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On a Theorem of Bombieri, Friedlander, and Iwaniec

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Abstract. In this article, we show to what extent one can improve a theorem of Bombieri, Friedlander, and Iwaniec by using Hooley's variant of the divisor switching technique. We also give an application of the theorem in question, which is a Bombieri-Vinogradov type theorem for the Tichmarsh divisor problem in arithmetic progressions.

1 Introduction

The Bombieri–Vinogradov theorem implies that on average over $q \leq x^{1/2-o(1)}$, the primes less than x are equidistributed in the residue classes a mod q, with $(a,q) = 1$. Specifically, we have for any $A > 0$ that

$$\sum_{q \leq Q} \max_{a:(a,q)=1} \left| \psi(x;q,a) - \frac{x}{\phi(q)} \right| \ll \frac{x}{(\log x)^A}, \tag{1.1}$$

where $Q = x^{1/2}/(\log x)^{A+5}$. One could ask if (1.1) still holds if we take $Q = x^\theta$, with $\theta > \frac{1}{2}$. This would be a major achievement, since it would imply bounded gaps between primes [12], that is

$$\liminf_n (p_{n+1} - p_n) < \infty.$$

The Elliot–Halberstam conjecture stipulates that we can take θ to be any real number less than 1. This conjecture is, however, very far from reach.

One way to get past the barrier of $Q = x^{1/2-o(1)}$ is to relax the condition on a. Indeed, in concrete problems, one often only needs the bound (1.1) for a fixed value of a. Sometimes, even the absolute values are not necessary. These variants were studied very closely in a series of groundbreaking articles by Fouvry and Iwaniec [9, 10], Fouvry [6–8], and Bombieri, Friedlander, and Iwaniec [1–3]. We will list the results of these authors by increasing order of uniformity.

By fixing a, one can go up to $Q = x^{\frac{1}{2}+\frac{1}{(\log\log x)^B}}$.

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Theorem 1.1 (Bombieri, Friedlander, and Iwaniec [2]) *Let $a \neq 0$, $x \geq y \geq 3$, and $Q^2 \leq xy$. Then there exists an absolute constant B such that*

$$\sum_{\substack{Q \leq q < 2Q \\ (q,a)=1}} \Big| \psi(x;q,a) - \frac{x}{\phi(q)} \Big| \ll x \Big(\frac{\log y}{\log x} \Big)^2 (\log\log x)^B.$$

The best known result was obtained shortly afterwards by the same authors, and shows that one can go up to $Q = x^{\frac{1}{2}+o(1)}$, whatever the nature of the $o(1)$ is.

Theorem 1.2 (Bombieri, Friedlander, and Iwaniec [3]) *Let $a \neq 0$ be an integer and let $A > 0$, $2 \leq Q \leq x^{3/4}$ be reals. Let $\mathcal{Q}$ be the set of all integers q, prime to a, from an interval $Q' < q \leq Q$. Then*

$$\sum_{q \in \mathcal{Q}} \Big| \pi(x;q,a) - \frac{\pi(x)}{\phi(q)} \Big|$$
$$\leq \Big\{ K\Big(\theta - \frac{1}{2}\Big)^2 \frac{x}{\log x} + O_A\Big(\frac{x(\log\log x)^2}{(\log x)^3} \Big) \Big\} \sum_{q \in \mathcal{Q}} \frac{1}{\phi(q)} + O_{a,A}\Big(\frac{x}{(\log x)^A} \Big),$$

where $\theta := \frac{\log Q}{\log x}$ and K is absolute.

Replacing the absolute values by a certain weight (see [1] for the definition of *well factorable*), we can take $Q = x^{4/7-\epsilon}$.

Theorem 1.3 (Bombieri, Friedlander, and Iwaniec [1]) *Let $a \neq 0$, $\epsilon > 0$ and $Q = x^{4/7-\epsilon}$. For any well factorable function $\lambda(q)$ of level Q and any $A > 0$ we have*

$$\sum_{(q,a)=1} \lambda(q) \Big(\psi(x;q,a) - \frac{x}{\phi(q)} \Big) \ll \frac{x}{(\log x)^A}. \tag{1.2}$$

Theorem 1.3 is an improvement of a result of Fouvry and Iwaniec [10], which showed that (1.2) holds with $\lambda(q)$ of level $Q = x^{9/17-\epsilon}$.

If we remove the weight $\lambda(q)$, we can take $Q = x/(\log x)^B$, which is even further than in the Elliot–Halberstam conjecture. This result was obtained independently by Fouvry [8] and Bombieri, Friedlander, and Iwaniec [1] (in stronger form).

Theorem 1.4 (Bombieri, Friedlander, and Iwaniec [1]) *Let $a \neq 0$, $\lambda < \frac{1}{10}$, and $R < x^{\lambda}$. For any $A > 0$ there exists $B = B(A)$ such that, provided $QR < x/(\log x)^B$, we have*

$$\sum_{\substack{r \leq R \\ (r,a)=1}} \Big| \sum_{\substack{q \leq Q \\ (q,a)=1}} \Big(\psi(x;qr,a) - \Lambda(a) - \frac{x}{\phi(qr)} \Big) \Big| \ll_{a,A,\lambda} \frac{x}{(\log x)^A}. \tag{1.3}$$

Remark 1.5 We subtracted $\Lambda(a)$ from $\psi(x;qr,a)$ in (1.3) because the arithmetic progression a mod qr contains the prime power p^e for all values of qr if $a = p^e$. This induces a negligible error term in (1.3) (for $B > A$).

In this article we focus on Theorem 1.4. We show in Corollary 2.2 that for any $A > 0$:

- If $a = \pm 1$, then Theorem 1.4 holds if $B(A) > A$, and is false if $B(A) \le A$.
- If $a = \pm p^e$, then Theorem 1.4 holds if $B(A) \ge A$, and is false if $B(A) < A$.
- If a has two or more distinct prime factors, then Theorem 1.4 holds if $B(A) > \frac{538}{743}A$.

One of the applications of Theorem 1.4 and of Fouvry's result [8] is the best known estimate for the Titchmarsh divisor problem. We will show that Theorem 1.4 yields a generalization of this result that is a Bombieri-Vinogradov type result for the Titchmarsh divisor problem in arithmetic progressions, up to level $Q = x^{1/10-\epsilon}$.

2 Statement of Results

Here is our main result.

Theorem 2.1 *Fix an integer $a \ne 0$, a positive real number $\lambda < \frac{1}{10}$, and an arbitrarily large real number C. We have for $R = R(x) \le x^{\lambda}$ and $M = M(x) \le (\log x)^C$ that*

$$\sum_{\substack{\frac{R}{2}<r\le R\\(r,a)=1}}\Big|\sum_{\substack{q\le \frac{x}{rM}\\(q,a)=1}}\Big(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\Big)-\frac{\phi(a)}{a}\frac{x}{rM}\mu(a,r,M)\Big| \ll_{a,C,\epsilon,\lambda} \frac{x}{M^{\frac{743}{538}-\epsilon}},$$

where the "average" is given by

$$\mu(a,r,M) := \begin{cases} -\frac{1}{2}\log M - C_5(r) & \text{if } a=\pm 1,\\ -\frac{1}{2}\log p & \text{if } a = \pm p^e,\\ 0 & \text{otherwise,}\end{cases}$$

with

$$C_5(r) := \frac{1}{2}\Big(\log 2\pi + 1 + \gamma + \sum_{p}\frac{\log p}{p(p-1)} + \sum_{p\mid r}\frac{\log p}{p}\Big).$$

We also have the following similar result.

$$\sum_{\substack{r\le R\\(r,a)=1}}\Big|\sum_{\substack{q\le \frac{x}{RM}\\(q,a)=1}}\Big(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\Big)-\frac{\phi(a)}{a}\frac{x}{RM}\mu(a,r,RM/r)\Big| \ll_{a,A,\epsilon,\lambda} \frac{x}{M^{\frac{743}{538}-\epsilon}}.$$

As a corollary, we get a more precise form of Theorem 1.4.

Corollary 2.2 *Fix an integer $a \ne 0$, a positive real number $\lambda < \frac{1}{10}$, and an arbitrarily large real number C. We have for $R = R(x) \le x^{\lambda}$ and $M = M(x) \le (\log x)^C$ that*

$$\sum_{\substack{r\le R\\(r,a)=1}}\Big|\sum_{\substack{q\le \frac{x}{RM}\\(q,a)=1}}\Big(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\Big)\Big| = \Big(\frac{\phi(a)}{a}\Big)^2\frac{x}{M}\nu(a,M) + O_{a,C,\epsilon,\lambda}\Big(\frac{x}{M^{\frac{743}{538}-\epsilon}}\Big),$$

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where

$$\nu(a,M) := \begin{cases} \frac{1}{2}\log M + C_6 + O\Big(\frac{\log(RM)}{R}\Big) & \textit{if } a = \pm 1, \\ \frac{1}{2}\log p + O\big(\frac{1}{R}\big) & \textit{if } a = \pm p^e, \\ 0 & \textit{otherwise,} \end{cases}$$

with

$$C_6 := C_5(1) + \frac{1}{2} + \frac{1}{2}\sum_p \frac{\log p}{p^2}.$$

Remark 2.3 If a has at most one prime factor, then for M and R both tending to infinity we have that

$$\nu(a,M) \sim \begin{cases} \frac{1}{2}\log M & \text{if } a = \pm 1, \\ \frac{1}{2}\log p & \text{if } a = \pm p^e. \end{cases}$$

(If R is bounded, then we should multiply by $\frac{a}{\phi(a)}\frac{\#\{r\leq R:(r,a)=1\}}{R}$ in the case $a = \pm p^e$, and by $\frac{\lfloor R\rfloor}{R}$ in the case $a = \pm 1$.)

Another corollary of our results (which actually follows from Theorem 1.4) is a Bombieri–Vinogradov type result for the Titchmarsh divisor problem in arithmetic progressions. For an integer $n \geq 1$, we define:

$$\tau(n) := \sum_{d\mid n} 1, \qquad\qquad n' := \prod_{p\mid n} p.$$

Theorem 2.4 *Fix an integer $a \neq 0$ and let $\lambda < \frac{1}{10}$ and C be two fixed positive real numbers. We have for $Q \leq x^{\lambda}$ that*

$$\sum_{\substack{q\leq Q\\(q,a)=1}} \Big| \sum_{|a|/q<m\leq x/q} \Lambda(qm+a)\tau(m) - M.T.\Big| \ll_{a,C,\lambda} \frac{x}{(\log x)^C}, \tag{2.1}$$

where the main term is

$$M.T. := \frac{x}{q}\bigg(C_1(a,q)\log x + 2C_2(a,q) + C_1(a,q)\log\Big(\frac{(q')^2}{eq}\Big)\bigg),$$

with $C_1(a,q)$ and $C_2(a,q)$ defined as in section 3.

A version of Theorem 2.4 was obtained independently by Felix [4], who also showed how to apply this result to a question related to Artin's primitive root conjecture. Using Theorem 2.4, one can give a slight improvement of [4, Theorem 1.5] replacing $O(\log\log x)$ by $c\log\log x + O(1)$, for some constant c.

Remark 2.5 Taking $Q = (\log x)^C$ in Theorem 2.4, we obtain a "Siegel–Walfisz theorem" for the Titchmarsh divisor problem, and one could ask if this is sufficient to give the bound (2.1) for $Q = x^{1/2}/(\log x)^B$, since it is known that the Bombieri–Vinogradov theorem holds with fairly general sequences satisfying a Siegel–Walfisz condition. If this were true, then, using the same ideas as in the proof of Proposition 5.1, it would yield the following improvement of a dyadic version of Theorem 1.4, valid for $L := (\log x)^{C+3}$ and $R = R(x) \leq x^{1/2}/(\log x)^{3C+5}$:

$$\sum_{\substack{\frac{R}{2}<r\leq R \\ (r,a)=1}} \Big| \sum_{\substack{q\leq \frac{x}{RL} \\ (q,a)=1}} \Big(\psi(x;qr,a) - \Lambda(a) - \frac{x}{\phi(qr)} \Big) \Big| \ll_{a,C} \frac{x}{(\log x)^C}. \tag{2.2}$$

In fact, any improvement of the level of distribution in (2.1) yields an improvement on the range of R in (2.2).

3 Notation

We will denote by γ the Euler–Mascheroni constant. We also define the following constants:

$$C_1(a,r) := \frac{\zeta(2)\zeta(3)}{\zeta(6)} \prod_{p|a}\Big(1 - \frac{p}{p^2-p+1}\Big) \prod_{p|r}\Big(1 + \frac{p-1}{p^2-p+1}\Big),$$

$$C_2(a,r) := C_1(a,r)\Big(\gamma - \sum_p \frac{\log p}{p^2-p+1} + \sum_{p|a} \frac{p^2\log p}{(p-1)(p^2-p+1)} - \sum_{p|r} \frac{(p-1)p\log p}{p^2-p+1}\Big),$$

$$C_3(a,r) := C_2(a,r) - C_1(a,r),$$

$$C_5(r) := \frac{1}{2}\Big(\log 2\pi + 1 + \gamma + \sum_p \frac{\log p}{p(p-1)} + \sum_{p|r} \frac{\log p}{p}\Big).$$

Moreover, for $i = 1, 2, 3$,

$$C_i(a) := C_i(a,1) \quad \text{and} \quad C_5 := C_5(1).$$

We denote by $\omega(n)$ the number of prime factors of n.

4 Preliminary Lemmas

We start with some elementary estimates.

Lemma 4.1 *Let f be a not identically zero multiplicative function and let g be an additive function, that is for $(m,n) = 1$, $f(mn) = f(m)f(n)$ and $g(mn) = g(m)+g(n)$ (in particular, $f(1) = 1$ and $g(1) = 0$). Then for a squarefree integer r we have that*

$$\sum_{d|r} f(d)g(d) = \prod_{p'|r}(1 + f(p')) \sum_{p|r} \frac{g(p)f(p)}{1+f(p)}.$$

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Proof We write

$$\sum_{d|r} f(d)g(d) = \sum_{d|r} f(d) \sum_{p|r} g(p) = \sum_{p|r} g(p) \sum_{\substack{d|r:\\ p|d}} f(d) = \sum_{p|r} g(p) \sum_{d|\frac{r}{p}} f(p)f(d)$$

$$= \sum_{p|r} g(p)f(p) \prod_{p'|\frac{r}{p}} (1+f(p')) = \sum_{p|r} \frac{g(p)f(p)}{1+f(p)} \prod_{p'|r} (1+f(p')). \qquad \blacksquare$$

Lemma 4.2 *Let a and r be coprime integers, with r squarefree. We have for* $i = 1, 2$ *that*

$$\frac{C_i(a,r)}{r} = \sum_{d|r} \mu(d) C_i(ad). \tag{4.1}$$

Proof By the definition of $C_1(a)$, we have

$$\sum_{d|r} \mu(d)C_1(ad) = C_1(a) \prod_{p|r} \Big(1 - \Big(1 - \frac{p}{p^2-p+1}\Big)\Big) = \frac{C_1(a,r)}{r}.$$

Moreover, by defining the multiplicative function $f(d) := \frac{\zeta(6)}{\zeta(2)\zeta(3)} \mu(d)C_1(d)$ we have

$$\sum_{d|r} \mu(d)C_2(ad)$$
$$= C_1(a) \sum_{d|r} f(d) \Big(\gamma - \sum_p \frac{\log p}{p^2-p+1} + \sum_{p|a} \frac{p^2 \log p}{(p-1)(p^2-p+1)}\Big)$$
$$+ C_1(a) \sum_{d|r} f(d) \sum_{p|d} \frac{p^2 \log p}{(p-1)(p^2-p+1)}$$
$$= C_2(a) \prod_{p|r} \frac{p}{p^2-p+1} + C_1(a) \sum_{d|r} f(d) \sum_{p|d} \frac{p^2 \log p}{(p-1)(p^2-p+1)}.$$

Applying Lemma 4.1, we get that this is

$$= C_2(a) \prod_{p|r} \frac{p}{p^2-p+1} + C_1(a) \prod_{p'|r} (1+f(p')) \sum_{p|r} \frac{p^2 \log p}{(p-1)(p^2-p+1)} \frac{f(p)}{1+f(p)}$$
$$= C_2(a) \prod_{p|r} \frac{p}{p^2-p+1} - C_1(a) \prod_{p'|r} \frac{p'}{(p')^2-p'+1} \sum_{p|r} \frac{(p-1)p \log p}{p^2-p+1}$$

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$$= C_1(a)\prod_{p|r}\frac{p}{p^2-p+1}\Big(\gamma-\sum_{p}\frac{\log p}{p^2-p+1}+\sum_{p|a}\frac{p^2\log p}{(p-1)(p^2-p+1)}$$
$$-\sum_{p|r}\frac{(p-1)p\log p}{p^2-p+1}\Big)$$
$$= \frac{C_2(a,r)}{r}. \qquad \blacksquare$$

Lemma 4.3 *Fix $r > 0$ and $a \neq 0$ two coprime integers. We have*

$$\sum_{\substack{n\le M\\(n,a)=1}}\frac{n}{\phi(n)} = C_1(a)M + O(2^{\omega(a)}\log M),$$

$$\sum_{\substack{n\le M\\(n,a)=1}}\frac{1}{\phi(n)} = C_1(a)\log M + C_2(a) + O\Big(2^{\omega(a)}\frac{\log M}{M}\Big),$$

$$\sum_{\substack{n\le M\\(n,a)=1}}\frac{rn}{\phi(rn)} = C_1(a,r)M + O\left(3^{\omega(ar)}\log(r'M)\right),$$

$$\sum_{\substack{n\le M\\(n,a)=1}}\frac{1}{\phi(rn)} = \frac{C_1(a,r)}{r}\log(r'M) + \frac{C_2(a,r)}{r} + O\Big(3^{\omega(ar)}\frac{\log(r'M)}{rM}\Big).$$

Proof For the first two estimates, see [5] or [11]. We now sketch a proof the last estimate. First we assume that r is squarefree, since if it is not we can write

$$\frac{1}{\phi(rn)} = \frac{r'}{r\phi(r'n)}.$$

Then we use the identity

$$\sum_{\substack{d|r\\(d,n)=1}}\mu(d) = \begin{cases}1 & \text{if } r \mid n,\\ 0 & \text{else,}\end{cases}$$

to write

$$\sum_{\substack{n\le M\\(n,a)=1}}\frac{1}{\phi(rn)} = \sum_{d|r}\mu(d)\sum_{\substack{n\le rM\\(n,ad)=1}}\frac{1}{\phi(n)}.$$

Now, substituting in the $r = 1$ estimate, we get that

$$\sum_{\substack{n\le M\\(n,a)=1}}\frac{1}{\phi(rn)} = \log(rM)\sum_{d|r}\mu(d)C_1(ad) + \sum_{d|r}\mu(d)C_2(ad) + O\Big(3^{\omega(ar)}\frac{\log(rM)}{rM}\Big).$$

The result follows by Lemma 4.2. The proof of the third estimate proceeds along the same lines. $\blacksquare$

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The following two lemmas give a more precise estimate, which is made possible by the extra weight $1 - n/M$, which appears naturally in the problem (see the proof of Proposition 5.1).

Lemma 4.4 *Let $a \neq 0$ be an integer and $M \geq 1$ be a real number.*
If $\omega(a) \geq 1$,

$$\sum_{\substack{n\leq M\\(n,a)=1}} \frac{1}{\phi(n)}\Big(1-\frac{n}{M}\Big) = C_1(a)\log M + C_3(a) + \frac{\phi(a)}{a}\frac{\Lambda(a)}{2M} + E(M,a). \tag{4.2}$$

If $a = \pm 1$,

$$\sum_{\substack{n\leq M\\(n,a)=1}} \frac{1}{\phi(n)}\Big(1-\frac{n}{M}\Big) = C_1(1)\log M + C_3(1) + \frac{1}{2}\frac{\log M}{M} + \frac{C_5}{M} + E(M,a). \tag{4.3}$$

There exists $\delta > 0$ such that the error term $E(M,a)$ satisfies

$$E(M,a) \ll_\epsilon \frac{\prod_{p\mid a}\big(1+\frac{1}{p^\delta}\big)}{M}\Big(\frac{a'}{M}\Big)^{\frac{205}{538}-\epsilon}. \tag{4.4}$$

Proof See [5, Lemma 5.9] (the constant $C_3(a)$ in this paper refers to $C_2(a)$ in [5]). Note that the different behaviour depending on the number of distinct prime factors of a comes from a certain Dirichlet series, which either has a pole (if $a = \pm 1$), is holomorphic but non-zero (if $a = \pm p^e$) or is zero (if a has two or more distinct prime factors) at the point $s = -1$. ■

Lemma 4.5 *Fix $r > 0$ and $a \neq 0$ two coprime integers.*
If $\omega(a) \geq 1$,

$$\sum_{\substack{n\leq M\\(n,a)=1}} \frac{1}{\phi(nr)}\Big(1-\frac{n}{M}\Big) = \frac{C_1(a,r)}{r}\log(r'M) + \frac{C_3(a,r)}{r} + \frac{\phi(a)}{a}\frac{\Lambda(a)}{2rM} + E(a,r,M).$$

If $a = \pm 1$,

$$\sum_{\substack{n\leq M\\(n,a)=1}} \frac{1}{\phi(nr)}\Big(1-\frac{n}{M}\Big) = \frac{C_1(1,r)}{r}\log(r'M) + \frac{C_3(1,r)}{r} + \frac{\log(r'M)}{2rM} + \frac{C_5}{rM} + E(a,r,M).$$

The error term satisfies

$$E(a,r,M) \ll \frac{\prod_{p\mid ar}\big(1+\frac{1}{p^\delta}\big)}{rM}\Big(\frac{a'}{M}\Big)^{\frac{205}{538}-\epsilon},$$

for some $\delta > 0$.

Proof We will use the estimates of Lemma 4.4 by proceeding as in the proof of Lemma 4.3. We can again assume that r is squarefree, and write

$$\sum_{\substack{n\le M\\(n,a)=1}} \frac{1}{\phi(nr)}\Big(1-\frac{n}{M}\Big) = \sum_{d\mid r}\mu(d)\sum_{\substack{n\le rM\\(n,ad)=1}}\frac{1}{\phi(n)}\Big(1-\frac{n}{rM}\Big),$$

in which we substitute the estimates of Lemma 4.4. If $\omega(a)\ge 2$, then $\omega(ad)\ge 2$ for all $d\mid r$, so we get

$$\begin{aligned}\sum_{\substack{n\le M\\(n,a)=1}} \frac{1}{\phi(rn)}\Big(1-\frac{n}{M}\Big) &= \sum_{d\mid r}\mu(d)\big(C_1(ad)\log(rM)+C_3(ad)+E(ad,1,rM)\big)\\ &= C_1(a,r)\log(rM)+C_3(a,r)+E(a,r,M)\end{aligned}$$

by Lemma 4.2. Here,

$$\begin{aligned}E(a,r,M) &\ll \sum_{d\mid r}\frac{\prod_{p\mid ad}\big(1+\frac{1}{p^\delta}\big)}{rM}\Big(\frac{a'd}{rM}\Big)^{\frac{205}{538}-\epsilon}\\ &= \frac{\prod_{p\mid a}\big(1+\frac{1}{p^\delta}\big)}{rM}\Big(\frac{a'}{rM}\Big)^{\frac{205}{538}-\epsilon}\sum_{d\mid r} d^{\frac{205}{538}-\epsilon}\prod_{p\mid d}\Big(1+\frac{1}{p^\delta}\Big)\\ &= \frac{\prod_{p\mid a}\big(1+\frac{1}{p^\delta}\big)}{rM}\Big(\frac{a'}{rM}\Big)^{\frac{205}{538}-\epsilon}\prod_{p\mid r}\Big(1+p^{\frac{205}{538}-\epsilon}\Big(1+\frac{1}{p^\delta}\Big)\Big)\\ &\ll \frac{\prod_{p\mid ar}\big(1+\frac{1}{p^\delta}\big)}{rM}\Big(\frac{a'}{M}\Big)^{\frac{205}{538}-\epsilon},\end{aligned}$$

where we might have to change the value of $\delta>0$.

If $\omega(a)=1$, then $\omega(ad)\ge 1$ for all $d\mid r$, so we get

$$\begin{aligned}&\sum_{\substack{n\le M\\(n,a)=1}} \frac{1}{\phi(rn)}\Big(1-\frac{n}{M}\Big)\\ &\quad= \sum_{d\mid r}\mu(d)\Big(C_1(ad)\log(rM)+C_3(ad)+\frac{\phi(ad)}{ad}\frac{\Lambda(ad)}{2rM}+E(ad,1,rM)\Big)\\ &\quad= \sum_{d\mid r}\mu(d)\big(C_1(ad)\log(rM)+C_3(ad)\big)+\frac{\phi(a)}{a}\frac{\Lambda(a)}{2rM}+E(a,r,M)\\ &\quad= C_1(a,r)\log(rM)+C_3(a,r)+\frac{\phi(a)}{a}\frac{\Lambda(a)}{2rM}+E(a,r,M).\end{aligned}$$

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If $a = \pm 1$, then we get

$$\sum_{\substack{n\le M\\(n,a)=1}} \frac{1}{\phi(rn)}\Big(1-\frac{n}{M}\Big) = \sum_{d\mid r}\mu(d)(C_1(ad)\log(rM)+C_3(ad)+E(ad,1,rM))$$
$$-\sum_{p\mid r}\frac{\phi(p)}{p}\frac{\Lambda(p)}{2rM}+\frac{\log(rM)}{2rM}+\frac{C_5}{rM}$$
$$= C_1(a,r)\log(rM)+C_2(a,r)+\frac{\log M}{2rM}+\frac{C_5(r)}{rM}+E(a,r,M). \blacksquare$$

5 Further Results and Proofs

Proposition 5.1 *Fix two positive real numbers $\lambda < \frac{1}{10}$ and D. Let $M = M(r,x)$ be an integer such that $1 \le M(r,x) \le (\log x)^D$. Then for $R = R(x) \le x^\lambda$ we have*

$$(5.1)\quad \sum_{\substack{R/2<r\le R\\(r,a)=1}} \Bigg| \sum_{\substack{q\le \frac{x}{rM}\\(q,a)=1}} \Big(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\Big)$$
$$-x\bigg(\frac{C_1(a,r)}{r}\log(r'M)+\frac{C_3(a,r)}{r}-\sum_{\substack{s\le M\\(s,a)=1}}\frac{1}{\phi(rs)}\Big(1-\frac{s}{M}\Big)\bigg)\Bigg| = O_{a,A,D,\lambda}\Big(\frac{x}{\log^A x}\Big).$$

We can remove the condition of M being an integer at the cost of adding the error term $O\big(x\frac{\log\log M}{M^2}\big)$.

Proof The proof follows closely that of [5, Proposition 6.1]. We start by splitting the sum over q as follows:

$$\sum_{\substack{q\le\frac{x}{rM}\\(q,a)=1}} = \sum_{\substack{q\le\frac{x}{RL}\\(q,a)=1}} + \sum_{\substack{\frac{x}{RL}<q\le\frac{x}{r}\\(q,a)=1}} - \sum_{\substack{\frac{x}{rM}<q\le\frac{x}{r}\\(q,a)=1}}.$$

We use Theorem 1.4 to bound the first of these sums by taking $L := (\log x)^{A+B+D+4}$, with $B = B(A)$ coming from that theorem:

$$\sum_{\substack{R/2<r\le R\\(r,a)=1}} \Bigg|\sum_{\substack{q\le\frac{x}{RL}\\(q,a)=1}}\Big(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\Big)\Bigg| \ll_{a,A,D,\lambda} \frac{x}{(\log x)^A}.$$

We study the two remaining sums in the same way, by writing

$$\sum_{\substack{\frac{x}{rP}<q\le\frac{x}{r}\\(q,a)=1}}\Big(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\Big) = \sum_{\substack{\frac{x}{rP}<q\le\frac{x}{r}\\(q,a)=1}}\ \sum_{\substack{|a|<n\le x\\ n\equiv a \bmod qr}}\Lambda(n) - x\sum_{\substack{\frac{x}{rP}<q\le\frac{x}{r}\\(q,a)=1}}\frac{1}{\phi(qr)},$$

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where we will take $P \le 2L$ to be either M or $\frac{RL}{r}$. The last term on the right is treated using Lemma 4.3:

$$\sum_{\substack{\frac{x}{rP}<q\le\frac{x}{r}\\(q,a)=1}} \frac{1}{\phi(qr)} = \frac{C_1(a,r)}{r}\log P + O\Big(3^{\omega(ar)}\frac{P\log x}{x}\Big). \tag{5.2}$$

As for the first term, we first remove the prime powers using [5, Lemma 5.3], which states that

$$\sum_{\substack{q\le x\\(q,a)=1}} \Bigg(\sum_{\substack{|a|<n\le x\\ n\equiv a \bmod q}} \Lambda(n) - \sum_{\substack{|a|<p\le x\\ p\equiv a \bmod q}} \log p\Bigg) \ll_\epsilon x^{\frac{1}{2}+\epsilon}.$$

(The set of moduli $\{qr : 1 \le q \le x/r\}$ is a subset of the set of all moduli $q \le x$.) We end up with the sum

$$\sum_{\substack{\frac{x}{rP}<q\le\frac{x}{r}\\(q,a)=1}} \sum_{\substack{|a|<p\le x\\ p\equiv a \bmod qr}} \log p. \tag{5.3}$$

We will now use Hooley's variant of the divisor switching technique (see [13]). Writing $p = a + qrs$, we see that we should sum over s rather than over q, since the bound $\frac{x}{rP} < q$ forces s to be very small. Note that since $(qr, a) = 1$, we have $(s, a) = (p - a, a) = (p, a) = 1$, because $p > |a|$. Hence (5.3) becomes, for $a > 0$,

$$= \sum_{\substack{1\le s<P-\frac{aP}{x}\\(s,a)=1}} \sum_{\substack{\frac{sx}{P}+a<p\le x\\ p\equiv a \bmod s}} \log p.$$

If we had $a < 0$, we would get additional terms that are

$$\ll \sum_{x<q\le x-a} \log x \ll |a| \log x.$$

Thus, up to an error $\ll \log x$, (5.3) is equal to

$$\begin{aligned}\sum_{\substack{1\le s<P-\frac{aP}{x}\\(s,a)=1}} \sum_{\substack{\frac{sx}{P}+a\le p\le x\\ p\equiv a \bmod sr}} \log p &= \sum_{\substack{1\le s<P-\frac{aP}{x}\\(s,a)=1}} \Bigg(\theta(x; sr, a) - \theta\Big(\frac{sx}{P} + a; sr, a\Big)\Bigg)\\ &= \sum_{\substack{1\le s<P-\frac{aP}{x}\\(s,a)=1}} \frac{x}{\phi(sr)}\Big(1 - \frac{s}{P}\Big) + E(r, a),\end{aligned} \tag{5.4}$$

where, by the Bombieri–Vinogradov theorem,

$$\begin{aligned}\sum_{\substack{R/2<r\le R\\(r,a)=1}} |E(r, a)| &\le \sum_{\substack{s\le 2L\\(s,a)=1}} \sum_{\substack{r\le R\\(r,a)=1}} \max_{y\le x}\Big|\theta(y; sr, a) - \frac{y}{\phi(sr)}\Big| + O_{a,A}\Big(\frac{x}{(\log x)^A}\Big)\\ &\le 2L \sum_{\substack{q\le 2RL\\(q,a)=1}} \max_{y\le x}\Big|\theta(y; q, a) - \frac{y}{\phi(q)}\Big| + O_{a,A}\Big(\frac{x}{(\log x)^A}\Big) \ll_A \frac{x}{(\log x)^A}.\end{aligned}$$

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We would like to replace the condition $s < P - \frac{aP}{x}$ by $s \leq x$ in the last sum appearing in (5.4). If P is an integer, this can be done without adding any error term, since the last term of the sum is $\frac{x}{\phi(sr)}\big(1 - \frac{P}{P}\big) = 0$. If $P \notin \mathbb{Z}$, then we need to add the error term $O\big(x\frac{\log\log P}{P^2\phi(r)}\big)$.

Putting all this together and using the triangle inequality, we get that the left-hand side of (5.1) is

$$
\begin{aligned}
(5.5)\quad &\leq \sum_{\substack{R/2<r\leq R\\(r,a)=1}} \Bigg| \sum_{\substack{s\leq \frac{RL}{r}\\(s,a)=1}} \frac{x}{\phi(sr)}\Big(1-\frac{s}{RL/r}\Big) - \sum_{\substack{s\leq M\\(s,a)=1}} \frac{x}{\phi(sr)}\Big(1-\frac{s}{M}\Big) - \sum_{\substack{\frac{x}{RL}<q\leq \frac{x}{rM}\\(q,a)=1}} \frac{x}{\phi(qr)} \\
&-x\Bigg(\frac{C_1(a,r)}{r}\log(r'M)+\frac{C_3(a,r)}{r}-\sum_{\substack{s\leq M\\(s,a)=1}}\frac{1}{\phi(sr)}\Big(1-\frac{s}{M}\Big)\Bigg)\Bigg| + O_{a,A,D,\lambda}\Big(\frac{x}{(\log x)^A}\Big),
\end{aligned}
$$

since M is an integer. If M is not an integer, we have to add an error term of size

$$\ll x \sum_{R/2<r\leq R} \frac{\log\log M}{\phi(r)M^2} \ll \frac{x\log\log M}{M^2}.$$

(We already used the fact that

$$x \sum_{R/2<r\leq R} \frac{\log\log(RL/r)}{\phi(r)(RL/r)^2} \ll \frac{x\log\log L}{L^2}$$

in (5.5).) Applying the triangle inequality once more gives that (5.5) is

$$
\begin{aligned}
&\leq x \sum_{\substack{R/2<r\leq R\\(r,a)=1}} \Bigg| \sum_{\substack{s\leq \frac{RL}{r}\\(s,a)=1}} \frac{1}{\phi(sr)}\Big(1-\frac{s}{RL/r}\Big) - \frac{C_1(a,r)}{r}\log\Big(\frac{r'RL}{r}\Big) - \frac{C_3(a,r)}{r}\Bigg| \\
&+ x \sum_{\substack{R/2<r\leq R\\(r,a)=1}} \Bigg| \sum_{\substack{\frac{x}{RL}<q\leq \frac{x}{rM}\\(q,a)=1}} \frac{1}{\phi(qr)} - \frac{C_1(a,r)}{r}\log\Big(\frac{RL}{rM}\Big)\Bigg| + O_{a,A,D,\lambda}\Big(\frac{x}{(\log x)^A}\Big),
\end{aligned}
$$

which by Lemma 4.3 and (5.2) is

$$
\begin{aligned}
&\ll_{a,A,D,\lambda} x \sum_{\substack{R/2<r\leq R\\(r,a)=1}} \frac{3^{\omega(r)}\log(RL)}{RL} + x \sum_{\substack{R/2<r\leq R\\(r,a)=1}} \frac{3^{\omega(r)}L\log x}{x} + \frac{x}{(\log x)^A} \\
&\ll \frac{x(\log R)^2}{RL} + \frac{x}{(\log x)^A} \ll \frac{x}{(\log x)^A}. \qquad \blacksquare
\end{aligned}
$$

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Proof of Theorem 2.4 Taking $M = 1$ in Proposition 5.1 and applying Lemma 4.3 and the triangle inequality, we get

$$\sum_{\substack{\frac{R}{2}<r\leq R\\ (r,a)=1}}\Big|\sum_{\substack{q\leq\frac{x}{r}\\ (q,a)=1}}(\psi(x;qr,a)-\Lambda(a))-\frac{x}{r}\Big(C_1(a,r)\log\Big(\frac{(r')^2x}{er}\Big)+2C_2(a,r)\Big)\Big| \ll_{a,A,\lambda}\frac{x}{\log^{A+1}x}.$$

Taking dyadic intervals, one can easily use this to show that the whole sum over $r\leq R$ is $\ll_{a,A}\frac{x}{\log^A x}$. The result follows if $a>0$ by exchanging the order of summation:

$$\sum_{\substack{q\leq\frac{x}{r}\\ (q,a)=1}}\sum_{\substack{|a|<n\leq x\\ n\equiv a \bmod qr}}\Lambda(n)=\sum_{\substack{|a|<n\leq x\\ n\equiv a \bmod r}}\Lambda(n)\sum_{\substack{q\leq\frac{x}{r}:\\ qr|n-a}}1 =\sum_{\substack{|a|<n\leq x\\ n\equiv a \bmod r}}\Lambda(n)\tau\Big(\frac{n-a}{r}\Big).$$

If $a<0$, then

$$\sum_{\substack{|a|<n\leq x\\ n\equiv a \bmod r}}\Lambda(n)\sum_{\substack{q\leq\frac{x}{r}:\\ qr|n-a}}1=\sum_{\substack{|a|<n\leq x\\ n\equiv a \bmod r}}\Lambda(n)\tau\Big(\frac{n-a}{r}\Big)-\sum_{\substack{|a|<n\leq x\\ n\equiv a \bmod r}}\Lambda(n)\sum_{\substack{\frac{x}{r}<q:\\ qr|n-a}}1.$$

(The last equality is exact if $a>0$; otherwise we have to add a negligible error term.) ∎

Proof of Theorem 2.1 For the first result, we take $M(r,x):=M(x)$ in Proposition 5.1. By Lemma 4.5, we have that

$$\sum_{\substack{\frac{R}{2}<r\leq R\\ (r,a)=1}}\Big|\frac{\phi(a)}{a}\frac{x}{rM}\mu(a,r,M)-x\Big(\frac{C_1(a,r)}{r}\log(r'M)+\frac{C_3(a,r)}{r}-\sum_{\substack{s\leq M\\ (s,a)=1}}\frac{1}{\phi(rs)}\Big(1-\frac{s}{M}\Big)\Big)\Big|$$
$$\leq x\sum_{\substack{\frac{R}{2}<r\leq R\\ (r,a)=1}}|E(a,r,M)|\ll_a\frac{x}{M^{\frac{205}{538}-\epsilon}}\sum_{\substack{\frac{R}{2}<r\leq R\\ (r,a)=1}}\frac{\prod_{p|r}\big(1+\frac{1}{p^\delta}\big)}{r}\ll\frac{x}{M^{\frac{205}{538}-\epsilon}}, \tag{5.6}$$

hence the result follows by the triangle inequality.

The second result is a bit more delicate, since we have the full range of r, and the innermost sum depends on R. For this reason, we need to go back to the proof of

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Proposition 5.1. We first split the sum over r into the two intervals $r \le R/(\log x)^B$ and $R/(\log x)^B < r \le R$, where we take $B = B(2A)$ as in Theorem 1.4, and we can assume that $B(2A) \ge 2A$. The first part of the sum is treated using this theorem:

$$\sum_{\substack{r \le \frac{R}{(\log x)^B} \\ (r,a)=1}} \Bigg| \sum_{\substack{q \le \frac{x}{RM} \\ (q,a)=1}} \Big(\psi(x; qr, a) - \Lambda(a) - \frac{x}{\phi(qr)} \Big) - \frac{\phi(a)}{a} \frac{x}{RM} \mu(a, r, M) \Bigg| \ll_{a,A,\lambda} \frac{x}{(\log x)^{2A}} + \frac{x}{(\log x)^B},$$

since $\frac{R}{(\log x)^B} \cdot \frac{x}{RM} = \frac{x}{M(\log x)^B} \le \frac{x}{(\log x)^B}$. For the rest of the sum, we argue as in the proof of Proposition 5.1. We split the sum over q as follows:

$$\sum_{\substack{q \le \frac{x}{RM} \\ (q,a)=1}} = \sum_{\substack{q \le \frac{x}{RL} \\ (q,a)=1}} + \sum_{\substack{\frac{x}{RL} < q \le \frac{x}{r} \\ (q,a)=1}} - \sum_{\substack{\frac{x}{RM} < q \le \frac{x}{r} \\ (q,a)=1}}.$$

Taking P to be either $\frac{R}{r}L$ or $\frac{R}{r}M$, we have that $P \le L(\log x)^B$ (instead of $P \le 2L$). The rest of the proof goes through, and we get that

(5.7)
$$\sum_{\substack{\frac{R}{L} < r \le R \\ (r,a)=1}} \Bigg| \sum_{\substack{q \le \frac{x}{RM} \\ (q,a)=1}} \Big(\psi(x; qr, a) - \Lambda(a) - \frac{x}{\phi(qr)} \Big) - x \Big(\frac{C_1(a,r)}{r} \log(r' RM/r) + \frac{C_3(a,r)}{r} - \sum_{\substack{s \le RM/r \\ (s,a)=1}} \frac{1}{\phi(rs)} \Big(1 - \frac{s}{RM/r} \Big) \Big) \Bigg| \ll_{a,A,D,\lambda} \frac{x}{(\log x)^{2A}} + E_2(x, M),$$

where $E_2(x, M)$ is the error coming from the fact that $\frac{R}{r}M$ is not an integer, which is

$$\ll x \sum_{\frac{R}{L} < r \le R} \frac{\log\log(RM/r)}{\phi(r) RM/r} \frac{1}{RM/r} \ll \frac{x}{(RM)^2} \sum_{\frac{R}{L} < r \le R} \frac{r^2 \log\log(RM/r)}{\phi(r)} \ll \frac{x \log\log M}{M^2}.$$

We finish the proof by applying Lemma 4.5 and the triangle inequality. ■

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Proof of Corollary 2.2 By the triangle inequality we have

$$\sum_{\substack{r\leq R\\(r,a)=1}}\Big|\frac{\phi(a)}{a}\frac{x}{RM}\mu(a,r,RM/r)\Big| \leq$$

$$\sum_{\substack{r\leq R\\(r,a)=1}}\Big|\sum_{\substack{q\leq \frac{x}{RM}\\(q,a)=1}}\Big(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\Big)-\frac{\phi(a)}{a}\frac{x}{RM}\mu(a,r,RM/r)\Big|$$

$$+\sum_{\substack{r\leq R\\(r,a)=1}}\Big|\sum_{\substack{q\leq \frac{x}{RM}\\(q,a)=1}}\Big(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\Big)\Big|,$$

hence by Theorem 2.1 we get the lower bound

$$\sum_{\substack{r\leq R\\(r,a)=1}}\Big|\sum_{\substack{q\leq \frac{x}{RM}\\(q,a)=1}}\Big(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\Big)\Big| \geq$$

$$\frac{\phi(a)}{a}\frac{x}{RM}\sum_{\substack{r\leq R\\(r,a)=1}}|\mu(a,r,RM/r)|-O_\epsilon\Big(\frac{x}{M^{\frac{743}{538}-\epsilon}}\Big),$$

since for M large enough, $\mu(a,r,RM/r)\leq 0$. For the upper bound, we write

$$\sum_{\substack{r\leq R\\(r,a)=1}}\Big|\sum_{\substack{q\leq \frac{x}{RM}\\(q,a)=1}}\Big(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\Big)\Big|$$

$$\leq \sum_{\substack{r\leq R\\(r,a)=1}}\Big|\sum_{\substack{q\leq \frac{x}{RM}\\(q,a)=1}}\Big(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\Big)$$

$$-\sum_{\substack{r\leq R\\(r,a)=1}}\frac{\phi(a)}{a}\frac{x}{RM}\mu(a,r,RM/r)\Big|+\sum_{\substack{r\leq R\\(r,a)=1}}\Big|\frac{\phi(a)}{a}\frac{x}{RM}\mu(a,r,M)\Big|$$

$$\leq \frac{\phi(a)}{a}\frac{x}{RM}\sum_{\substack{r\leq R\\(r,a)=1}}|\mu(a,r,RM/r)|+O_\epsilon\Big(\frac{x}{M^{\frac{743}{538}-\epsilon}}\Big).$$

The result follows by the definition of $\mu(a,r,RM/r)$. Note that if $a=\pm1$, then we

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have

$$\begin{aligned}2\sum_{\substack{r\le R\\(r,a)=1}}|\mu(a,r,RM/r)|\\&=\sum_{r\le R}\Big(\log(RM/r)+2C_5+\sum_{p\mid r}\frac{\log p}{p}\Big)\\&=(R+O(1))\Big(\log M+1+2C_5+O\Big(\frac{\log R}{R}\Big)\Big)+\sum_{p\le R}\frac{\log p}{p}\Big\lfloor\frac{R}{p}\Big\rfloor,\end{aligned}$$

by Stirling's approximation. The last sum can be handled without much effort:

$$\begin{aligned}\sum_{p\le R}\frac{\log p}{p}\Big\lfloor\frac{R}{p}\Big\rfloor&=R\sum_{p\le R}\frac{\log p}{p^2}+O\Big(\sum_{p\le R}\frac{\log p}{p}\Big)\\&=R\Big(\sum_{p}\frac{\log p}{p^2}+O\Big(\frac{1}{R}\Big)\Big)+O(\log R).\end{aligned}$$

Hence,

$$\sum_{\substack{r\le R\\(r,a)=1}}|\mu(a,r,RM/r)|=R\left(\frac{1}{2}\log M+C_6\right)+O(\log(RM)).\qquad\blacksquare$$

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