ON THE JOINT SPECTRA OF DOUBLY COMMUTING n-TUPLES OF SEMI-NORMAL OPERATORS

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Let H be a complex Hilbert space. For any operator (bounded linear transformation) T on H, we denote the spectrum of T by $\sigma(T)$. Let $T=(T_1,\ldots,T_n)$ be an n-tuple of commuting operators on H. Let $\mathrm{Sp}(T)$ be the Taylor joint spectrum of T. We refer the reader to [**8**] for the definition of $\mathrm{Sp}(T)$. A point $\nu=(\nu_1,\ldots,\nu_n)$ of $\mathbb{C}^n$ is in the joint approximate point spectrum $\sigma_\pi(T)$ of T if there exists a sequence $\{x_k\}$ of unit vectors in H such that

$$\|(T_j-\nu_j)x_k\|\to 0 \quad \text{as} \quad k\to\infty \quad \text{for} \quad j=1,\ldots,n.$$

A point $\nu=(\nu_1,\ldots,\nu_n)$ of $\mathbb{C}^n$ is in the joint approximate compression spectrum $\sigma_\delta(T)$ of T if there exists a sequence $\{x_k\}$ of unit vectors in H such that

$$\|(T_j-\nu_j)^*x_k\|\to 0 \quad \text{as} \quad k\to\infty \quad \text{for} \quad j=1,\ldots,n.$$

A point $\nu=(\nu_1,\ldots,\nu_n)$ of $\mathbb{C}^n$ is in the joint point spectrum $\sigma_p(T)$ of T if there exists a non-zero vector x in H such that $(T_j-\nu_j)x=0$ for all j, $1\leq j\leq n$.

Consult [**4**] for further details regarding the notions of $\sigma_\pi(T)$, $\sigma_\delta(T)$ and $\sigma_p(T)$. It is well known that $\sigma_\pi(T)\cup\sigma_\delta(T)\subseteq \mathrm{Sp}(T)$.

LEMMA 1 (S. K. Berberian [**1**]). *Let $B(H)$ be the *-algebra of all bounded operators on H. Then there exists an extension space K of H and a faithful *-homomorphism of $B(H)$ into $B(K): S\to S^0$ such that*

$$\sigma_\pi(S)=\sigma_\pi(S^0)=\sigma_p(S^0).$$

Furthermore, if $T=(T_1,\ldots,T_n)$ is an n-tuple of commuting operators on H then

$$\sigma_\pi(T_1,\ldots,T_n)=\sigma_\pi(T_1^0,\ldots,T_n^0)=\sigma_p(T_1^0,\ldots,T_n^0).$$

See [**2**] or [**5**, Proposition 3.2] for a proof.

An n-tuple $T=(T_1,\ldots,T_n)$ of operators is said to be *hyponormal* if $T_j^*T_j-T_jT_j^*\geq 0$ for $j=1,\ldots,n$. We say that $T=(T_1,\ldots,T_n)$ is *semi-normal* if each T_j is semi-normal (T_j or T_j^* is hyponormal) for j, $1\leq j\leq n$.

LEMMA 2 [**3**, Corollary 3.8]. *Let $T=(T_1,\ldots,T_n)$ be a doubly commuting n-tuple of hyponormal operators (i.e. $T_iT_j=T_jT_i$ for all i, j and $T_iT_j^*=T_j^*T_i$ for all $i\neq j$, and $T_i^*T_i\geq T_iT_i^*$). Then*

$$\mathrm{Sp}(T)=\sigma_\delta(T). \tag{1}$$

Furthermore, if $T=(T_1,T_2)$ is a doubly commuting pair of operators with T_1 and T_2^* being hyponormal then

$$(\nu_1,\nu_2)\in \mathrm{Sp}(T_1,T_2) \text{ if and only if } (\nu_1,\nu_2^*)\in \mathrm{Sp}(T_1,T_2^*)=\sigma_\delta(T_1,T_2^*). \tag{2}$$

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The following result was proved by Putman [**6**]. See also [**7**].

LEMMA 3. *Let T be a semi-normal operator with $T=A+iB$.*

(i) *If $\lambda\in\sigma(A)$, there exists a real number μ and a sequence $\{x_k\}$ of unit vectors such that*

$$\|(A-\lambda)x_k\|\to 0 \quad \text{and} \quad \|(B-\mu)x_k\|\to 0,$$

that is, $\lambda+i\mu\in\sigma(T)$. Similarly, if $\mu\in\sigma(B)$ then there exists a real number λ and $\{x_k\}$ in H with $\|x_k\|=1$ such that

$$\|(A-\lambda)x_k\|\to 0 \quad \text{and} \quad \|(B-\mu)x_k)\|\to 0.$$

(ii) *Let λ and μ be real numbers. If $\lambda+i\mu\in\sigma(T)$ then $\lambda\in\sigma(A)$ and $\mu\in\sigma(B)$.*

We generalize this result to doubly commuting n-tuples of semi-normal operators.

THEOREM. *Let $T=(T_1,\ldots,T_n)$ be a doubly commuting n-tuple of semi-normal operators with $T_j=A_j+iB_j$, $j=1,\ldots,n$. Write $A=(A_1,\ldots,A_n)$ and $B=(B_1,\ldots,B_n)$.*

(i) *If $\lambda=(\lambda_1,\ldots,\lambda_n)\in\mathrm{Sp}(A)$ then there exists $\mu=(\mu_1,\ldots,\mu_n)\in\mathbb{R}^n$ and a sequence $\{x_k\}$ of unit vectors in H such that*

$$\|(A_j-\lambda_j)x_k\|\to 0 \quad \text{and} \quad \|(B_j-\mu_j)x_k\|\to 0, \qquad j=1,\ldots,n,$$

that is, $\nu=\lambda+i\mu=(\lambda_1+i\mu_1,\ldots,\lambda_n+i\mu_n)\in\mathrm{Sp}(T)$.

An analogous result holds for $\mathrm{Sp}(B)$.

(ii) *Let λ_j and μ_j be real numbers for j, $j=1,\ldots,n$. If $\lambda+i\mu=(\lambda_1+i\mu_1,\ldots,\lambda_n+i\mu_n)\in\mathrm{Sp}(T)$ then $\lambda=(\lambda_1,\ldots,\lambda_n)\in\mathrm{Sp}(A)$ and $\mu=(\mu_1,\ldots,\mu_n)\in\mathrm{Sp}(B)$.*

Proof. It is clear from Lemma 2 (2) and the proof of the theorem given below that there is no loss of generality in assuming that $T=(T_1,\ldots,T_n)$ is hyponormal.

(i) Here we give the proof for $\mathrm{Sp}(A)$. The proof for $\mathrm{Sp}(B)$ is similar. Furthermore, since A is an n-tuple of commuting self-adjoint operators, it is well known that $\mathrm{Sp}(A)=\sigma_\pi(A)$. (Consult [**4**] or Lemma 2.) Thus $\lambda\in\sigma_\pi(A)$. By Lemma 1, we have $\sigma_\pi(A)=\sigma_\pi(A^0)=\sigma_p(A^0)$. Hence $\lambda=(\lambda_1,\ldots,\lambda_n)\in\sigma_p(A^0)$. Set

$$M=\{f\in K:(A_j^0-\lambda_j)f=0, \qquad j=1,\ldots,n\}.$$

We show that M is a reducing subspace for $B_1^0,\ldots,B_n^0$. Since T_j^0 is hyponormal, we have $T_j^{0*}T_j^0-T_j^0T_j^{0*}=2i(A_j^0B_j^0-B_j^0A_j^0)\geq 0$. Set $C_j=i(A_j^0B_j^0-B_j^0A_j^0)$. Thus $C_j\geq 0$. But $(A_j^0-\lambda_j)B_j^0-B_j^0(A_j^0-\lambda_j)=-iC_j$. Therefore, for $f\in M$, $-i(C_jf,f)=((A_j^0-\lambda_j)B_j^0f,f)-(B_j^0(A_j^0-\lambda_j)f,f)=0$. Since $C_j\geq 0$, it follows that $C_jf=0$. This implies that $(A_j^0-\lambda_j)B_j^0f=0$. If $i\neq j$, $(A_j^0-\lambda_j)B_i^0=B_i^0(A_j^0-\lambda_j)$. Thus $(A_j^0-\lambda_j)B_i^0f=0$ for $f\in M$. Hence M is a reducing subspace for $B_1^0,\ldots,B_n^0$. Thus there exists $\mu=(\mu_1,\ldots,\mu_n)\in\sigma_\pi(B_{1|_M}^0,\ldots,B_{n|_M}^0)=\sigma_p(B_{1|_M}^0,\ldots,B_{n|_M}^0)$. This implies that there exists a non-zero vector $f\in M$ such that $B_j^0f=\mu_jf$, $j=1,\ldots,n$. Hence $(\lambda,\mu)\in\sigma_p(A^0,B^0)=\sigma_\pi(A,B)$. Therefore, there exists $\{x_k\}$ in H with $\|x_k\|=1$ such that

$$\|(A_j-\lambda_j)x_k\|\to 0 \quad \text{and} \quad \|(B_j-\mu_j)x_k\|\to 0, \qquad j=1,\ldots,n.$$

(ii) We will prove this part of the theorem by the method of induction. For $n=1$, see Lemma 3(ii). However, we give here a simple proof. If $\nu=\lambda+i\mu\in\sigma(T)$, where T is hyponormal, then there exists a real number μ' such that $\nu'=\lambda+i\mu'\in\partial\sigma(T)$. Here $\partial\sigma(T)$ denotes the boundary of $\sigma(T)$. But $\partial\sigma(T)\subseteq\sigma_\pi(T)$. This means that there exists a sequence $\{x_k\}$ of unit vectors in H such that $((T-\nu')^*(T-\nu')x_k, x_k)\to 0$. But $(T-\nu')^*(T-\nu')=(A-\lambda)^2+(B-\mu)^2+C$. Since $C\geqslant 0$, this implies that $\|(A-\lambda)x_k\|\to 0$. Thus $\lambda\in\sigma(A)$. Next we assume that the theorem is true for $(n-1)$-tuples $(T_1,\ldots,T_{n-1})$. Moreover, if we denote the complex conjugate of a complex number ν by ν^*, then $\mathrm{Sp}(T)=\sigma_\delta(T)=\sigma_\pi(T^*)^*=\sigma_\pi(T^{*0})^*=\sigma_p(T^{*0})^*$. Thus $\nu^*=(\nu_1^*,\ldots,\nu_n^*)\in\sigma_p(T^{*0})$. Hence there exists $f\in K$ such that $(T_j^{*0}-\nu_j^*)f=0$ for $j=1,\ldots,n$. Let $E=\{f\in K:(T_n^{*0}-\nu_j^*)f=0\}$. Since $T_1^0,\ldots,T_{n-1}^0$ are doubly commuting, it is clear that E is a reducing subspace of $T_1^0,\ldots,T_{n-1}^0$. Thus $(\nu_1^*,\ldots,\nu_{n-1}^*)\in\sigma_p(T^{*0}_{1|_E},\ldots,T^{*0}_{n-1|_E})$. This implies that $(\nu_1,\ldots,\nu_{n-1})\in\mathrm{Sp}(T^0_{1|_E},\ldots,T^0_{n-1|_E})$. Note that $(T^0_{1|_E},\ldots,T^0_{n-1|_E})$ is a doubly commuting $(n-1)$-tuple of hyponormal operators. Thus by the assumption, it follows that $(\lambda_1,\ldots,\lambda_{n-1})\in\sigma_\pi(A^0_{1|_E},\ldots,A^0_{n-1|_E})=\sigma_p(A^0_{1|_E},\ldots,A^0_{n-1|_E})$. Therefore, there exists a non-zero vector f_0 such that

$$A_j^0f_0=\lambda_jf_0\quad (j=1,\ldots,n-1)\quad\text{and}\quad T_n^{*0}f_0=\nu_n^*f_0. \tag{3}$$

Let $N=\{f\in K:A_j^0f=\lambda_jf,\ j=1,\ldots,n-1\}$. Thus, by equation (3), we have $\nu_n\in\sigma(T^0_{n|_N})$. Set $\nu_n=\lambda_n+i\mu_n$. Clearly $T^0_{n|_N}$ is hyponormal. But $\partial\sigma(T^0_{n|_N})\subseteq\sigma_\pi(T^0_{n|_N})$. Thus there exists a real number μ_n' such that $\nu_n'=\lambda_n+i\mu_n'\in\partial\sigma(T^0_{n|_N})\subseteq\sigma_\pi(T^0_{n|_N})=\sigma_p(T^0_{n|_N})$. This means that there exists $f\in N$ such that $(T_n^0-\nu_n')f=0$. But $T^0_{n|_N}$ is hyponormal and hence $T^0_{n||_N}-\nu_n'$ is hyponormal. Thus $0=(T^0_{n|_N}-\nu_n')^*(T^0_{n|_N}-\nu_n')f=(A^0_{n|_N}-\lambda_n)^2f+B^0_{n|_N}-\mu_n')^2f+C_nf$. Since $C_n\geqslant 0$, this implies that $(A^0_{n|N}-\lambda_n)f=0$. Therefore, we have $(A_j^0-\lambda_j)f=0$, for $j=1,\ldots,n$. Hence $\lambda\in\sigma_p(A_1^0,\ldots,A_n^0)=\sigma_\pi(A_1,\ldots,A_n)$. Similarly, one shows that $(\mu_1,\ldots,\mu_n)\in\sigma_\pi(B_1,\ldots,B_n)$. This proves the theorem.

Thus we have shown:

$$\mathrm{Sp}(A)=\{\mathrm{Re}\,\nu:\nu\in\mathrm{Sp}(T)\}$$

and

$$\mathrm{Sp}(B)=\{\mathrm{Im}\,\nu:\nu\in\mathrm{Sp}(T)\},$$

where $T=(T_1,\ldots,T_n)$ is an n-tuple of doubly commuting semi-normal operators, and $\mathrm{Re}\,\nu=(\mathrm{Re}\,\nu_1,\ldots,\mathrm{Re}\,\nu_n)$ and $\mathrm{Im}\,\nu=(\mathrm{Im}\,\nu_1,\ldots,\mathrm{Im}\,\nu_n)$.

Corollary. *Let $T=(T_1,\ldots,T_n)$ be a commuting n-tuple of semi-normal operators. If $\mathrm{Sp}(T)\subseteq\mathbb{R}^n$ then T_j is self-adjoint for each j, $j=1,\ldots,n$.*

Proof. Suppose that $T_j=A_j+iB_j$, $B_j\neq 0$ for some j. Then there exists a real number $\mu_j\neq 0$ such that $\mu_j\in\sigma(B_j)$. Then by the above theorem, there exists $\lambda_j\in\mathbb{R}$ such that $\lambda_j+i\mu_j\in\sigma(T_j)$. See also [**5**]. But, by the projection property of Taylor's joint spectra, we have $P_j(\mathrm{Sp}(T))=\sigma(T_j)$, where P_j is the projection onto the jth co-ordinate. Thus we have $\lambda_j+i\mu_j\in\sigma(T_j)=P_j(\mathrm{Sp(T)})$. This contradicts the fact that $\mathrm{Sp}(T)\subseteq\mathbb{R}^n$. Thus T_j is self-adjoint for each j, $j=1,\ldots,n$.

In the passing, we remark that our proof of the theorem also gives a simpler proof of Putman's theorem [Lemma 3].

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