient map. Form the complete pullback diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & X^{**}/X & = & X^{**}/X & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \kappa(X^{**}) & \longrightarrow & \ell_1(I) & \xrightarrow{\pi} & X^{**} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle \delta} & & \\
0 & \longrightarrow & \kappa(X^{**}) & \longrightarrow & \mathrm{PB} & \longrightarrow & X & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We need from the reader a leap of faith here and belief that the space PB is complemented in its bidual (the justification comes in Lemma 10.4.1). Form the diagonal exact sequence

$$0 \longrightarrow \mathrm{PB} \longrightarrow \ell_1(I) \times X \longrightarrow X^{**} \longrightarrow 0$$

Multiplying by a complement V of PB in PB^{**}, we get the sequence

$$0 \longrightarrow \mathrm{PB}^{**} \longrightarrow V \times \ell_1(I) \times X \longrightarrow X^{**} \longrightarrow 0 \qquad \square$$

To obtain specific examples, just set $X = \mathsf{JT}_*$, the natural predual of the James–Tree space JT. It is well known (see, e.g., [102]) that JT_* is uncomplemented in its bidual JT^* and that $\mathsf{JT}^*/\mathsf{JT}_*$ is a non-separable Hilbert space. All this yields the non-trivial exact sequence

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$$0 \longrightarrow \mathrm{PB}^{**} \longrightarrow (\mathrm{PB}^{**}/\mathrm{PB}) \times \ell_1(I) \times \mathsf{JT}_* \longrightarrow \ell_2(I) \longrightarrow 0$$

whose middle space cannot be an ultrasummand, let alone a dual space.

The more elaborate counterexample in Section 10.5 alluded to earlier will consist of an exact sequence of Banach spaces $0 \longrightarrow R \longrightarrow \diamond \longrightarrow U \longrightarrow 0$ in which R is reflexive, U is an ultrasummand and $\diamond$ is not an ultrasummand. The other two possible configurations of Banach spaces lead to ultrasummands:

2.12.4 *In any of the sequences of Banach spaces* $0 \longrightarrow R \longrightarrow U \longrightarrow \diamond \longrightarrow 0$ *or* $0 \longrightarrow U \longrightarrow \diamond \longrightarrow R \longrightarrow 0$*, the space* $\diamond$ *is an ultrasummand.*

Proof In the first situation, we can assume that R is a subspace of U, hence of U^{**}. Observe the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & U & \xrightarrow{\pi} & \diamond & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \delta_U} & & \downarrow{\scriptstyle \delta_\diamond} & & \\ 0 & \longrightarrow & R & \longrightarrow & U^{**} & \xrightarrow{\pi^{**}} & \diamond^{**} & \longrightarrow & 0 \end{array}$$

If P is a projection along δ_U, then $P|_R = \mathbf{1}_R$, and so it induces an operator $\overline{P}\colon \diamond^{**} \longrightarrow \diamond$. This operator is a projection along $\delta_\diamond$ because $\overline{P}\delta_\diamond\pi = \overline{P}\pi^{**}\delta_U = \pi P\delta_U = \pi$ implies $\overline{P}\delta_\diamond = \mathbf{1}_\diamond$ since π is surjective. The second situation was already treated in the introduction to this chapter. □

The 3-Space Problem for the Dunford–Pettis Property

The DPP is not a 3-space property [102]. Moreover, a careful using of the pullback construction establishes that counterexamples are almost ubiquitous.

Proposition 2.12.5 *Every Banach space is a complemented subspace of a twisted sum of two Banach spaces with the Dunford–Pettis property.*

Proof Let X be any Banach space. Consider an embedding $\jmath\colon X \longrightarrow C(K)$; fix, then, a quotient $\pi\colon \ell_1(I) \longrightarrow C(K)$ and draw the pullback diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\pi & \longrightarrow & \ell_1(I) & \longrightarrow & C(K) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle \jmath} & & \\ 0 & \longrightarrow & \ker\pi & \longrightarrow & \mathrm{PB} & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

The diagonal pullback sequence $0 \longrightarrow \mathrm{PB} \longrightarrow \ell_1(I) \times X \longrightarrow C(K) \longrightarrow 0$ proves the assertion: the space PB, as any space with the Schur property does, has the DPP as well as any $C(K)$ space. □

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The Johnson–Lindenstrauss Spaces

Let $\mathcal{M}$ be an almost disjoint family of subsets of $\mathbb{N}$. The Nakamura–Kakutani sequence 2.2.10 it generates is the perfect example of a pullback sequence. Indeed, the inclusion $C_0(\wedge_{\mathcal{M}}) \longrightarrow \ell_\infty$ induces an isometry $\iota\colon c_0(\mathcal{M}) \longrightarrow \ell_\infty/c_0$ that generates the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/c_0 & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle \iota} & & \\ 0 & \longrightarrow & c_0 & \longrightarrow & C_0(\wedge_{\mathcal{M}}) & \longrightarrow & c_0(\mathcal{M}) & \longrightarrow & 0 \end{array} \tag{2.37}$$

Now pick the size $\mathfrak{c}$ family $\mathcal{D}$ of branches of the dyadic tree. Let $\iota_p\colon \ell_p(\mathcal{D}) \longrightarrow c_0(\mathcal{D})$ denote the canonical inclusion and form the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/c_0 & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle \iota} & & \\ 0 & \longrightarrow & c_0 & \longrightarrow & C_0(\wedge_{\mathcal{D}}) & \longrightarrow & c_0(\mathcal{D}) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle \iota_p} & & \\ 0 & \longrightarrow & c_0 & \longrightarrow & \mathsf{JL}_p & \longrightarrow & \ell_p(\mathcal{D}) & \longrightarrow & 0 \end{array} \tag{2.38}$$

The lower pullback space / sequence

$$\mathsf{JL}_p = \left\{(\xi, z)\colon \xi \in \ell_\infty, z \in \ell_p(\mathcal{D})\colon \xi + c_0 = \iota\,\iota_p\,(z)\right\}$$

is called the Johnson–Lindenstrauss space / sequence.

2.12.6 *The Johnson–Lindenstrauss sequence is non-trivial.*

Indeed, as in 2.2.10, no injective operator $\ell_p(\mathcal{D}) \longrightarrow \ell_\infty$ exists. In particular, JL_p is not WCG. The spaces JL_p were obtained in [225] and have the following surprising property:

Lemma 2.12.7 *JL_p and ℓ_∞ do not have isomorphic non-separable subspaces. Moreover, $\mathsf{JL}_p^* \simeq \ell_1 \times \ell_p^*(\mathcal{D})$ with duality given by $\langle (y^*, z^*), (y, z)\rangle = y^*(y) + z^*(z)$.*

Proof Let us show first that JL_p cannot have a countable norming set of functionals, which means that it cannot be a subspace of ℓ_∞. To this end, let (y_n^*, z_n^*) be a sequence of norm 1 functionals. Observe that $(1_{M_\alpha}, e_\alpha) \in \mathsf{JL}_p$. Consider the set M_0 formed by all $\gamma \in \mathcal{D}$ appearing in the support of all z_n^*.

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Take a sequence γ such that $\gamma(k) \notin M_0$ for all k and consider, for each $\gamma(k)$, the element $(1_{M_{\gamma(k)}}, e_{\gamma(k)})$. Take $N \in \mathbb{N}$ such that

$$\left\| \left(\frac{1}{\sqrt{k}} \sum_{k=1}^{N} 1_{M_{\gamma(k)}}, \frac{1}{\sqrt{k}} \sum_{k=1}^{N} e_{\gamma(k)} \right) \right\| = 1$$

This yields

$$\left| \left\langle (y_n^*, z_n^*), \left(\frac{1}{\sqrt{k}} \sum_{k=1}^{N} 1_{M_{\gamma(k)}}, \frac{1}{\sqrt{k}} \sum_{k=1}^{N} e_{\gamma(k)} \right) \right\rangle \right| = \left\langle y_n^*, \frac{1}{\sqrt{k}} \sum_{k=1}^{N} 1_{M_{\gamma(k)}} \right\rangle \leq \frac{1}{\sqrt{k}}$$

and thus the functionals cannot norm the space. A few minor changes make the previous argument work for arbitrary non-separable subspaces of JL_p that contain the canonical copy of c_0. If, however, X is non-separable but does not contain that c_0, then form the exact sequence $0 \longrightarrow X \longrightarrow [X+c_0] \longrightarrow c_0 \longrightarrow 0$ and use a straightforward 3-space argument to get that if X is a subspace of ℓ_∞ then $[X + c_0]$ must also be a subspace of ℓ_∞, which has been shown to be impossible. □

Proposition 8.7.18 asserts that the spaces $C_0(\wedge_{\mathcal{M}})$ can be different or equal depending on which cardinal axioms are assumed. It is our belief that different $C_0(\wedge_{\mathcal{M}})$ generate different JL_p spaces, but what happens after taking new pullbacks is unclear. Moving on in a different direction, Yost [461] constructs a twisted sum of c_0 and $\ell_p(\aleph_1)$ with properties completely different to those of the Johnson–Lindenstrauss space(s):

2.12.8 *For each $1 < p < \infty$, there exists an exact sequence*

$$0 \longrightarrow c_0 \longrightarrow \mathscr{Y}_p \longrightarrow \ell_p(\aleph_1) \longrightarrow 0$$

in which $\mathscr{Y}_p$ is a subspace of ℓ_∞.

Pick a continuous surjection $\gamma\colon \mathbb{N}^* \longrightarrow B^*_{\ell_p(\aleph_1)}$ and form the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & c_0 & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/c_0 & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle \gamma^\circ} & & \\
0 & \longrightarrow & c_0 & \longrightarrow & \mathrm{PB} & \longrightarrow & C(B^*_{\ell_p(\aleph_1)}) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle \delta} & & \\
0 & \longrightarrow & c_0 & \longrightarrow & \mathscr{Y}_p & \longrightarrow & \ell_p(\aleph_1) & \longrightarrow & 0
\end{array}$$

The map γ exists by Parovičenko's theorem since the dual unit ball of $\ell_p(\aleph_1)$ is, with its weak topology, a compact space of weight $\aleph_1$. Thus, γ° as well as $\gamma^\circ\delta$ is an isometry, which is what makes $\mathscr{Y}_p$ isometric to a subspace of ℓ_∞. The

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lower sequence cannot split since $c_0 \times \ell_p(\aleph_1)$ is not a subspace of ℓ_∞. A more careful analysis of Johnson–Lindenstrauss spaces can be found in [460; 396].

Twisted Sums of c_0 and ℓ_∞

2.12.9 *There exist non-trivial twisted sums of c_0 and ℓ_∞.*

Pick a weakly compact operator $\tau\colon \ell_\infty \longrightarrow \ell_\infty/c_0$ with non-separable range and form the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/c_0 & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle \tau} & & \\ 0 & \longrightarrow & c_0 & \longrightarrow & \mathsf{CC}(\tau) & \longrightarrow & \ell_\infty & \longrightarrow & 0 \end{array} \tag{2.39}$$

It is clear that the lower pullback sequence does not split since τ cannot be lifted to ℓ_∞ because any weakly compact operator $\ell_\infty \longrightarrow \ell_\infty$ has separable range. There are different ways to get such a τ:

- A composition $\tau = \imath \jmath_2 \pi$, where $\imath\colon c_0(I) \longrightarrow \ell_\infty/c_0$ is an embedding, $\jmath_2\colon \ell_2(I) \longrightarrow c_0(I)$ is the canonical inclusion and $\pi\colon \ell_\infty \longrightarrow \ell_2(I)$ is a quotient map. Such a quotient map exists when $\aleph_1 \leq |I| \leq \mathfrak{c}$: (a) observe that $\ell_1(\mathfrak{c})$ is a subspace of ℓ_∞ – a proof can be seen in [22, Claim 3, p. 138]; (b) any quotient map $Q\colon \ell_1(\mathfrak{c}) \longrightarrow \ell_2(\mathfrak{c})$ must be 2-summing [153, Theorem 3.1]; (c) extend Q to ℓ_∞. Thus, the lower sequence in the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & \mathsf{JL}_2 & \longrightarrow & \ell_2(I) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & c_0 & \longrightarrow & \mathsf{CC}(\imath \jmath_2 \pi) & \longrightarrow & \ell_\infty & \longrightarrow & 0 \end{array}$$

  is non-trivial. It is likely that different choices of τ generate different spaces $\mathsf{CC}(\tau)$. It is not known if there is a twisted sum of c_0 and ℓ_∞ that is not isomorphic to a $\mathscr{C}$-space.
- Use [300, Section 4].

Twisted Sums of $\mathscr{C}$-Spaces

A twisted sum of two $\mathscr{C}$-spaces can fail to have Pełczyński's property (V), as we show in Section 10.5, thus it does not have even to be isomorphic to a $\mathscr{C}$-space. The first example of such a phenomenon appears in [102, 3.5] and is worked out in detail in [22, 2.2.6]. The reason we talk about this here is because it is based on a wonderfully clever construction of Benyamini [39] whose bricks and mortar are, however, simple pullback. Indeed, the counterexample depends on Benyamini's argument that given a multiplication operator $x \mapsto \theta x$, the pullback space PB_θ in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/c_0 & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle\theta} & & \\ 0 & \longrightarrow & c_0 & \longrightarrow & \mathrm{PB}_\theta & \longrightarrow & \ell_\infty/c_0 & \longrightarrow & 0 \end{array} \tag{2.40}$$

which is clearly a renorming of ℓ_∞, is such that $\|u\|\|u^{-1}\|\|P\| \geq \theta^{-1}$ for every compact K, every embedding $u\colon \mathrm{PB}_\theta \longrightarrow C(K)$ and every projection $P\colon C(K) \longrightarrow u[\mathrm{PB}_\theta]$. The rest is oil steadily flowing down an inclined plane: both $\ell_\infty(\mathbb{N}, \mathrm{PB}_{1/n})$ and $c_0(\mathbb{N}, \mathrm{PB}_{1/n})$ are twisted sums of two $\mathscr{C}$-spaces, for instance

$$0 \longrightarrow c_0(\mathbb{N}, c_0) \longrightarrow c_0(\mathbb{N}, \mathrm{PB}_{1/n}) \longrightarrow c_0(\mathbb{N}, C(\mathbb{N}^*)) \longrightarrow 0$$

thus they cannot be complemented in any $\mathscr{C}$-space. Whether an anodyne space such as $c_0(I)$ can play the role of $C(\mathbb{N}^*)$ is a topic to discuss around the bonfire.

2.13 The Device

As we know, given an embedding $\jmath\colon Y \longrightarrow A$ and an operator $\tau\colon Y \longrightarrow B$, the pushout space can be understood as a superspace of B such that τ admits an extension $A \longrightarrow \mathrm{PO}$. Can we do the same with a family $(\imath_i\colon Y_i \longrightarrow A_i)_{i\in I}$ of embeddings and a family $(\tau_i\colon Y_i \longrightarrow B_i)_{i\in I}$ of operators? Under some reasonable restrictions, we certainly can. First condition: I must be a set, and all spaces involved must be p-normed spaces for some fixed p. Otherwise, the amalgamation of the spaces becomes complicated. Second condition: the operators must be uniformly bounded. Otherwise, their amalgamation becomes a nice linear map, not an operator. Third condition: when we work with embeddings (or quotients), they must be uniformly open. Otherwise, their amalgamation becomes a nice operator, not an embedding (or quotient). Having accepted those conditions, we can paste all embeddings into one single embedding $\prod \jmath_i\colon \ell_p(I, Y_i) \longrightarrow \ell_p(I, A_i)$, form the operator $\oplus\tau_i\colon \ell_p(I, Y_i) \longrightarrow B$ and obtain the pushout in $p\mathbf{B}$:

$$\begin{array}{ccc} \ell_p(I, Y_i) & \xrightarrow{\ \prod \jmath_i\ } & \ell_p(I, A_i) \\ {\scriptstyle\oplus\tau_i}\downarrow & & \downarrow{\scriptstyle\overline{\oplus\tau_i}} \\ B & \xrightarrow{\ \overline{\prod \jmath_i}\ } & \mathrm{PO} \end{array} \tag{2.41}$$

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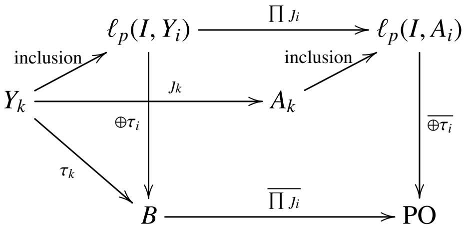
Observe that $\overline{\oplus\tau_i}$ provides an extension $\overline{\oplus\tau_i}|_{A_k}$ of each τ_k since the following diagram is commutative:



The iteration of this construction yields a rather flexible device to construct spaces with additional properties. Examples? Consequences? *Il catalogo è questo*: the p-Gurariy spaces, the p-Kadets spaces, the Bourgain–Pisier spaces, the Kubiś space, many other spaces of universal disposition, Lindenstrauss and $\mathscr{L}_{\infty,\lambda}$-envelopes etc. All these topics and examples will be treated from various angles in this book. Let us now present a detailed account of how this technique works. Even if the Device looks like a Turing machine, it actually works more like a Thermomix: with adequately chosen ingredients (data), a recipe and continuous attention will produce a wholesome (quasi-) Banach space.

Ingredients and recipe

- Pick a scalar $0 < p \leq 1$ and a p-Banach space X_0. These indicate the p-Banach category in which we will be working and the 'initial' object.
- Fix an ordinal μ. This indicates the length of the iteration and, to some extent, the size of the space to be obtained. Usually, μ is a limit ordinal.
- Construct an inductive system of p-Banach spaces $(X_\alpha)_{0\leq\alpha\leq\mu}$ by transfinite induction on α, starting with X_0. We will do that as follows: first assume that $(X_\alpha)_{0\leq\alpha<\beta}$ has been constructed for all $\alpha < \beta$. If β is a limit ordinal, we set $X_\beta = \lim_{\alpha<\beta} X_\alpha$, namely the completion of $\bigcup_{\alpha<\beta} X_\alpha$. If $\beta = \alpha + 1$, we will perform a pushout as described before, which we explain now in detail. We need to add two new ingredients: a uniformly bounded family $\mathfrak{J}_\alpha$ of embeddings between p-Banach spaces and a uniformly bounded family of operators $\mathfrak{L}_\alpha$ with values in X_α. Both $\mathfrak{J}_\alpha$ and $\mathfrak{L}_\alpha$ have to be *sets*. The space $X_{\alpha+1}$ that we will construct next will allow us to extend any operator $u \in \mathfrak{L}_\alpha$ through any embedding $v \in \mathfrak{J}_\alpha$ whenever this makes sense. Consider the set $I_\alpha = \{(u, v) \in \mathfrak{L}_\alpha \times \mathfrak{J}_\alpha : \operatorname{dom}(u) = \operatorname{dom}(v)\}$ and the ℓ_p-sums $\ell_p(I_\alpha, \operatorname{dom}(v))$ and $\ell_p(I_\alpha, \operatorname{cod}(v))$. There is an obvious operator $\prod \mathfrak{J}_\alpha : \ell_p(I_\alpha, \operatorname{dom}(v)) \longrightarrow \ell_p(I_\alpha, \operatorname{cod}(v))$ sending $(x_{(u,v)})_{(u,v)\in I_\alpha}$ to $(v(x_{(u,v)}))_{(u,v)\in I_\alpha}$. There is another obvious operator $\bigoplus \mathfrak{L}_\alpha : \ell_p(I_\alpha, \operatorname{dom}(u)) \longrightarrow X_\alpha$ sending $(x_{(u,v)})_{(u,v)\in I_\alpha}$ to

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$\sum_{(u,v)\in I_\alpha} u(x_{(u,v)})$. The notation is slightly imprecise because those operators depend, not only on $\mathfrak{L}_\alpha$ and $\mathfrak{J}_\alpha$, but also on I_α. Now we take the pushout

$$\begin{array}{ccc} \ell_p(I_\alpha, \operatorname{dom}(u)) = \ell_p(I_\alpha, \operatorname{dom}(v)) & \xrightarrow{\prod \mathfrak{J}_\alpha} & \ell_p(I_\alpha, \operatorname{cod}(v)) \\ {\scriptstyle \bigoplus \mathfrak{L}_\alpha}\downarrow & & \downarrow \\ X_\alpha & \longrightarrow & \mathrm{PO} \end{array} \tag{2.42}$$

set $\mathrm{PO} = X_{\alpha+1}$ and use the lower arrow to embed X_α into $X_{\alpha+1}$.

- The recipe is finished by iterating the construction until μ. The output will be a p-Banach space X_μ plus an isometry $X_0 \longrightarrow X_\mu$.

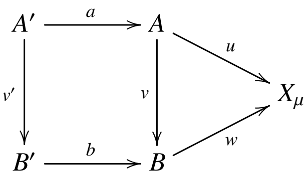
Some Device constructions are Fraïssé limits, a topic to which Chapter 6 is entirely devoted. As a product demo of the Device in action, let us construct p-Banach spaces U having the following extension property, which is usually said 'to be of universal disposition for separable spaces' (SUD): given an isometry $v\colon A \longrightarrow B$ between separable p-Banach spaces and an isometry $u\colon A \longrightarrow U$, there is an isometry $w\colon B \longrightarrow U$ such that $u = wv$. To be able to proceed, let us confirm that there is a *set*, $\mathscr{S}_p = \{\ell_p/Y\colon Y \text{ is a closed subspace of } \ell_p\}$, containing an isometric copy of every separable p-Banach space. In fact, $|\mathscr{S}_p| = \mathfrak{c}$.

2.13.1 Recipe for Spaces of Separable Universal Disposition

- Work in the category $p\mathbf{B}$; pick as X your favourite p-Banach space of dimension up to $\mathfrak{c}$ and pick any ordinal $\mu \le \mathfrak{c}$ of uncountable cofinality, say ω_1.
- Set $X_0 = X$; once X_α is constructed, fix $\mathfrak{L}_\alpha$ as the set of isometries $u\colon A \longrightarrow X_\alpha$, with domain in $\mathscr{S}_p$; the set $\mathfrak{J} = \mathfrak{J}_\alpha$ is the same for all α: the set of all isometries with domain and codomain in $\mathscr{S}_p$.

Let us observe the output space X_μ in some detail. It is of SUD: let $v\colon A \longrightarrow B$ an isometry between separable p-Banach spaces and let $u\colon A \longrightarrow X_\mu$ be an isometry. We can assume that v belongs to $\mathfrak{J}$ since there are surjective isometries $a\colon A' \longrightarrow A$ and $b\colon B' \longrightarrow B$ with A', B' in $\mathscr{S}_p$, and so $v' = b^{-1}va$ belongs to $\mathfrak{J}$. Now, letting $u' = ua$, it is clear that if w' is an isometry such that

$u' = w'v'$ then $w = w'b^{-1}$ is the required extension of u. Don't believe us, just watch:

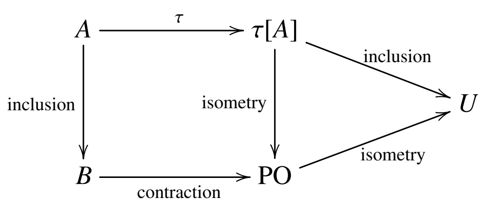


As μ has uncountable cofinality, $X_\mu = \bigcup_{\alpha<\mu} X_\alpha$, since this last space is already complete. It follows that $u[A] \subset X_\alpha$ for some $\alpha < \mu$ and so $u \in \mathfrak{L}_\alpha$, when u is interpreted as an isometry $A \longrightarrow X_\alpha$. Therefore, the pair (u, v) belongs to I_α, and thus there is $w\colon B \longrightarrow X_{\alpha+1}$ such that $wv = \imath_{\alpha,\alpha+1}u$, where $\imath_{\alpha,\alpha+1}\colon X_\alpha \longrightarrow X_{\alpha+1}$ is the inclusion map. So, X_μ is of SUD, and, since $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$, $\dim X_\mu = \mathfrak{c}$ (actually $\dim X_1$ is already $\mathfrak{c}$ by the proposition in Note 6.5.1). The interested chef can find inspiration for their own elaborations and variations on this recipe in either [22, Chapter 3] or Proposition 7.3.2. Here, we prove:

Proposition 2.13.2

(a) *Any p-Banach space of SUD is* 1*-separably injective in p***B**.
(b) *Any p-Banach space of SUD contains isometric copies of all p-Banach spaces of dimension up to* $\aleph_1$.
(c) [CH] *All p-Banach spaces of SUD with dimension* $\aleph_1$ *are isometric.*

Proof (a) Let B be a separable p-Banach space and let $\tau\colon A \longrightarrow U$ be a contractive operator, where A is a subspace of B. If U is of separable universal disposition, there is a commutative diagram



and so τ has a 1-extension to B. (b) Every p-Banach space of density $\aleph_1$ or less can be written as the union of a continuous ω_1-chain of separable subspaces: $X = \bigcup_{\alpha<\omega_1} X_\alpha$. This means that $X_\alpha \subset X_\beta$ for $\alpha < \beta$ and that $X_\beta = \overline{\bigcup_{\alpha<\beta} X_\alpha}$ when β is a limit ordinal. We may assume that $X_0 = 0$. If U is of SUD, we can construct a compatible system ($u_\beta|_{X_\alpha} = u_\alpha$ for $\alpha < \beta < \omega_1$) of isometries $u_\alpha\colon X_\alpha \longrightarrow U$ as follows: u_0 must be 0; then, assuming that u_α has been defined for all $\alpha < \beta$, we define u_β by continuity if β is a limit ordinal and using the SUD of U when $\beta = \alpha + 1$ to extend $u_\alpha\colon X_\alpha \longrightarrow U$ to

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an isometry $u_{\alpha+1}: X_{\alpha+1} \longrightarrow U$. After that, just define $u: X \longrightarrow U$ by declaring $u(x) = u_\alpha(x)$ if $x \in X_\alpha$. (c) Assume U and V are of SUD and density $\aleph_1$ and write them as $U = \bigcup_{\alpha<\omega_1} U_\alpha$ and $V = \bigcup_{\beta<\omega_1} V_\beta$, where (U_α) and (V_β) are continuous ω_1-chains of separable subspaces with $U_0 = V_0 = 0$. Set $U_{\omega_1} = U$ and $V_{\omega_1} = V$. Consider the set S of all triples (α, β, f), where $\alpha, \beta \in [0, \omega_1]$ and $f: U_\alpha \longrightarrow V_\beta$ is a surjective isometry. Declare $(\alpha, \beta, f) \leq (\gamma, \delta, g)$ if $\alpha \leq \gamma, \beta \leq \delta$ and $g|_{U_\alpha} = f$. The set S is not empty since $(0, 0, 0) \in S$. A maximal element (α, β, f) exists by Zorn's lemma since every chain admits an upper bound. We end the proof by showing that $\alpha = \beta = \omega_1$. Otherwise, we inductively define sequences $\alpha \leq \alpha_1 \leq \alpha_2 \leq \dots$ and $\beta \leq \beta_1 \leq \beta_2 \leq \dots$ and isometries $f_n: U_{\alpha_n} \longrightarrow V_{\beta_n}$ and $f_n^{-1}: V_{\beta_n} \longrightarrow U_{\alpha_{n+1}}$ with $f_{n+1}|_{U_{\alpha_n}} = f_n$ in the obvious way: assuming α_n, β_n, f_n have been obtained, extend f_n^{-1} to an isometry $g_n: V_{\beta_n} \longrightarrow g_n[V_{\beta_n}]$ and set α_{n+1} to be the smallest ordinal such that $g_n[V_{\beta_n}] \subset U_{\alpha_{n+1}}$ and then extend f_n to an isometry $f_{n+1}: U_{\alpha_{n+1}} \longrightarrow f_{n+1}[U_{\alpha_{n+1}}]$ and set β_{n+1} to be the smallest ordinal such that $f_{n+1}[U_{\alpha_{n+1}}] \subset V_{\beta_{n+1}}$. Set $\alpha' = \sup_n \alpha_n$ and $\beta' = \sup_n \beta_n$ and let the continuity of the chain produce a surjective isometry $f': U_{\alpha'} \longrightarrow V_{\beta'}$ extending f, in flagrant contradiction of the alleged maximality. □

The Bourgain–Pisier Construction

Bourgain and Pisier showed in [52] that, for each $\lambda > 1$, every separable Banach space X can be embedded into some $\mathscr{L}_{\infty,\lambda}$-space $\mathscr{L}_\infty^{\mathsf{BP}}(X)$ in such a way that the corresponding quotient space $\mathscr{L}_\infty^{\mathsf{BP}}(X)/X$ has the Schur property and the RNP. Their construction is a clever iteration of pushouts in which the embeddings are no longer isometries. Assume that $X = \overline{\bigcup_{n\geq1} X_n}$, where (X_n) is a chain of finite-dimensional subspaces. Fix $\lambda > 1$ and then take $\eta \in (\lambda^{-1}, 1)$. Set $a(1) \in \mathbb{N}$ such that there is a subspace $S_1 \subset \ell_\infty^{a(1)}$ and an isomorphism $u_1: S_1 \longrightarrow X_1$ with $\|u_1\| \leq \eta$ and $\|u_1^{-1}\| \leq \lambda$. Form the consecutive pushouts

$$\begin{array}{ccc} S_1 & \xrightarrow{\text{inclusion}} & \ell_\infty^{a(1)} \\ {\scriptstyle u_1}\downarrow & & \downarrow{\scriptstyle \overline{u_1}} \\ X_1 & \longrightarrow & \mathrm{PO}_1 \end{array} \qquad \begin{array}{ccc} X_1 & \longrightarrow & \mathrm{PO}_1 \\ {\scriptstyle \text{inclusion}}\downarrow & & \downarrow \\ X_2 & \longrightarrow & \mathrm{PO}_1' \end{array}$$

and continue inductively: set $a(n) \in \mathbb{N}$ such that there is $S_n \subset l_\infty^{a(n)}$ and an isomorphism $u_n: S_n \longrightarrow \mathrm{PO}_n'$ with $\|u_n\| \leq \eta$ and $\|u_n^{-1}\| \leq \lambda$ and form new pushouts

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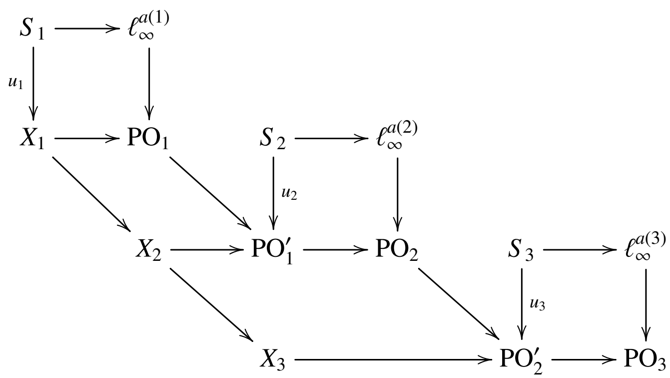
$$\begin{array}{ccc} S_n & \xrightarrow{\text{inclusion}} & \ell_\infty^{a(n)} \\ {\scriptstyle u_n}\downarrow & & \downarrow{\scriptstyle \overline{u_n}} \\ \mathrm{PO}'_n & \longrightarrow & \mathrm{PO}_{n+1} \end{array}$$

The space $\mathscr{L}_\infty^{\mathsf{BP}}(X) = \lim \mathrm{PO}_n$ is an $\mathscr{L}_{\infty,\lambda}$-space since it is the inductive limit of the spaces PO_n, which are λ-isomorphic to $\ell_\infty^{a(n)}$. A diagram might illuminate the construction



(2.43)

The trickier part of the proof is showing that $\mathscr{L}_\infty^{\mathsf{BP}}(X)/X$ has the Schur and RNP. Both properties follow from an imaginative idea. An isometry $C \longrightarrow D$ is η-*admissible*, where $0 < \eta < 1$, if it can be placed in a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{b} & B \\ {\scriptstyle c}\downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in which b is an isometry and $\|c\| \leq \eta$. The following result [52, Theorem 1.6], not proved here, is the key to the argument:

2.13.3 *Let* $0 \leq \eta < 1$. *The direct limit of a sequence of* η-*admissible isometries has the Schur and RNP.*

To be able to apply the result to our situation, let us complete Diagram (2.43) with new pushouts into X:

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$$\begin{array}{ccc} S_1 & \longrightarrow & \ell_\infty^{a(1)} \\ {\scriptstyle u_1}\downarrow & & \downarrow \\ X_1 & \longrightarrow & \mathrm{PO}_1 \\ \downarrow & & \downarrow \\ X & \longrightarrow & E_n \end{array}$$

Here $X \longrightarrow E_1$ is η-admissible. Now observe that E_1 contains PO_2', thus

$$\begin{array}{ccc} S_2 & \longrightarrow & \ell_\infty^{a(2)} \\ {\scriptstyle u_2}\downarrow & & \downarrow \\ \mathrm{PO}_1' & \longrightarrow & \mathrm{PO}_2 \\ \downarrow & & \downarrow \\ E_1 & \longrightarrow & E_2 \end{array}$$

yields the new η-admissible map $E_1 \longrightarrow E_2$. Continue this way and consider the direct limit $Y = \lim_n E_n$. This Y is a limit of η-admissible maps, but not between finite-dimensional spaces. However, $Y/X = \lim_n E_n/X$ is a limit of η-admissible maps between finite-dimensional spaces, and thus Y/X is a Schur space with the RNP, as is its subspace $\mathscr{L}_\infty^{\mathsf{BP}}(X)/X$ since, obviously, Y contains $\mathscr{L}_\infty^{\mathsf{BP}}(X)$ and those properties pass to subspaces [155, III. Theorem 2]. The properties of the embedding $X \longrightarrow \mathscr{L}_\infty^{\mathsf{BP}}(X)$ will be studied between Propositions 10.6.9 and 10.6.11. López-Abad extends the Bourgain–Pisier construction to arbitrary Banach spaces in [340].

2.14 Extension and Lifting of Operators

The extension of operators is one of the most classical problems. It consists of determining when, given an operator $\tau\colon Y \longrightarrow E$ and an isomorphic embedding $\jmath\colon Y \longrightarrow X$, there exists an extension $T\colon X \longrightarrow E$; i.e. an operator T such that $T\jmath = \tau$. The lifting problem is entirely dual in its formulation, although maybe not in the results we get: determine when, given a quotient operator $\rho\colon X \longrightarrow Z$ and an operator $\tau\colon E \longrightarrow Z$, there exists a lifting for τ; i.e. an operator $T\colon E \longrightarrow X$ such that $\rho T = \tau$. Homological techniques provide tools to treat these questions (which is what we are doing throughout

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this book). As a rule, operators between quasi-Banach spaces do not extend. The simplest example to mention is the identity operator: it cannot be extended unless the subspace is complemented. The extension problem admits a natural formulation in homological terms: an operator $\tau\colon Y \longrightarrow E$ and an embedding $\jmath\colon Y \longrightarrow Z$ form the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\ \jmath\ } & Z & \longrightarrow & X & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\tau} & & \downarrow & & \| & & \\
0 & \longrightarrow & E & \xrightarrow{\ \bar\jmath\ } & \mathrm{PO} & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

The splitting criterion for pushout sequences Lemma 2.6.3 yields that τ can be extended to Z if and only if the lower sequence is trivial. If we consider instead the question of when all operators $Y \longrightarrow E$ can be extended through the embedding $\jmath$ then we see that this happens if and only if the restriction operator $j^\circ\colon \mathfrak{L}(Z,E) \longrightarrow \mathfrak{L}(Y,E)$ is surjective. A different question, considered throughout this book, is that of when all operators $Y \longrightarrow E$ can be extended through any embedding $Y \longrightarrow Z$ such that $Z/Y = X$. As a rule, operators between quasi-Banach spaces cannot be lifted (a quotient map can be lifted if and only if its kernel is complemented). The lifting problem admits a natural formulation in homological terms: a quotient map $\rho\colon Z \longrightarrow X$ and an operator $\tau\colon E \longrightarrow X$ can be assembled in a pullback diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & Z & \xrightarrow{\ \rho\ } & X & \longrightarrow & 0\\
 & & \| & & \uparrow & & \uparrow{\scriptstyle\tau} & & \\
0 & \longrightarrow & Y & \longrightarrow & \mathrm{PB} & \longrightarrow & E & \longrightarrow & 0
\end{array}$$

for which we know (Lemma 2.8.3) that τ can be lifted through ρ if and only if the pullback sequence splits. That all operators $E \longrightarrow X$ can be lifted to Z through ρ can be reformulated as follows: the operator $\rho_\circ\colon \mathfrak{L}(E,Z) \longrightarrow \mathfrak{L}(E,X)$ is surjective. We will also consider the question whether all operators $E \longrightarrow X$ can be lifted to Z through any quotient $Z \longrightarrow X$ whose kernel is Y.

Extension: $\mathscr{A}$-Trivial Sequences

The following notation unifies different ideas scattered through the literature and, most importantly, is useful. Let $\mathscr{A}$ be a class of quasi-Banach spaces.

Definition 2.14.1 An emdedding $\jmath\colon Y \longrightarrow Z$ is $\mathscr{A}$-trivial if, for every $A \in \mathscr{A}$, every operator $\tau\colon Y \longrightarrow A$ has an extension $T\colon Z \longrightarrow A$. If this can be achieved with $\|T\| \le \lambda\|\tau\|$, we say that $\jmath$ is $(\lambda, \mathscr{A})$-trivial.

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This is really a property of the inclusion of $\jmath[Y]$ in Z, but some flexibility is convenient here. These definitions extend to short exact sequences by declaring sequences to be $\mathscr{A}$-trivial when their embeddings are. If the reader wonders whether $\mathscr{A}$-trivial sequences and homology like to mingle, just observe that given a fixed space A, the (contravariant; see Chapter 4 for details) functor $\mathfrak{L}(\cdot, A)$ takes each exact sequence $0 \longrightarrow Y \xrightarrow{\jmath} Z \xrightarrow{\rho} X \longrightarrow 0$ into the exact sequence

$$0 \longrightarrow \mathfrak{L}(X,A) \xrightarrow{\rho^\circ} \mathfrak{L}(Z,A) \xrightarrow{\jmath^\circ} \mathfrak{L}(Y,A)$$

(open end!); $\jmath^\circ$ is surjective (so the diagram can be completed with a right 0) if and only if the sequence is A-trivial in the obvious sense. Chapter 8 is devoted to $\mathscr{C}$-trivial sequences. The behaviour of $\mathscr{A}$-trivial sequences under pullback/pushout constructions is as follows:

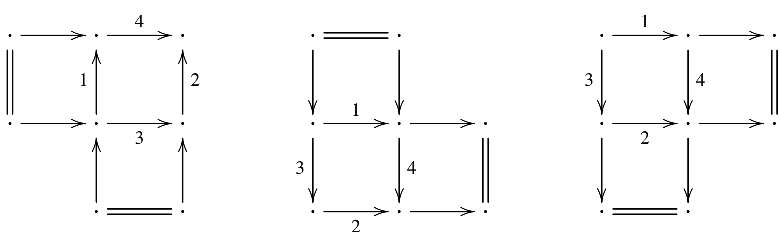
Proposition 2.14.2 *Pullbacks and pushouts preserve $\mathscr{A}$-trivial sequences.*

Proof The case of pullbacks is obvious. Assume we have a pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \overline{\alpha}} & & \| & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\overline{\jmath}} & \mathrm{PO} & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

in which the upper row is A-trivial. Given an operator $\tau\colon Y' \longrightarrow A$, the composition $\tau\alpha$ extends to an operator $T\colon Z \longrightarrow A$ through $\jmath$. As $T\jmath = \tau\alpha$ the universal property of the pushout yields an operator $\gamma\colon \mathrm{PO} \longrightarrow A$ such that $\gamma\overline{\jmath} = \tau$ and $\gamma\overline{\alpha} = T$. □

Lemma 2.14.3 *In each of the following commutative diagrams, in which any three aligned points represent a short exact sequence of quasi-Banach spaces,*



the sequences 1 *and* 3 *are $\mathscr{A}$-trivial if and only if the sequences* 2 *and* 4 *are $\mathscr{A}$-trivial.*

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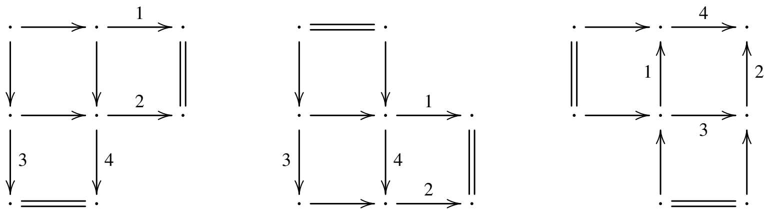
Proof The proof is a mere diagram chase keeping Proposition 2.14.2 in mind. A more homologically flavoured proof can be given. Fix $A \in \mathscr{A}$ and apply the functor $\mathfrak{L}(\cdot, A)$. This transforms the three previous diagrams into



with open right ends now since, we already mentioned, $\mathfrak{L}(\cdot, A)$ is not necessarily exact at the subspace. Which is why we have passed the identification numbers to the 'quotient' maps. It is now easy to check that 1 and 3 are surjective if and only if 2 and 4 are surjective. □

Lifting: M-Ideals

According to the Yellow Book of M-idealism [209, Definition I.1.1], a closed subspace J of a Banach space X is an M-ideal if the annihilator $J^\perp = \{x^* \in X^* : \langle x^*, x\rangle = 0\ \forall\, x \in J\}$ is an L-summand in X^*, which means that it is the range of an L-projection. Just in case, recall that a projection $P \in \mathfrak{L}(X)$ is called an L-projection if $\|x\| = \|P(x)\| + \|x - P(x)\|$ for all $x \in X$, and it is called an M-projection if $\|x\| = \max(\|P(x)\|, \|x - P(x)\|)$ for all $x \in X$. The simplest examples of M-ideals are the ideals $J = \{f \in C(K) : f|_S = 0\}$ for some closed $S \subset K$ in a $C(K)$-space, as a direct consequence of the Riesz representation of $C(K)^*$. But there are many more:

Lemma 2.14.4 *If $(X_n)_n$ is an increasing sequence of subspaces of X with $X = \overline{\bigcup_n X_n}$ then $c_0(\mathbb{N}, X_n)$ is an M-ideal in $c(\mathbb{N}, X_n)$. If $(A_i)_{i\in I}$ is a family of Banach spaces and $\mathfrak{U}$ is a free ultrafilter on I then $c_0(I, A_i)$ and $c_0^{\mathfrak{U}}(I, A_i)$ are M-ideals in $\ell_\infty(I, A_i)$.*

Proof For the first part, define the desired L-projection in the form

$$\langle P\mu, (x_n)\rangle = \lim_n \langle \mu, (0, \ldots, 0, x_n, x_{n+1}, \ldots)\rangle$$

for $\mu \in c(\mathbb{N}, X_n)^*$ and $(x_n)_n \in c(\mathbb{N}, X_n)$, which makes sense because μ is applied to a weakly Cauchy sequence. P is a projection onto $c_0(\mathbb{N}, X_n)^\perp$, and if one picks $\varepsilon > 0$ and normalised sequences $(y_n)_n$ and $(z_n)_n$ such that $\langle P\mu, (y_n)\rangle > \|P\mu\| - \varepsilon$ and $\langle \mu - P\mu, (z_n)\rangle > \|\mu - P\mu\| - \varepsilon$, then

$$\|\mu\| \geq \sup_n \langle \mu, (z_1, \ldots, z_n, y_{n+1}, y_{n+2}, \ldots)\rangle \geq \|P\mu\| + \|\mu - P\mu\| - 2\varepsilon.$$

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The second part requires a different approach avoiding duality since the dual of $\ell_\infty(I, A_i)$ is unmanageable. It turns out [209, Theorem I.2.2] that J is an M-ideal in X if and only if, for every finite family of closed balls $B(x^k, r_k)$ in X such that $B(x^k, r_k) \cap J \neq \emptyset$ for all k and every $\varepsilon > 0$, we have

$$\bigcap_k B(x^k, r_k) \neq \emptyset \qquad \Longrightarrow \qquad \bigcap_k B(x^k, r_k + \varepsilon) \cap J \neq \emptyset.$$

Let us check this condition for $c_0^{\mathfrak{U}}(I, A_i)$, the case of $c_0(I, A_i)$ being simpler. Let $B(x^k, r_k)$ be the corresponding balls, and take $x = (x_i)$ in their intersection. Also, for each k, pick $y^k \in B(x^k, r_k) \cap c_0^{\mathfrak{U}}(I, A_i)$. Now, given $\varepsilon > 0$, as $\|y_i^k\| \longrightarrow 0$ along $\mathfrak{U}$, we may find I_ε in $\mathfrak{U}$ such that $\|y_i^k\| \leq \varepsilon$ for all k and all $i \in I_\varepsilon$. If we define $y = (y_i)$, setting $y_i = 0$ for $i \in I_\varepsilon$ and $y_i = x_i$ otherwise, it is clear that $y \in \bigcap_k B(x^k, r_k + \varepsilon) \cap c_0^{\mathfrak{U}}(I, A_i)$. □

The following remarkable connection between M-ideals and lifting properties was discovered by Ando [11] and, almost simultaneously and independently, in a slightly weaker form, by Choi and Effros [138].

Theorem 2.14.5 *Let J be an M-ideal in the Banach space Z, and let Y be a separable Banach space with the λ-AP. Every operator $T\colon Y \longrightarrow Z/J$ admits a lifting $L\colon Y \longrightarrow Z$ such that $\|L\| \leq \lambda\|T\|$.*

The result can be seen as a wide generalisation of the Borsuk–Dugundji theorem. A complete proof can be found in [209, Theorem II.2.1]. What we present here is a modulo ε proof that suffices for our qualitative purposes. A few comments on the $\varepsilon = 0$ case can be found at the end of this section. A simple observation to warm up is that if Q is an M-projection on a Banach space X then

$$\|Q(x) + (\mathbf{1}_X - Q)(y)\| \leq \max\{\|x\|, \|y\|\}. \tag{2.44}$$

The crucial step in the proof of Theorem 2.14.5 is the following magical

Lemma 2.14.6 *Let J be an M-ideal in the Banach space Z. Let E be a finite-dimensional space, and let F be a 1-complemented subspace of E. Let $T\colon E \longrightarrow Z/J$ be an operator. Then, for every lifting $L_F\colon F \longrightarrow Z$ of $T|_F$ and every $\varepsilon > 0$, there is a lifting $L_E\colon E \longrightarrow Z$ of T such that $L_E = L_F$ on F and $\|L_E\| < (1 + \varepsilon)\max(\|T\|, \|L_F\|)$.*

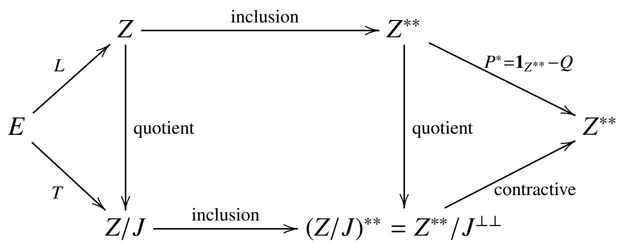
Proof The lifting we are looking for is a point of small norm in the space $\mathfrak{L}(E, Z)$. The action takes place in the bidual $\mathfrak{L}(E, Z)^{**}$, identified with $\mathfrak{L}(E, Z^{**})$ thanks to Dean's identity [145]. The simplest linear functionals on $\mathfrak{L}(E, Z)$ have the form $S \longmapsto \langle z^*, S(e)\rangle$, for fixed $z^* \in Z^*, e \in E$, and these generate the dual of $\mathfrak{L}(E, Z)$. Thus, typical neighbourhoods of zero in the weak topology of $\mathfrak{L}(E, Z)$ and the weak* topology of $\mathfrak{L}(E, Z^{**})$ are given, respectively, by $\{S \in \mathfrak{L}(E, Z) : |\langle x^*, Se\rangle| \le 1\}$ and $\{S \in \mathfrak{L}(E, Z^{**}) : |\langle x^*, Se\rangle| \le 1\}$, where $x^* \in X^*$ and $e \in E$. We consider the following subspaces of $\mathfrak{L}(E, Z)$:

- $W = \{S \in \mathfrak{L}(E, Z) : S[E] \subset J\} = \mathfrak{L}(E, J)$.
- $V = \{S \in W : S|_F = 0\}$.

Clearly, $W^{\perp\perp} = \{S \in \mathfrak{L}(E, Z^{**}) : S[E] \subset J^{\perp\perp}\} = \mathfrak{L}(E, J^{\perp\perp})$ and $V^{\perp\perp} = \{S \in W^{\perp\perp} : S|_F = 0\}$. The operator $Q_\circ$ sending T to QT is a weak* continuous M-projection on $\mathfrak{L}(E, Z)^{**}$ of range $W^{\perp\perp}$; in fact, W is an M-ideal in $\mathfrak{L}(E, Z)$, but this fact is not to be used in the ensuing argument. Let $T \in \mathfrak{L}(E, Z/J)$ be an operator and let $L_F : F \longrightarrow Z$ be a lifting of $T|_F$. Fix a contractive projection $\pi : E \longrightarrow F$. Let $L \in \mathfrak{L}(E, Z)$ be *any* lifting of T such that $L|_F = T_F$ and, using the two projections at hand, write

$$L = (\mathbf{1}_{Z^{**}} - Q)L + QL = (\mathbf{1}_{Z^{**}} - Q)L + QL\pi + QL(\mathbf{1}_E - \pi). \tag{2.45}$$

Obviously, $QL(\mathbf{1}_E - \pi) \in V^{\perp\perp}$. With an eye on (2.44), we want to bound the other two chunks: we have $QL\pi = QL_F\pi$, so $\|QL\pi\| \le \|L_F\|$. As for the first summand, since $\mathbf{1}_{Z^{**}} - Q = P^*$ vanishes on $J^{\perp\perp}$, we have a commutative diagram



and so $\|(\mathbf{1}_{Z^{**}} - Q)L\| \le \|T\|$. Thus, $\|(\mathbf{1}_{Z^{**}} - Q)L + QL\pi\| \le \max(\|(\mathbf{1}_{Z^{**}} - Q)L\|, \|QL\pi\|) \le \max(\|T\|, \|L_F\|)$. Hence, letting $r = \max(\|T\|, \|L_F\|)$,

$$L \in rB_{\mathfrak{L}(E,Z^{**})} + V^{\perp\perp} = \overline{rB_{\mathfrak{L}(E,Z)}}^{\text{weak*}} + \overline{V}^{\text{weak*}} = \overline{rB_{\mathfrak{L}(E,Z)} + V}^{\text{weak*}},$$

and since $L \in \mathfrak{L}(E, Z)$, the weak* topology of $\mathfrak{L}(E, Z^{**})$ restricted to $\mathfrak{L}(E, Z)$ is its weak topology, and the weak and norm closures of convex sets coincide, $L \in \overline{rB_{\mathfrak{L}(E,Z)} + V}$. Therefore, given $\varepsilon > 0$, there exist $L' \in \mathfrak{L}(E, Z)$ with $\|L'\| \le r$ and $S \in V$ such that $\|L - L' - S\| < \varepsilon$, and so $L_E = L - S$ is a lifting of T that extends L_F, with $\|L_E\| \le r + \varepsilon$, which is enough. □

Proof of Theorem 2.14.5 It suffices to prove the result assuming that Y has a 1-FDD by Lemma 2.2.20. Write $Y = \overline{\bigcup_n Y_n}$ for an increasing chain of finite-dimensional subspaces $(Y_n)_{n\ge 0}$ with $Y_0 = 0$ and Y_n 1-complemented in Y_{n+1}. Fix $\varepsilon > 0$, and pick a sequence $(\varepsilon_n)_n$ such that $\prod_n(1 + \varepsilon_n) < 1 + \varepsilon$. Use the preceding lemma inductively to find a sequence of operators $L_n \colon Y_n \longrightarrow Z$ such that (a) L_n is a lifting of $T|_{Y_n}$; (b) L_{n+1} is an extension of L_n and (c) $\|L_n\| \le \prod_{1\le k\le n}(1+\varepsilon_k)$. These operators define the required lifting of T. □

The hypotheses on Y of Theorem 2.14.5 cannot be removed without asking something else. The separability is necessary: consider the family of Hilbert spaces $(\ell_2^n)_{n\ge 1}$, a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and the exact sequence defining the ultraproduct $0 \longrightarrow c_0^{\mathcal{U}}(\mathbb{N}, \ell_2^n) \longrightarrow \ell_\infty(\mathbb{N}, \ell_2^n) \longrightarrow [\ell_2^n]_{\mathcal{U}} \longrightarrow 0$. Since $[\ell_2^n]_{\mathcal{U}}$ is a Hilbert space of density $\mathfrak{c}$, every subspace Y has 1-AP, but if Y is non-separable then the inclusion $Y \longrightarrow [\ell_2^n]_{\mathcal{U}}$ cannot be lifted to $\ell_\infty(\mathbb{N}, \ell_2^n)$ because this space can be separated by a countable family of functionals, something that Y cannot. The BAP is also necessary, as shown by Proposition 2.2.19. But one could simply ask that Z/J has BAP (it is in this form that the result will be used in Section 10.1) or, in general, that the operator $T\colon Y \longrightarrow Z/J$ factorises through a space with BAP. Another variation of the result is possible by asking for J to be a Lindenstrauss space: in that case, $J^{\perp\perp}$ is an 1-injective Banach space, and $QL_F \colon F \longrightarrow J^{\perp\perp}$ has an extension $\Lambda\colon E \longrightarrow J^{\perp\perp}$ with $\|\Lambda\| \le \|L_F\|$. Now use the decomposition $L = ((\mathbf{1}_{Z^{**}} - Q)L + \Lambda) + (QL - \Lambda)$ instead of (2.45) and proceed as in the proof. The final estimate in this case is better: $\|L\| = \|T\|$. The *lifting* Theorem 2.14.5 has the following application to the *extension* of operators:

Corollary 2.14.7 *Let J be an M-ideal in A, and let $\tau\colon Y \longrightarrow J$ be an operator, where Y is a subspace of the Banach space Z such that Z/Y is separable. We denote the inclusion maps by $\jmath\colon Y \longrightarrow Z$ and $\imath\colon J \longrightarrow A$. Assume that*

- *there is an operator $\tau_A \colon Z \longrightarrow A$ such that $\tau_A \jmath = \imath\tau$, with $\|\tau_A\| \le \mu\|\tau\|$,*
- *Z/Y or A/J has the λ-AP.*

Then τ has a $\mu(1+\lambda)$ extension $T\colon Z \longrightarrow J$.

Proof Assume $\|\tau\| \le 1$, and display all the information at hand in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & Z/Y & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \tau} & & \big\downarrow{\scriptstyle \tau_A} & & \big\downarrow{\scriptstyle \tau'} & & \\
0 & \longrightarrow & J & \xrightarrow{\imath} & A & \xrightarrow{\pi} & A/J & \longrightarrow & 0
\end{array}$$

where τ' is induced by the fact that $\pi\tau_A\jmath = \pi\imath\tau = 0$. The operator τ' has a lifting $L\colon Z/Y \longrightarrow A$ with $\|T\| \le \lambda\|\tau'\| \le \lambda\mu$. The required extension is $T = \tau_A - L\rho$: this operator takes values in J since $\pi T = \pi\tau_A - \pi L\rho = \pi\tau_A - \tau'\rho = 0$. Obviously, $T|_Y = \tau$ and, finally, $\|T\| \le \|\tau_A\| + \|L\rho\| \le \mu(1+\lambda)$. □

The pervasive chronicle of the quest for Sobczyk's theorem, as related in this book, departs from Sections 1.7 and 1.8.4, crosses through this section here with the obtention of the following optimal vector-valued version and will meander through 5.2.5 and its consequences, to arrive at 10.1 in a rather satisfactory conclusion.

2.14.8 Vector-valued Sobczyk's theorem *Let Y be a subspace of a separable Banach space Z such that Z/Y has the λ-AP. Let $\tau\colon Y \longrightarrow c_0(\mathbb{N}, E_n)$ be an operator. If each $\pi_n\tau\colon Y \longrightarrow E_n$ admits a μ-extension to Z then τ admits a $\mu(1+\lambda)$-extension to Z.*

Proof Let $\imath\colon c_0(\mathbb{N}, E_n) \longrightarrow \ell_\infty(\mathbb{N}, E_n)$ denote the canonical inclusion. Since each $\pi_n\tau$ admits a λ-extension to Z, the operator $\imath\tau\colon Y \longrightarrow \ell_\infty(\mathbb{N}, E_n)$ admits a μ-extension $T\colon Z \longrightarrow \ell_\infty(\mathbb{N}, E_n)$ as in the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & Z/Y & \longrightarrow & 0\\
 & & \big\downarrow{\scriptstyle \tau} & & \big\downarrow{\scriptstyle T} & & \big\downarrow & & \\
0 & \longrightarrow & c_0(\mathbb{N}, E_n) & \xrightarrow{\ \imath\ } & \ell_\infty(\mathbb{N}, E_n) & \longrightarrow & \ell_\infty(\mathbb{N}, E_n)/c_0(\mathbb{N}, E_n) & \longrightarrow & 0
\end{array}$$

Since Z/Y has the λ-AP, τ admits a $\mu(1+\lambda)$-extension $Z \longrightarrow c_0(\mathbb{N}, E_n)$. □

We are not that interested in removing the ε appearing in the proof of Theorem 2.14.5. But a sharp proof in the simplest conceivable case of rank one operators is definitely worthwhile:

2.14.9 *M*-ideals are proximinal *If J is an M-ideal in Z then for every $z \in Z$, there is $y \in J$ such that $\|z - y\| = \|z + J\|$.*

Proof It is clear that if J is an M-ideal in Z and $J \subset X \subset Z$ then J is an M-ideal in X. Hence, it suffices to prove the result when J is a hyperplane of Z. Write $J = \ker\rho$, where ρ is normalised in Z^*, and observe that what one must show is that ρ attains the norm on B_Z. Write $Z^* = J^\sharp \oplus_1 J^\perp$, where $J^\sharp = \{f \in Z^*\colon \|f\| = \|f|_J\|\}$. By the Bishop–Phelps theorem, not every norm-attaining functional can be in $J^\sharp$. Pick a normalised, norm-attaining $f \in Z^*\backslash J^\sharp$ and then $z \in Z$ such that $\langle f, z\rangle = 1$. Obviously, $z \notin J$; writing $f = g + h$, with $g \in J^\sharp$ and h non-zero in $J^\perp$, we have $1 = \langle f, z\rangle \le |\langle g, z\rangle| + |\langle g, z\rangle| \le \|g\| + \|h\| = 1$, hence h attains the norm at z, and so does ρ. □

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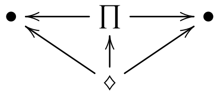
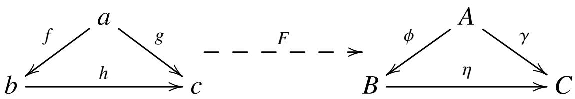
2.15 Notes and Remarks

2.15.1 Categorical Limits

The simplest diagram is formed by just two points $\bullet \quad \bullet$ and no arrows, except identities that we will not draw. The limit of this diagram in a category is an object $\prod$ of the category and two arrows $\bullet \longleftarrow \prod \longrightarrow \bullet$ with the universal property: for any other object $\diamond$ yielding a similar diagram $\bullet \longleftarrow \diamond \longrightarrow \bullet$ there is a unique arrow $\diamond \longrightarrow \prod$ filling a commutative diagram



Anyone who does not find it evident that the limit of diagram $\bullet \quad \bullet$ is the product (and its colimit, the coproduct $\coprod$, usually named the direct sum in topological surroundings) in the time a chestnut takes to drop from a stool can skip this section with no permanent harm. Let's nonetheless pretend we already know what a category and a functor are. An abstract diagram **D** made with points and arrows is itself a category, a *small* category for what it is worth, that can be depicted by means of a directed graph. For instance, the diagram on the left in



represents the not-so-entertaining category having three objects a, b, c and the following sets of morphisms: $\mathrm{Hom}(a, b) = \{f\}$, $\mathrm{Hom}(a, c) = \{g, hf\}$, $\mathrm{Hom}(b, c) = \{h\}$. The other sets of morphisms are either empty or consist of the corresponding identities. Given another category **C**, a functor $F\colon \mathbf{D} \longrightarrow \mathbf{C}$ means just filling **D** with objects and arrows of **C**, as in the preceding diagram. Think of Banach spaces and operators instead of points and arrows, as represented on the right side of the drawing. The limit $\lim F$ of F is an object of **C** together with a family of arrows $(\alpha_d\colon F(d) \longrightarrow \lim F)_d$ parametrised by the points of **D** satisfying the following conditions:

- Compatibility: if $s\colon d \longrightarrow e$ is an arrow of **D** then $\alpha_d = \alpha_e F(s)$.
- Universality: for any other object $X \in \mathbf{C}$ and a system of arrows $(\alpha_d\colon F(d) \longrightarrow X)_d$ with the same property, there is a unique arrow $\xi\colon \lim F \longrightarrow X$ such that $\xi\alpha_d = \beta_d$ for all d.

The colimit (also called inverse limit) is defined by considering 'arrows from' instead of 'arrows into'. The universal mapping property of these objects

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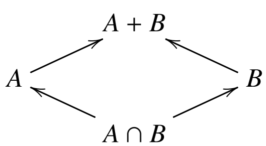
guarantees that they are unique, up to isomorphism in the corresponding category. It is obvious that even the simplest infinite diagram cannot have a limit in **B** since the fact that operators have norms prevents it. The groundbreaking result in this regard is from Semadeni and Zidenberg [431]: *Every diagram in* $\mathbf{B}_1$ *admits a limit and a colimit*. All mathematical constructions are (co)limits, or so the Eilenberg–MacLane programme [165] says.

2.15.2 How to Draw More Diagrams

Working with diagrams is simple, extremely rewarding and a little addictive. Powerful, too, sometimes. This is so because the mere drawing of an exact sequence $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ encodes much more information than its Banach space counterpart 'Y is a subspace of Z in such a way that Z/Y is isomorphic to X'. It also contains the assertions '$Y \longrightarrow Z$ is the kernel of $Z \longrightarrow X$' and '$Z \longrightarrow X$ is the cokernel of $Y \longrightarrow X$'. In turn, the former of these two assertions contains more information than $Y = \{x \in Z\colon \rho x = 0\}$, while the latter contains more information than $X \simeq Z/Y$. And the amount of information grows exponentially with each new arrow we add. Working with diagrams requires us to follow some rules too. The rule that drawings should be *complete*, meaning they should start and end in 0, means that if yours is not then something has been overlooked. And the rule is subtly demoed when there is some (categorical) construction that makes the diagram complete. For instance, an operator $T\colon A \longrightarrow B$ is an incomplete diagram $A \longrightarrow B$, and thus kernels and cokernels are there to complete it as $0 \longrightarrow \ker T \longrightarrow A \longrightarrow B \longrightarrow \mathrm{coker} T \longrightarrow 0$. To complete more complex diagrams, one will need more complex constructions. Let us see some.

The diamond lemma. This is an elementary result in linear algebra: if A and B are linear subspaces of V then the quotients $(A + B)/A$ and $B/(A \cap B)$ are isomorphic. The name comes from the figure



where the arrows mean containment. To transform this linear isomorphism into a linear *homeomorphism* when working in **Q** requires an additional hypothesis to ensure all the spaces are complete: if A, B are complete, then $A + B$ is com-

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plete, assuming that either A or B is finite-dimensional or A and B are *totally incomparable* (do not have infinite-dimensional isomorphic subspaces) [410].

The snake lemma. To complete the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\jmath} & B & \xrightarrow{\rho} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & A' & \xrightarrow{\jmath'} & B' & \xrightarrow{\rho'} & B' & \longrightarrow & 0
\end{array}$$

with exact rows, we can start by drawing the sequence of kernels and cokernels. It is easy to check that one gets the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker\alpha & \longrightarrow & \ker\beta & \longrightarrow & \ker\gamma & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{coker}\alpha & \longrightarrow & \operatorname{coker}\beta & \longrightarrow & \operatorname{coker}\gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2.46}$$

But, still, this is not complete: have you seen the open end at the topmost right corner? And the open start at the lowest left corner? The snake lemma says that the diagram can be completed with a *connecting morphism* $\omega\colon\ \ker\gamma \longrightarrow$

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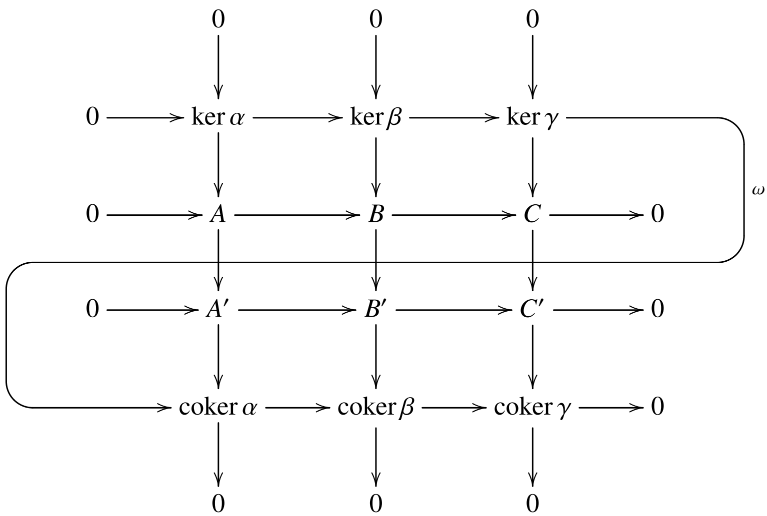
cokerα, whose construction is mere diagram chasing (promise), yielding a ‘long exact sequence’



In applications, it shall be expected that α, β and γ shall be either embeddings or quotient maps, so that either ker or coker will be 0. This means that the sequence of quotient spaces (case of embeddings) or kernels (case of quotient maps) in the completed diagram is also exact. In particular, we have verified that the test Diagram (2.1) in the introduction exercise of the chapter was correctly drawn.

More pullback and pushout diagrams. Diagram (2.32) = (2.33) is the diagram we obtain by completing

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\ & & & & \uparrow{\scriptstyle\beta} & & & & \\ & & & & Z' & & & & \end{array}$$

via pullback when β is a quotient map to obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle\underline{\beta}} & & \uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle\overline{\beta}} & & \\ 0 & \longrightarrow & \mathrm{PB} & \longrightarrow & Z' & \longrightarrow & Z'/\mathrm{PB} & \longrightarrow & 0 \end{array}$$

If, however, β is an embedding then no simple outcome exists. Indeed, imagine for simplicity’s sake that $\jmath, \beta$ are natural inclusions. Then $\mathrm{PB} = Y \cap Z'$.

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The diamond lemma yields that, if $Y + Z'$ is closed then $Z'/\mathrm{PB} = Z'/(Y \cap Z') = (Y + Z')/Y$ is a subspace of X. But if not then $\overline{\beta}$ is just an injective operator. Assuming that $Y + Z'$ is closed, the complete diagram is

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Y/\mathrm{PB} & \longrightarrow & Z/Z' & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle \beta} & & \uparrow & & \\
0 & \longrightarrow & \mathrm{PB} & \longrightarrow & Z' & \longrightarrow & Z'/\mathrm{PB} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2.47}$$

Completing the dual diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \beta} & & & & \\
 & & & & Z' & & & &
\end{array}$$

via a pushout, when β is moreover surjective, yields

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker \underline{\beta} & \xrightarrow{\underline{\jmath}} & \ker \beta & \xrightarrow{\underline{\rho}} & \ker \overline{\beta} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota_-} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota^-} & & \\
0 & \longrightarrow & Y & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \underline{\beta}} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \overline{\beta}} & & \\
0 & \longrightarrow & \ker \overline{\rho} & \longrightarrow & Z' & \xrightarrow{\overline{\rho}} & \mathrm{PO} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where ι_-, ι, ι^- are the corresponding inclusions. The reader might doubt that $\underline{\beta}$ is surjective. But take $x_1 \in \ker \overline{\rho}$ and find $x \in Z$ such that $\beta x = x_1$. This and the commutativity of the pushout square imply $\overline{\beta}\rho x = \overline{\rho}\beta x = 0$, and thus

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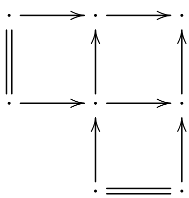
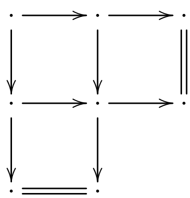
$(0, \rho x) \in \Delta(\beta, \rho)$. Thus, there exists $x' \in X$ such that $(0, \rho x) = (\beta x', \rho x')$, thus giving $x - x' \in Y$ and $\beta(x') = 0$. Therefore, $\beta(x - x') = \beta(x) = x_1$. The equality $\ker \underline{\beta} = \ker \underline{\rho}$ is standard, and the exactness of the upper sequence of kernels is due to the snake lemma. An attentive reader should have realised that we have already encountered this diagram: turn Diagram (2.47) upside down, and there it is! In fact, the top-most left corner space $\ker \underline{\beta}$ is the pullback space of the two operators $\imath, \jmath$ that point at Z; to see this, assume that for some space A and arrows $a \colon A \longrightarrow \ker \underline{\beta}$ and $b \colon A \longrightarrow Y$, we have $\jmath b = \imath a$. Then $\beta \jmath b = \beta \imath a = 0$, and thus $\underline{\beta} b = \beta \jmath b = 0$, which means that b factorises through $\ker \underline{\beta}$ as $b = \imath_- u$ for some operator $u \colon A \longrightarrow \ker \underline{\beta}$. The other equality $\underline{\jmath} u = a$ just follows from $\imath \underline{\jmath} u = \jmath \imath_- u = \jmath b = \imath a$ and the injectivity of $\imath$. When β is an embedding then $\underline{\beta}$ is an embedding, and we get – turn the diagram upside down – Diagram (2.47) again.

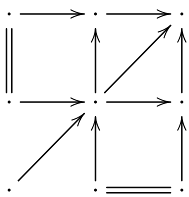
2.15.3 Amalgamation of Sequences

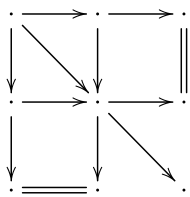
The much less frequently used multiple pullback also exists [110; 365]. Other diagonals concealed in complete pullback / pushout diagrams, such as





exist too:





And they exist in complete multiple pushput / pullback diagrams as well. The left-upwards diagonal can be understood as the product of the two (or many) data sequences $0 \longrightarrow Y_i \longrightarrow \diamond_i \longrightarrow X \longrightarrow 0$ in the appropriate category, and it will typically have the form $0 \longrightarrow \ell_\infty(I, Y_i) \longrightarrow \mathrm{PB} \longrightarrow X \longrightarrow 0$; while the right-downwards diagonal can be understood as the coproduct of the two (or many) data sequences $0 \longrightarrow Y \longrightarrow \diamond_i \longrightarrow X_i \longrightarrow 0$ and will typically have the form $0 \longrightarrow Y \longrightarrow \mathrm{PO} \longrightarrow \ell_p(I, X_i) \longrightarrow 0$ in $p\mathbf{B}$.

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2.15.4 Categories of Short Exact Sequences

We dealt with categories, we deal with exact sequences and we will deal with commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & 0 \end{array} \tag{2.48}$$

And, call us Ishmael, we are just one whale away from sinking into the ocean of forming our own category of exact sequences. Indeed, the preceding diagram can be regarded as a 'morphism of sequences', an idea that is implicit in the notion of isomorphic sequences. A natural first attempt is to set exact sequences as objects and triples of arrows (α, β, γ), as in Diagram (2.48), as morphisms. If one does so, *isomorphic* objects are isomorphic exact sequences, and we can keep sailing. But we would also like to consider as objects equivalence classes of exact sequences, in which case triples of arrows are no longer well suited (after all, *where* should one define β?). So, let us declare that morphisms in this category are pairs (α, γ) such that $[\alpha\, \mathsf{z}] = [\mathsf{z}'\gamma]$, as we did in fact, in (2.28), where it was shown that if (α, β, γ) represents a morphism from z to z' then $[\alpha\, \mathsf{z}] = [\mathsf{z}'\gamma]$. With these objects and morphisms, our class of isomorphisms has changed [110, Proposition 3.1]: two objects are *isomorphic in this category* if and only if they have isomorphic multiples (understanding that the multiples of a sequence z are the sequences $E \times \mathsf{z}$ and $\mathsf{z} \times E$). There are other possibilities for forming categories of short exact sequences, such as: following the uses of the theory of complexes in homology and considering *homotopic* triples [213] – a slightly different form of equivalence that is pointless to define here: we will encounter it but one more time, long after all this is over, and the outcome of that meeting will not be satisfactory for either of us – or by fixing the start or the end spaces in exact sequences [110] and determining equality in terms of pullback/pushout.

Sources

The (arguably) more d(iag)ramatic than necessary example leading to Diagram (2.1) appeared in [459], albeit with a far more innocuous purpose. The notion of isomorphic sequences in 2.1.6 appears perhaps for the first time in [65] and [107]. The analysis of the Foiaş–Singer sequences is, broadly speaking, as in [73], even though these ideas can be traced back to Ditor [158; 159]. Proposition 2.2.5 provides (many) continuous surjections $\Delta \longrightarrow [0, 1]$ without averaging operators. Typical examples are Cantor's dyadic expansion

$\varepsilon \in \{0,1\}^{\mathbb{N}} \longmapsto \sum_n \varepsilon_n 2^{-n-1} \in [0,1]$ and Lebesgue's ternary one $\{-1,0,1\}^{\mathbb{N}} \longrightarrow [-1,1]$ given by $\varepsilon \longmapsto \sum 2\varepsilon_n 3^{-n}$; see [430, 8.3.2], [458, III.D.Ex. 4], [5, Proposition 4.4.6]. The crux in Milutin's theorem is to prove that surjections $\Delta \longrightarrow [0,1]$ admitting averaging operators do exist: each of them provides a complemented copy of $C[0,1]$ in $C(\Delta)$. This was shown by Milutin in [364] with a rather involved construction (see also [377, Lemma 5.5]); some simplifications are available: see [5, Lemma 4.4.7] or [458, III.D.18 Proposition]. Argyros and Arvanitakis stablish a clean criterion for a surjection $\Delta \longrightarrow [0,1]$ to admit an averaging operator and conclude that all maps $\varphi_r(\varepsilon) = (1-r)\sum_{n\geq 1} \varepsilon_n r^{n-1}$ admit one when $r \in (\frac{1}{2}, 1)$; see [15, Theorems 2 and 12]. Proposition 2.2.19 (and its proof) is from Lusky [347]. Whether or not it could be attributed to Pełczyński [382] is left to the reader's opinion. Read from an expert about the origin and development of the notion of categorical limit in [350, p.76]. Assertion 2.7.1 is due to Ortyński [372, Theorem 2]. There is a continued long-standing tradition in functional analysis of reinventing the pushout: we can mention, in chronological order, Gurariy [203], Dierolf [149], Kalton [247; 248], Kisliakov [294], Lusky [346], Pisier [389] and Kuchment [313], but admission to this club is still open. A very welcome more recent way to join the club is to reinvent the pushout in a different related category, such as the different *complemented pushouts* of [116] modeled on ideas of Garbulińska and W. Kubiś [184; 308], or to work in the category of Banach spaces and *pairs* (see Chapter 6 and Section 10.7). The Device presented in Section 2.13 is, in a more or less recognisable form, in Kalton [248, Lemma 4.2], Lusky [346, Lemma] (applied to an infinite set of operators) and Pisier [389, Corollary 2.3] (applied to *three* operators and including estimates for the norms of the involved operators). The parallel principles appeared in [121], while the diagonal principles appeared in [109] with the declared purpose of understanding the Lindenstrauss–Rosenthal theorem. Lemma 2.12.2 is from Benyamini and Lindenstrauss [41] and obtained with the purpose of doing what the title says. The 3-space problem for the Dunford–Pettis property has seen a few false positive answers, a few interesting partial solutions and a counterexample [102]. The general result in Proposition 2.12.5 is from [124]. Vogt's duality problem was treated by Díaz, Dierolf, Domański and Fernández in [150], providing a nice counterexample in the context of Fréchet spaces. Reformulated in diagrams, their solution can be read as follows: everything stems from the existence of a Fréchet–Montel space FM admitting ℓ_1 as a quotient (see [303]). Let N be a subspace of ℓ_1 that is not complemented in its bidual (say, the kernel of a quotient map $\ell_1 \longrightarrow L_1$). Form the pullback diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathsf{FM}/\mathrm{PB} & = & \ell_1/N & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K & \longrightarrow & \mathsf{FM} & \longrightarrow & \ell_1 & \longrightarrow & 0\\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K & \longrightarrow & \mathrm{PB} & \longrightarrow & N & \longrightarrow & 0\\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

with diagonal pullback sequence $0 \longrightarrow \mathrm{PB} \longrightarrow \mathsf{FM} \times N \longrightarrow \ell_1 \longrightarrow 0$. Since PB is a closed subspace of FM, it is itself a Fréchet–Montel space, hence reflexive and thus a dual space; as is ℓ_1. On the other hand, N is not complemented in its bidual, so the same happens to $\mathsf{FM} \times N$. It is clear that this approach cannot work for Banach spaces. The Banach space solution is from [65]. The construction in 2.12.9 appeared in [67]; we taught it to Nigel on the blackboard one fine day when he was curious 'about all this funny diagram stuff', and he used it in [276, Proposition 6.3], graciously calling the resulting space CC. The material on M-ideals is taken from [209] with the exception of the gorgeus proof of 2.14.9, which is due to Indumathi and Lalithambigai [215]. The disquisitions on categories of short exact sequences are taken from [365], later developed in [107; 110].

We conclude with a remark: homology is not category theory. It is just a part. Therefore, homological Banach space theory is not the same as categorical Banach space theory, even if Manuel González finds a logical contradiction in this sentence. Probably Categorical Banach Space Theory has still to be created for good; see [387, 6.9.7.1] or [92]. There are sound arguments to maintain that (quasi-) Banach spaces is a very interesting test category to work with even if you care only about category theory. Even if it is not an Abelian category (whatever that means), it is a forgiving place to work, since it is 'almost Abelian' and *exact* in Quillen's sense, which amounts to saying, more or less, that, one way or another, most homological and categorical constructions can be used there.

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