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CONTINUITY OF THE SUPERPOSITION OPERATOR ON ORLICZ-SOBOLEV SPACES

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We give sufficient conditions for a homogeneous superposition operator to be a continuous mapping between Orlicz-Sobolev spaces. This extends a result of Marcus and Mizel concerning mappings between Sobolev spaces.

1. INTRODUCTION

Let Ω be a domain in $\mathbb{R}^N$, let $\mathcal{M}(\Omega)$ be the space of real measurable functions defined on Ω, and, for a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$, define the *homogeneous superposition operator* $T_f : \mathcal{M}(\Omega) \to \mathcal{M}(\Omega)$ by

$$T_f u = f \circ u, \quad u \in \mathcal{M}(\Omega). \tag{1.1}$$

In [**9**], Marcus and Mizel show that (under certain conditions) T_f is continuous as an operator from the Sobolev space $W^{1,p}(\Omega)$ to the Sobolev space $W^{1,r}(\Omega)$. Here we consider the continuity of T_f on Orlicz-Sobolev spaces, and show that results analogous to those of Marcus and Mizel (for $1 < p < \infty$) hold for Orlicz-Sobolev spaces.

We remark that in both [**5**] and [**8**], results were obtained for the non-homogeneous superposition operator T_f acting on vector-valued functions $u = (u_1, \cdots, u_m)$, where T_f is defined by

$$(T_f u)(x) = f(x, u(x))$$

for $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$. An example, given in Section 6, suggests that we cannot expect to obtain theorems on continuity of the type found here and in [**9**] in the general case.

2. PRELIMINARIES

ORLICZ SPACES. We shall use the properties of N-functions and Orlicz spaces as given in [**7**]. A brief summary of most of the definitions and theorems we need can also be found in [**2**]. To set the notation, and for later reference, we list a few properties below.

(i) An N-function M satisfies the Δ_2*-condition* (or $M \in \Delta_2$) if there exists a constant $u_0 \geqslant 0$ and a real-valued function k_M such that

$$M(lu) \leqslant k_M(l)M(u) \text{ for } u \geqslant u_0, \quad l > 1.$$

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(ii) If M is an N-function, the *Orlicz Space* $L_M = L_M(\Omega)$, is the set of all functions u, measurable on Ω, such that there exists a constant $\lambda > 0$ such that $\int_\Omega M(\lambda u) < \infty$.

(iii) We shall use either the *Luxemburg norm*

$$|||u|||_M = |||u|||_{M,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega M(u/\lambda) \leqslant 1\right\}$$

or the equivalent *Orlicz norm*

$$\|u\|_M = \sup_{\int_\Omega \widetilde{M}(v)\leqslant 1} \left\{\int_\Omega |\, u\, v\,|\right\}$$

according to convenience. (Here $\widetilde{M}(v) = \sup_{u \geqslant 0}\{u \mid v \mid - M(u)\}$ denotes the N-function complementary to M.)

We have the following expressions for the norms of the characteristic function of a measurable set $A \subset \Omega$:

$$\|\chi_A\|_{P,\Omega} = \begin{cases} 0, & \text{if } m_N(A) = 0, \\ m_N(A)\widetilde{P}^{-1}[1/m_N(A)], & m_N(A) \neq 0 \end{cases}$$

$$|||\chi_A|||_{P,\Omega} = \begin{cases} 0, & \text{if } m_N(A) = 0, \\ 1/P^{-1}[1/m_N(A)], & m_N(A) \neq 0, \end{cases}$$

where $m_N(\cdot)$ denotes Lebesgue measure in $\mathbb{R}^N$.

(iv) Let P and M be N-functions, where $M \in \Delta_2$.

(α) Suppose $P \circ M^{-1}$ is an N-function. If $u \in L_P(\Omega)$, then $M \circ u \in L_{P\circ M^{-1}}(\Omega)$, and

$$|||M(u)|||_{P\circ M^{-1}} \leqslant \text{ const } (1 + k_M(|||u|||_P)).$$

If further $P \circ (M')^{-1}$ is an N-function, and M' is strictly increasing, then

$$|||M'(u)|||_{P\circ(M')^{-1}} \leqslant \text{ const } (1 + k_{M'}(|||u|||_P)).$$

(See [5].)

(v) If R and P are N-functions, we write $R \preccurlyeq P$ if there exist constants u_0 and k such that $R(u) \leqslant P(ku)$ for all $u \geqslant u_0$; and $R \prec P$ if $\lim_{u\to\infty} R(\lambda u)/P(u) = 0$ for all $\lambda > 0$.

(vi) For $R \prec P$, the *multiplicator space* $L_R : L_P$ is defined as the set of all functions v on Ω such that $uv \in L_R$ for all $u \in L_P$. We define the N-function $R : P$ by

$$(R : P)(u) = \sup_{v\geqslant 0}\{R(uv) - P(v)\}$$

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and then

$$L_R : L_P = L_{R:P}.$$

(See [1].)

(vii) Let P be a strictly convex N-function satisfying the Δ_2-condition. If both $u_n \to u$ $E_{\tilde{P}}(\Omega)$-weakly and $|||u_n|||_{P,\Omega} \to |||u|||_{P,\Omega}$, then $|||u_n - u|||_{P,\Omega} \to 0$ as $n \to \infty$. (See [6].)

(viii) If $P \in \Delta_2$, then $u \in L_P(\Omega)$ has an *absolutely continuous* L_P *norm*; that is, given, $\varepsilon > 0$, there exists $\delta > 0$ such that, for every measurable set $E \subset \Omega$ with $m_N(E) < \delta$, we have

$$\|u\chi_E\|_{P,\Omega} < \varepsilon.$$

(The same result holds with $|||\cdot|||$ instead of $\|\cdot\|$.)

ORLICZ-SOBOLEV SPACES. We shall use the definitions and properties of Orlicz-Sobolev spaces as given in [2].

(ix) The *Sobolev conjugate* N-function P_* of an N-function P is defined by

$$P_*^{-1}(s) = \int_0^{|s|} P^{-1}(t)t^{-1-1/N}dt$$

where it is assumed that, if necessary, $P(t)$ is redefined for small values of t (giving an equivalent N-function) so that

$$\int_0^1 P^{-1}(t)t^{-1-1/N}dt < \infty.$$

(x) The *Orlicz-Sobolev* space $W^1L_P(\Omega)$ is defined as the set of all functions u in $L_P(\Omega)$ whose distributional derivatives $\partial_i u$ also belong to $L_P(\Omega)$.

A norm $\|u\|_{1,P,\Omega} = \|u\|_{1,P}$ may be defined on $W^1L_P(\Omega)$ by

$$\|u\|_{1,P} = \max\{\|u\|_P, \|\partial_1 u\|_P, \cdots, \|\partial_N u\|_P\}.$$

(xi) If Ω is a bounded domain in $\mathbb{R}^N$ satisfying the cone condition, we have the following continuous imbeddings:

(a) if $P_*^{-1}(\infty) = \infty$, $W^1L_P(\Omega) \to L_{P_*}(\Omega)$;
(b) if $P_*^{-1}(\infty) < \infty$, $W^1L_P(\Omega) \to L_\infty(\Omega) \cap C(\Omega)$.

(xii) Let $A(\Omega)$ denote the set of all functions u, measurable on Ω, such that for almost all lines τ parallel to any coordinate axis x_i, $i = 1, \cdots, N$, u is locally absolutely continuous on $\tau \cap \Omega$. The Orlicz-Sobolev spaces $W^1L_P(\Omega)$ may be given an alternative characterisation in terms of the class $A(\Omega)$, as follows:

Let Ω be a bounded domain in $\mathbb{R}^N$ with the cone property, and let P be an N-function. A function u defined on Ω is in $W^1L_P(\Omega)$ if and only if there exists $\widetilde{u} \in A(\Omega)$ such that

(a) $\widetilde{u} = u$ almost everywhere in Ω;
(b) $\partial\widetilde{u}/\partial x_i \in L_P(\Omega)$, $(i = 1, \cdots, N)$.

Further $\partial\widetilde{u}/\partial x_i = \partial_i u$ almost everywhere in Ω. See [4] and [5].

3. Superposition Operators Mapping $W^1L_P(\Omega)$ into $W^1L_P(\Omega)$

Let Ω be a bounded open subset of $\mathbb{R}^N$, let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly Lipschitz function, and let $f^* : \mathbb{R} \to \mathbb{R}$ be any Borel function (which may be taken to be bounded) such that $f^* = f'$ almost everywhere in $\mathbb{R}$. By using the arguments in Section 2 of [8] (replace their Lemma 1.5 by our 2(xii)) we can show that 2.1 in [9] holds for Orlicz spaces; that is,

$$u \in W^1L_P(\Omega) \Rightarrow \left\{ \begin{array}{c} f \circ u \in W^1L_P(\Omega) \\ \partial_i(f \circ u) = (f^* \circ u)\partial_i u \quad \text{almost everywhere in } \Omega. \end{array} \right\}$$

We can now modify the proof of Theorem 1 in [9] to obtain the following:

Theorem 3.1. *Let Ω and f be as above, and let P be a strictly convex N-function satisfying the Δ_2-condition. Then the mapping $T_f : W^1L_P(\Omega) \to W^1L_P(\Omega)$ is continuous.*

Proof: Let $u_n \to u$ in $W^1L_P(\Omega)$. By proceeding as in the proof of Theorem 1 $(p > 1)$ in [9], we find that $f \circ u_n \to f \circ u$ in $L_P(\Omega)$ and that

$$\partial_i(f \circ u_n) \to \partial_i(f \circ u) \qquad E_{\widetilde{P}}(\Omega) - \text{ weakly.} \tag{3.1}$$

If f^* is the characteristic function of a Borel set, $g^*(t) = f^*(t) - 1/2$, and $g(t) =$

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$f(t) - t/2$, then

$$\begin{aligned}
\|\partial_i(g\circ u_n)\|_{P,\Omega} &= \sup_{\int \widetilde{P}(v)\leqslant 1} \int_\Omega |[g^*(u_n)\partial_i u_n]v| \\
&= \sup_{\int \widetilde{P}(v)\leqslant 1} \int_\Omega \frac{1}{2}\,|(\partial_i u_n)v| \\
&= \frac{1}{2}\,\|\partial_i u_n\|_{P,\Omega} \to \frac{1}{2}\,\|\partial_i u\|_{P,\Omega} \\
&= \sup_{\int \widetilde{P}(v)\leqslant 1} \int_\Omega \left|\left(\frac{1}{2}\partial_i u\right)v\right| \\
&= \sup_{\int \widetilde{P}(v)\leqslant 1} \int_\Omega |[g^*(u)\partial_i u]v| \\
&= \|\partial_i(g\circ u)\|_{P,\Omega}\,. \qquad (3.2)
\end{aligned}$$

By 2.1 (vii), (3.1) and (3.2), $\partial_i(g\circ u_n) \to \partial_i(g\circ u)$ in $L_P(\Omega)$, and so $\partial_i(f\circ u_n) \to \partial_i(f\circ u)$ in $L_P(\Omega)$, in this case.

The proof of Theorem 3.1 can now be completed as in the proof of Theorem 1 $(p > 1)$ in [**9**]. ∎

4. Vitali's Theorem

We shall need a version of Vitali's convergence theorem for Orlicz spaces. The following is adequate for our requirements.

Theorem 4.1. *(Vitali). Let $\Omega \subset \mathbb{R}^N$ have finite measure, let P be an N-function, let $\{f_n\}$ be a sequence in $L_P(\Omega)$, and let f be a measurable function on Ω. Then the following two conditions are sufficient for the convergence of $\{f_n\}$ to f in $L_P(\Omega)$:*

(i) *$\{f_n\}$ converges to f in measure.*

(ii) *For each $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $A \subset \mathbb{R}^N$ and $m_N(A) < \delta$, then*

$$|||f_n\chi_A|||_{P,\Omega} < \varepsilon \quad \textit{for all } n\in\mathbb{N},$$

where $\mathbb{N}$ denotes the set of all positive integers.

If $P \in \Delta_2$, then (i) and (ii) are necessary for $\{f_n\}$ to converge to f in $L_P(\Omega)$.

Proof: It is convenient to introduce the following notation: for $m \geqslant 1$, $n \geqslant 0$, and $\alpha > 0$, let

$$A_{mn} = A_{mn}(\alpha) = \{x\in\Omega : |f_m(x) - f_n(x)| \geqslant \alpha\}. \qquad (4.1)$$

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Observe that

$$\|\!|f_m - f_n|\!\|_{P,\Omega} \geqslant \alpha \,\|\!|\chi_{A_{mn}}|\!\|_{P,\Omega}\,. \tag{4.2}$$

(a) **Sufficiency**. Suppose (i) and (ii) hold. Let $\varepsilon > 0$, let δ be as in (ii), and let $\alpha = \varepsilon / \|\!|\chi_\Omega|\!\|_{P,\Omega}$. From (i), there exists a positive integer $K = K(\varepsilon)$ such that if $m, n > K$

$$m_N(A_{mn}) < \delta. \tag{4.3}$$

From (ii) and (4.3),

$$\|\!|f_i \chi_{A_{mn}}|\!\|_{P,\Omega} < \varepsilon \tag{4.4}$$

for all $i \in \mathbb{N}$ and $m, n > K$.

Thus, using (4.2) and (4.4), we have

$$\begin{aligned}\|\!|f_m - f_n|\!\|_{P,\Omega} &\leqslant \left\|\!\left|(f_m - f_n)\chi_{\Omega\setminus A_{mn}}\right|\!\right\|_{P,\Omega} + \|\!|f_m\chi_{A_{mn}}|\!\|_{P,\Omega} + \|\!|f_n\chi_{A_{mn}}|\!\|_{P,\Omega} \\ &\leqslant \alpha \left\|\!\left|\chi_{\Omega\setminus A_{mn}}\right|\!\right\|_{P,\Omega} + \varepsilon + \varepsilon \leqslant 3\varepsilon\end{aligned}$$

for m, $n > K$.

Therefore $\{f_n\}$ is Cauchy in $L_P(\Omega)$ and converges to a limit $\tilde{f}$ in $L_P(\Omega)$. From (4.2), with $f_0 = \tilde{f}$, it follows that $\{f_n\}$ converges to $\tilde{f}$ in measure. Hence $\tilde{f} = f$ almost everywhere in Ω, and so $\{f_n\}$ converges to f in L_P, as required.

(b) **Necessity**. Now suppose $P \in \Delta_2$ and $\|\!|f_m - f|\!\|_{P,\Omega} \to 0$.

From (4.2), with $f_0 = f$, it follows that $\{f_m\}$ converges to f in measure. Thus (i) holds.

We now show that (ii) holds. Let $\varepsilon > 0$, and choose K so that

$$\|\!|f_m - f|\!\|_{P,\Omega} < \varepsilon/2 \quad \text{for } m > K. \tag{4.5}$$

From 2 (viii) we may choose $\delta > 0$ so that

$$\|\!|f\chi_A|\!\|_{P,\Omega} < \frac{\varepsilon}{2} \tag{4.6}$$

and

$$\|\!|f_m\chi_A|\!\|_{P,\Omega} < \varepsilon, \quad m = 1, 2, \cdots, K \tag{4.7}$$

if $m_N(A) < \delta$. From (4.5) and (4.6),

$$\begin{aligned}\|\!|f_m\chi_A|\!\|_{P,\Omega} &\leqslant \|\!|(f_m - f)\chi_A|\!\|_{P,\Omega} + \|\!|f\chi_A|\!\|_{P,\Omega} \\ &< \varepsilon \quad \text{if } m > K.\end{aligned} \tag{4.8}$$

From (4.7) and (4.8), (ii) follows. ∎

The following lemma is needed in our application of Vitali's theorem - Lemma 4.3. Lemma 4.3 is used in Theorem 5.4.

Lemma 4.2. *If* $|||u\chi_A|||_{Q,\Omega} \leqslant 1$ *for measurable* $A \subset \Omega$, *and if* M *and* Q *are* N*-functions such that* $Q \circ M^{-1}$ *is an* N*-function, then*

$$|||M \circ \chi_A|||_{Q\circ M^{-1},\Omega} \leqslant |||u\chi_A|||_{Q,\Omega} \tag{4.9}$$

Proof: Since $|||u\chi_A|||_{Q,\Omega} \leqslant 1$,

$$|||u\chi_A|||_{Q,\Omega} = \inf\left\{0 < \lambda \leqslant 1 : \int_\Omega Q\left(\frac{u\chi_A}{\lambda}\right) \leqslant 1\right\}.$$

For $0 < \lambda \leqslant 1$, $M(u/\lambda) \geqslant M(u)/\lambda$, so

$$Q \circ M^{-1} \circ M(u/\lambda) \geqslant Q \circ M^{-1}(M(u)/\lambda),$$

$$\int_A Q(u/\lambda) \geqslant \int_A Q \circ M^{-1}(M(u)/\lambda),$$

$$\inf\left\{0 < \lambda \leqslant 1 : \int_A Q(u/\lambda) \leqslant 1\right\} \geqslant \inf\left\{0 < \lambda \leqslant 1 : \int_A Q \circ M^{-1}(M(u)/\lambda) \leqslant 1\right\}.$$

(4.9) follows readily from the last inequality. □

Lemma 4.3. *Let* Q, M, *and* R *be* N*-functions such that*

(i) $Q \in \Delta_2$;
(ii) $Q \circ M^{-1}$ *is an* N*-function;*
(iii) $R \preccurlyeq Q \circ M^{-1}$.

Let $\Omega \subset \mathbb{R}^N$ *have finite measure, and let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that there exist constants* a, b *and* c *such that*

(iv) $|f(t)| \leqslant c + a|t| + bM(t)$ *for all* $t \in \mathbb{R}$.

If $u_n \to u$ *in* $L_Q(\Omega)$, *then* $f(u_n) \to f(u)$ *in* $L_R(\Omega)$.

Proof: Since $u_n \to u$ in $L_Q(\Omega)$, by Theorem 4.1 there exists $\eta > 0$ such that if $m_N(A) < \eta$ then

$$|||u_n\chi_A|||_{A,\Omega} \leqslant 1 \quad \text{for all } n \in \mathbb{N}. \tag{4.10}$$

From (iv),

$$\begin{aligned} |||f(u_n)\chi_A|||_{R,\Omega} &\leqslant c\,|||\chi_A|||_{R,\Omega} + a\,|||u_n\chi_A|||_{R,\Omega} \\ &\quad + b\,|||M(u_n)\chi_A|||_{R,\Omega}. \end{aligned} \tag{4.11}$$

From Lemma 4.2, (4.11), and (iii), if $m_N(A) < \eta$

$$|||M(u_n)\chi_A|||_{R,\Omega} \leqslant \text{ const } |||u_n\chi_A|||_{Q,\Omega}.$$

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Since (iii) implies that $R \preccurlyeq Q$, for $m_N(A) < \eta$, we have

$$\|\|f(u_n)\chi_A\|\|_{R,\Omega} \leqslant D[\|\|\chi_A\|\|_{Q,\Omega} + \|\|u_n\chi_A\|\|_{Q,\Omega}] \tag{4.12}$$

for all $n \in \mathbb{N}$ and some constant D.

We may partition Ω into a finite number of subsets Ω_i such that $m_N(\Omega_i) < \eta$. From (4.12), $f(u_n) \in L_Q(\Omega_i)$, and hence $f(u_n) \in L_Q(\Omega)$.

Since $u_n \to u$ in measure and f is continuous, $f(u_n) \to f(u)$ in measure.

We choose δ_1 so that

$$\|\|\chi_A\|\|_{A,\Omega} < \frac{\varepsilon}{2D} \quad \text{for } m_N(A) < \delta_1.$$

Using Vitali's theorem, we choose δ_2 so that

$$\|\|u_n\chi_A\|\|_{P,\Omega} < \frac{\varepsilon}{2D} \quad \text{for all } n \in \mathbb{N} \text{ and } m_N(A) < \delta_2.$$

Thus, by (4.12),

$$\|\|f(u_n)\chi_A\|\|_{R,\Omega} < \varepsilon \quad \text{for all } n \in \mathbb{N},$$

if $m_N(A) < \delta = \min(\eta, \delta_1, \delta_2)$.

By Vitali's theorem, $f(u_n) \to f(u)$ in $L_R(\Omega)$, as required. □

5. Superposition Operators Defined by Functions Bounded by N-Functions

We first give a lemma, corresponding to Lemma 3 in [**9**].

Lemma 5.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ satisfying the cone condition, and let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function. Let P and R be N-functions. Suppose that one of the following sets of conditions holds:*

(a) $P_*^{-1}(\infty) < \infty$ and $R = P$;

or

(b) $P_*^{-1}(\infty) = \infty$ and there exists an N-function $M \in \Delta_2$ such that M' is strictly increasing and

$$| f'(t) | \leqslant a + bM'(t), \tag{5.1}$$

*where a and b are constants. Suppose further that $P_* \circ (M')^{-1}$ and $P_* \circ M^{-1}$ are the principal parts of N-functions (completed, if necessary, by the procedure described in [**5**], page 16) and that*

$$R \prec P, \tag{5.2}$$

$$R : P \preccurlyeq P_* \circ (M')^{-1}, \tag{5.3}$$

$$R \preccurlyeq P_* \circ M^{-1}. \tag{5.4}$$

$$\textit{If } u \in W^1L_P(\Omega), \textit{ then } f \circ u \in W^1L_R(\Omega) \tag{5.5}$$

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and

$$\partial_i(f\circ u) = (f^*\circ u)\partial_i u \quad \textit{almost everywhere in } \Omega. \tag{5.6}$$

PROOF: Case (a) of the theorem follows from the Orlicz-Sobolev imbedding theorem 2(xi)(b).

Now suppose the hypotheses (b) hold. Let $v = f\circ u$. By 2(xii), there exists $\widetilde{u}\in A(\Omega)$ such that $\widetilde{u} = u$ almost everywhere in Ω, and then $\widetilde{v} = f\circ\widetilde{u} = v$ almost everywhere in Ω. From (5.1), (5.2) and (5.3),

$$\begin{aligned}\left|\!\left|\!\left|\frac{\partial\widetilde{v}}{\partial x_i}\right|\!\right|\!\right|_{R,\Omega} &\leqslant \left|\!\left|\!\left| a + bM'(|\widetilde{u}|)\frac{\partial\widetilde{u}}{\partial x_i}\right|\!\right|\!\right|_{R,\Omega}\\ &\leqslant \text{ const }\left[\left|\!\left|\!\left|\frac{\partial\widetilde{u}}{\partial x_i}\right|\!\right|\!\right|_{P,\Omega} + |\!|\!|M'(|\widetilde{u}|)|\!|\!|_{R:P,\Omega}\left|\!\left|\!\left|\frac{\partial\widetilde{u}}{\partial x_i}\right|\!\right|\!\right|_{P,\Omega}\right]\\ &\leqslant \text{ const }\left[|\!|\!|\widetilde{u}|\!|\!|_{1,P,\Omega} + |\!|\!|M'(|\widetilde{u}|)|\!|\!|_{P_*\circ(M')^{-1},\Omega}\,|\!|\!|\widetilde{u}|\!|\!|_{1,P,\Omega}\right].\end{aligned}$$

From the Orlicz-Sobolev imbedding theorem 2(xi)(a), $|\!|\!|u|\!|\!|_{P_*,\Omega} < \infty$, so by 2(iv), $|\!|\!|M'(|\widetilde{u}|)|\!|\!|_{P_*\circ(M')^{-1},\Omega} < \infty$. Thus we have

$$\left|\!\left|\!\left|\frac{\partial\widetilde{v}}{\partial x_i}\right|\!\right|\!\right|_{R,\Omega} = |\!|\!|\partial_i f(u)|\!|\!|_{R,\Omega} < \infty. \tag{5.7}$$

Integrating the inequality (5.1), we obtain

$$|f(t) - f(0)| \leqslant a\,|t| + bM(t).$$

Therefore, using (5.2) and (5.4),

$$\begin{aligned}|\!|\!|f(u) - f(0)|\!|\!|_{R,\Omega} &\leqslant a\,|\!|\!|u|\!|\!|_{R,\Omega} + b\,|\!|\!|M(u)|\!|\!|_{R,\Omega}\\ &\leqslant \text{ const } [|\!|\!|u|\!|\!|_{P,\Omega} + |\!|\!|M(u)|\!|\!|_{P_*\circ M^{-1},\Omega}]\\ &< \infty \qquad (5.8)\end{aligned}$$

from 2 (iv), because $|\!|\!|u|\!|\!|_{P_*,\Omega<\infty}$. (5.7) and (5.8) establish (5.5), and then (5.6) follows from 2(xii). ∎

EXAMPLES OF N-FUNCTIONS SATISFYING THE CONDITIONS OF LEMMA 5.1.

EXAMPLE 5.2. Suppose $q > 0$, and for $1 < r < p < N$, define $r:p$ by

$$\frac{1}{r:p} = \frac{1}{r} - \frac{1}{p}$$

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and p_* by

$$\frac{1}{p_*} = \frac{1}{p} - \frac{1}{N}.$$

Now suppose that

$$\frac{q}{p_*} \leqslant \frac{1}{r:p}. \tag{5.9}$$

Let $R(t) = 1/r\,|t|^r$, $P(t) = 1/p\,|t|^p$, and $M(t) = 1/(q+1)\,|t|^{q+1}$. Then $P_*^{-1}(\infty) = \infty$, $R \prec P$, $M \in \Delta_2$ and $M'(t)$ is strictly increasing for $t > 0$. An elementary calculation (see 2(ix)) shows that

$$(R:P)(t) = 1/(r:p)\,|t|^{r:p}.$$

From (5.9), we have

$$R:P \preccurlyeq P_* \circ (M')^{-1}.$$

It follows from (5.9) that

$$\frac{1}{r} \leqslant \frac{q+1}{p_*}$$

from which we obtain

$$R \preccurlyeq P_* \circ M^{-1}.$$

This example shows that our Lemma 5.1 contains Lemma 3.1 (for $r < p < N$) in [**8**] as a particular case. For $p > N$, and $r = p$, we obtain Marcus and Mizel's result by defining P as before.

NOTE 1. There appears to be a typographical error in [**9**]. In the statement of Lemma 3, part (b), the case $p < N$ "$1/r \leqslant (q+1)/p$" should read "$1/r \geqslant (q+1)/p_*$".

NOTE 2. In the case of power functions (5.4) follows from (5.3). It would be interesting to know if the same holds in the general case.

EXAMPLE 5.3. Suppose M, P, and R are N-functions such that

(i) $M'(u)$ is continuous and strictly increasing for $u > 0$,
(ii) $M \in \Delta_2$,
(iii) $P_*^{-1}(\infty) = \infty$,
(iv) $R^{-1}(t) = M' \circ P_*^{-1}(t) \cdot P^{-1}(t)$ for large t,
(v) $P_* \circ (M')^{-1}$ is an N-function.

Since $R \prec P$ is equivalent to $\lim_{t\to\infty} (P^{-1}(t))/(R^{-1}(t)) = 0$, (5.2) follows from (iv). It was shown in [5] (Lemma 3.3) that (i), (iii), (iv) and (v) imply that

$$|||uv|||_R \leqslant \text{ const } |||v|||_{P_*\circ(M')^{-1},\Omega}\,|||u|||_{P,\Omega}$$

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whence (see 2(vi))

$$L_{P_*\circ(M')^{-1}}(\Omega) \subseteq L_{R:P}(\Omega),$$

and so (see [7], Theorem 13.1)

$$R : P \preccurlyeq P_* \circ (M')^{-1}.$$

Thus (5.3) holds.

It was also shown in [5] that (i), (iii), (iv) and (v) imply that $P_* \circ M^{-1}$ is the principal part of an N-function and that $R_* \sim P_* \circ M^{-1}$. (5.4) then follows from the fact that $R \prec R_*$. (See [3], Lemma 4.14.)

We can construct particular cases of N-functions satisfying (i) – (v) of Example 5.3 by taking M and P to be power functions, as in Example 5.2. (iv) then defines R as a power function.

It was shown in [5] that there exist N-functions satisfying (i) – (v), for which only P was a power function: for $1 < p < N$ define $P(t)$ by $P(t) = c\,|t|^p$ (c a constant), for $q > 0$ define M by $M(t) = |t|^{q+1}(ln\,|t| + 1)$ for large t and define R by (iv) for large t. For the details, see [5].

Theorem 5.4 below corresponds to Theorem 2 in [9].

Theorem 5.4. *Let f, M, P, R and Ω be as in Lemma 5.1, and suppose further that $P \in \Delta_2$ and is strictly convex. Then the mapping $T_f : W^1L_P(\Omega) \to W^1L_R(\Omega)$ is continuous.*

Proof: Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz and u_n converge to u in $W^1L_P(\Omega)$. We need to show that

$$T_f u_n \to T_f u \text{ in } W^1L_R(\Omega). \tag{5.10}$$

We consider the case $\|\|u_n\|\|_{L_\infty(\Omega)}$ is unbounded; the case $\|\|u_n\|\|_{L_\infty(\Omega)}$ is bounded follows similarly to that in Theorem 2 of [9].

Suppose that f, M, P and R are as in Lemma 5.1 (b), and that $P \in \Delta_2$ and is strictly convex.

Let $w \in W^1L_P(\Omega)$ and $c > 0$. Following [9], we use the notation

$$S_i w = (f^* \circ w)\partial_i w, \quad i = 1, \cdots, N$$
$$A_c(w) = \{x \in \Omega : |w(x)| > c\}$$
$$w_c = h_c \circ w$$

where $h_c(t) = t$ for $|t| \leqslant c$, $h_c(t) = +c(-c)$ for $t > c(< -c)$. Then

$$\begin{aligned} S_i w_c &= 0 \quad \text{almost everywhere in } A_c(w) \\ S_i w_c &= S_i w \quad \text{almost everywhere in } \Omega \setminus A_c(w). \end{aligned} \tag{5.11}$$

For the details, see the proof of Theorem 2 in [9].
Since $u_n \to u$ in $L_{P_*}(\Omega)$, Lemma 4.3 shows that $T_f u_n \to T_f u$ in $L_R(\Omega)$. By Theorem 3.1, $u_{n,c} \to u_c$ in $W^1 L_P(\Omega)$, and so $T_f u_{n,c} \to T_f u_c$ in $W^1 L_P(\Omega)$. This implies that

$$S_i u_{n,c} \to S_i u_c \text{ in } L_R(\Omega) \tag{5.12}$$

because $R \prec P$.

Because u_n converges in $W^1 L_P(\Omega)$, the sequence $|||u_n|||_{P_*,\Omega}$ is bounded, from the Orlicz-Sobolev inequality, and therefore so is the sequence $|||M'(u_n)|||_{P_* \circ (M')^{-1},\Omega}$, from 2(iv). Then

$$\begin{aligned} |||S_i u_{n,c} - S_i u_n|||_{R,\Omega} &\leqslant \left|\left|\left| S_i u_n \chi_{A_c(u_n)} \right|\right|\right|_{R,\Omega} \\ &\leqslant C_1 \, |||a + bM'(|u_n|)|||_{R:P,\Omega} \left|\left|\left| \partial_i u_n \chi_{A_c(u_n)} \right|\right|\right|_{P,\Omega} \\ &\leqslant C_2 \, |||a + bM'(|u_n|)|||_{P_* \circ (M')^{-1},\Omega} \left|\left|\left| \partial_i u_n \chi_{A_c(u_n)} \right|\right|\right|_{P,\Omega} \\ &\leqslant C_3 \left|\left|\left| \partial_i u_n \chi_{A_c(u_n)} \right|\right|\right|_{P,\Omega}, \end{aligned} \tag{5.13}$$

where C_1, C_2 and C_3 are constants. Similar arguments show that

$$|||S_i u_c - S_i u|||_{R,\Omega} \leqslant C_4 \left|\left|\left| \partial_i u \chi_{A_c(u)} \right|\right|\right|_{P,\Omega} \tag{5.14}$$

where C_4 is a constant.

Let $\varepsilon > 0$. By Vitali's Theorem and 2(viii) we may choose $\delta > 0$ so that both

$$C_3 \, |||\partial_i u_n \chi_A|||_{P,\Omega} < \varepsilon$$

and

$$C_4 \, |||\partial_i u \chi_A|||_{P,\Omega} < \varepsilon$$

if $m_N(A) < \delta$. As shown in [9], we may choose $n(\delta)$ and $c(\delta)$ so that $m_N(A_c(u)) < \delta$ and $m_N(A_c(u_n)) < \delta$ for $n \geqslant n(\delta)$, $c \geqslant c(\delta)$. We thus have

$$|||S_i u_{n,c} - S_i u_n|||_{R,\Omega} < \varepsilon \tag{5.15}$$

and

$$|||Su_c - Su|||_{R,\Omega} < \varepsilon \tag{5.16}$$

for $n \geqslant n(\delta)$ and $c \geqslant c(\delta)$.

(5.12), (5.15) and (5.16) show that

$$S_i u_n \to S_i u \text{ in } L_R(\Omega). \tag{5.17}$$

The theorem follows from (5.10) and (5.17).

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6. Nonhomogeneous Superposition Operators

It is natural to ask if any of the continuity results for homogeneous superposition operators extend to the nonhomogeneous case. We construct $f : \mathbb{R}^2 \to \mathbb{R}$ which is uniformly Lipschitz continuous and a sequence u_i converging to 0 in $W^{1,\infty}[-\pi, \pi]$ such that $T_f(u_i)$ does not converge in $W^{1,2}[-\pi, \pi]$ to $T_f(0) = 0$, where

$$T_f(u)(x) = f(x, u(x)).$$

Then $T_f : W^{1,\infty}[-\pi, \pi] \to W^{1,2}[-\pi, \pi]$. If f is independent of x, then T_f is continuous in $W^{1,2}[-\pi, \pi]$. In view of these observations it is difficult to see what necessary and sufficient conditions can be given for nonhomogeneous operators.

Example 6.1. For each natural number n set $f(x, 2^{-n}) = 2^{-n} \sin 2^n x$ and $f(x, 0) = 0$. Thus $|f_x(x, 2^{-n})| = |\cos 2^n x| \leqslant 1$ for all n. Assuming $n < m$

$$\begin{aligned} \left|f(x, 2^{-n}) - f(x, 2^{-m})\right| &\leqslant \left|2^{-n} \sin 2^n x\right| + \left|2^{-m} \sin 2^m x\right| \\ &\leqslant 2^{-n+1} \\ &\leqslant 4\left|2^{-n} - 2^{-m}\right| \\ \text{and } \left|f(x, 2^{-n}) - f(x, 0)\right| &= \left|2^{-n} \sin 2^n x\right| = \left|2^{-n} - 0\right|. \end{aligned}$$

Thus f is uniformly Lipschitz on the closed set

$$G = \left\{(x, y) \in [-\pi, \pi] \times \mathbb{R} : y = 2^{-n}, \text{ some } n, \text{ or } y = 0\right\}.$$

Extend f to $[-\pi, \pi] \times \mathbb{R}$ with the same Lipschitz constant. Let $u_n = 2^{-n}$ so u_n converges to 0 in $W^{1,\infty}[-\pi, \pi]$. We show $T_f(u_n)$ does not converge in $W^{1,2}[-\pi, \pi]$ to $T_f(0) = 0$. Now $|f(x, u_n)| \leqslant 2^{-n}$ and $\dfrac{d}{dx} f(x, u_n) = \sin 2^n x$ so $\dfrac{d}{dx} f(x, u_n)$ converges weakly to 0 in $L_2[-\pi, \pi]$ and $\left\|\dfrac{d}{dx} f(x, u_n)\right\|_{L_2} = \pi$. Thus $\dfrac{d}{dx} f(x, u_n)$ does not converge in $L_2[-\pi, \pi]$ and thus $T_f(u_n)$ does not converge in $W^{1,2}[-\pi, \pi]$.

Remark. This counterexample also shows that necessary and sufficient conditions for the continuity of homogeneous superposition operators on vector valued Sobolev spaces are likely to be complicated. Just set $T_f(u_1, u_2)(x) = f(u_1(x), u_2(x))$. Thus $T_f : W^{1,\infty}[-\pi, \pi] \times W^{1,\infty}[-\pi, \pi] \to W^{1,2}[-\pi, \pi]$. Then f is uniformly Lipschitz continuous and $(u_{1n}, u_{2n}) = (x, 2^{-n})$ converges in $W^{1,\infty}[-\pi, \pi] \times W^{1,\infty}[-\pi, \pi]$ to $(x, 0)$ but $T_f(u_{1n}, u_{2n})$ does not converge to $T_f(x, 0)$.

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