ANZIAM J. 58(2017), 386–396
doi:10.1017/S1446181117000220

ON MULTI-ASSET SPREAD OPTION PRICING IN A WICK–ITÔ–SKOROHOD INTEGRAL FRAMEWORK

XIANGXING TAO ✉ [1] and YAFENG SHI [2]

(Received 14 July, 2016; accepted 16 October, 2016; first published online 24 May 2017)

Abstract

We provide an elementary method for exploring pricing problems of one spread options within a fractional Wick–Itô–Skorohod integral framework. Its underlying assets come from two different interactive markets that are modelled by two mixed fractional Black–Scholes models with Hurst parameters, $H_1 \neq H_2$, where $1/2 \leq H_i < 1$ for $i = 1, 2$. Pricing formulae of these options with respect to strike price $K = 0$ or $K \neq 0$ are given, and their application to the real market is examined.

2010 *Mathematics subject classification*: primary 60H30; secondary 42B20, 91G20, 91G80.

Keywords and phrases: fractional stochastic equation, spread option, Black–Scholes market, fractional Itô formula.

1. Introduction

A spread option is an option whose pay-off depends on the price spread between two correlated underlying assets with values at time t denoted by $S_1(t)$ and $S_2(t)$. Considering the European-type options, the pay-off for a spread option of strike K is $[S_1(T) - S_2(T) - K)]^+$ for a call. For a detailed review of different spread option types, we refer to the work of Carmona and Durrelman [4].

Spread options are popular derivative contracts which are widely traded both on organized exchanges and over the counter in equity, fixed income, foreign exchanges, energy and commodity markets, in order to speculate and manage basic risk. Early work on spread option pricing by Ravindran [16] and Shimko [18] assumed that prices evolve according to a bivariate geometric Brownian motion with constant volatility. This framework is tractable, but it cannot capture the implied volatility smiles. Dempster and Hong [6] advocated models that capture volatility skews on the

[1]School of Science, Zhejiang University of Science and Technology, Hangzhou, 310023, PR China;
e-mail: xxtao@zust.edu.cn.
[2]School of Science, Ningbo University of Technology, Ningbo, 315211, PR China;
e-mail: shiyafonglf@126.com.

two assets by introducing stochastic volatility to the price processes. In a Black–Scholes (B–S) framework (bivariate geometric Brownian motion framework or its extended form), a large number of methods have been developed to solve the problem of spread option pricing by numerical approaches and analytical approximations (see, for example, [3, 5, 7, 11, 12, 16]).

As mentioned above, spread options write on various assets. With increasing risk derived from different markets that needs to be hedged, the spread options that write on assets coming from different markets become ubiquitous. For instance, an investor may keep a long position in one market and a short position in another market. One of the important ways of managing the risk is by writing a spread option on the difference between the long and the short position of their portfolio. On the other hand, the evolution of prices may vary with different markets. Therefore, we focus on the spread options which write on the assets coming from different markets.

It seems unreasonable to use a uniform B–S framework to describe price dynamics of various underlying assets, and a more flexible framework needs to be created. More and more evidence shows that the B–S framework is incompatible with real market data. As summarised by Rostek [17], concerning the stochastic process of Brownian motion, the main criticism drawn from empiricism is at least two-fold. On one hand, real market distributions have been shown to be non-Gaussian (see, for example, Fama [9]), and the debate of recent years has put a great deal of effort into correcting this problem. On the other hand, the processes of observable market values seem to exhibit serial correlation [13]. There has been much less effort to get a grip on this problem by factoring in aspects of persistence. Fractional Brownian motion (fBm) has often been considered to map this kind of behaviour. This is a Gaussian stochastic process that is able to easily capture long-range dependencies as well as self-similarity, and it is an extension of classical Brownian motion and parsimony. It is natural to replace Brownian motion with fBm in the usual financial models which have been around for some time [10, 14, 15, 19]. Additionally, note that within the fractional B–S framework, the Hurst index may vary with different markets. For example, as Bianchi et al. [2] recently estimated, the Hurst index exhibits differences among the DJIA (Dow Jones Industrial Average), the FTSE 100 and the N225 (Nikkei 225). Therefore, we discuss the pricing problem in an fBm context, and use Hurst parameters to characterize different markets of underlying assets.

In our model, the prices of underlying assets follow a mixed geometric fBm, whose Hurst parameters are H_1 and H_2, respectively. Clearly, it is parsimonious that we characterize the fractional market by Hurst parameters. It is well known that the Hurst parameter stands for the intensity of long-range dependence of a fractional B–S market; the larger the Hurst parameter, the more intense is the long-range dependence of the fractional B–S market. So one advantage of our model is that it allows the implementer to calibrate the Hurst parameter to render the model as a good match with real markets. Since fBm is an extension of classical Brownian motion, our model includes the classical model. Moreover, a pricing formula is derived in the paper for spread options within a fractional Wick–Itô–Skorohod (FWIS) integral framework.

Since the model is better aligned with a real market, the pricing formula that we have derived is more practicable. Finally, some figures depicting the price of spread options versus the lifetime of the options are drawn with diverse values of parameters. These figures may give some intuition and insight into the pricing formula.

In the next section, we give some notation and definitions and present the underlying asset price model. In Section 3, the pricing formulae of the spread option towards strike price $K = 0$ and $K \neq 0$ are presented. In Section 4, the behaviour of the pricing formulae of the spread option is displayed in various situations. Conclusions are drawn in Section 5.

2. Fractional Wick–Itô–Skorohod integral framework

It is well known that using the pathwise integral concept, the B–S model based on fBm is not free of arbitrage. To avoid this, we discuss the spread option pricing problem within a fWIS integral framework developed by Biagini et al. [1]. We show that the fractional B–S model is free of arbitrage within this framework.

Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$, equipped with the inner product

$$\langle f, g\rangle_H = H(2H-1)\int_{\mathbb{R}^2} f(s)\, g(t)|t-s|^{2H-2}\, ds\, dt, \quad f, g \in \mathcal{S}(\mathbb{R}),$$

and let the completion of $\mathcal{S}(\mathbb{R})$ be denoted by $L^2_H(\mathbb{R})$. If $f, g \in L^2_H(\mathbb{R})$, then we define the Wick-product by

$$\Big(\int_{\mathbb{R}} f\, dB^{(H)}\Big) \diamond \Big(\int_{\mathbb{R}} g\, dB^{(H)}\Big) = \Big(\int_{\mathbb{R}} f\, dB^{(H)}\Big) \cdot \Big(\int_{\mathbb{R}} g\, dB^{(H)}\Big) - \langle f, g\rangle_H.$$

Moreover, we define the Wick-exponential for $f \in L^2_H(\mathbb{R})$ by

$$\exp^{\diamond}(<\omega, f>) = \exp(<\omega, f> - \tfrac{1}{2}\|f\|^2_H).$$

DEFINITION 2.1. Let $(F_t, t \in [0, T])$ be a stochastic process such that $F \in \mathcal{L}_\phi(0, T)$. Then its fWIS integral, $\int_0^T F_s\, dB^{(H)}_s$, is defined as

$$\int_0^T F_s\, dB^{(H)} = \lim_{|\pi_n|\to 0} \sum_{i=0}^{n} F_{t_i} \diamond (B^{(H)}_{t_{i+1}} - B^{(H)}_{t_i}),$$

where $\pi_n = \{t^{(n)}_i \mid 0 = t^{(n)}_0 < t^{(n)}_1 < \cdots < t^{(n)}_{n-1} < t^{(n)}_n = T\}$ is the division of the time interval $[0, T]$, $|\pi_n| = \max_{1\le i\le n}\{|t^{(n)}_i - t^{(n)}_{i-1}|\}$ and $H > 1/2$.

In this paper, our study is confined to two interactive financial markets, and each market consists of a risky asset (for example, a stock) and a risk-free asset (for example, a bank account). The risk-free asset is denoted by $S_0(t)$ and we assume that it satisfies

$$dS_0(t) = rS_0(t)\, dt, \quad S_0(0) = 1, \quad 0 \le t \le T \tag{2.1}$$

https://doi.org/10.1017/S1446181117000220 Published online by Cambridge University Press

with constant risk-free interest rate $r > 0$. The risky assets from each market are denoted by $S_i(t)$. It is assumed that $S_i(t)$ follows a mixed geometric fBm, that is, $S_1(t)$ and $S_2(t)$, respectively, obey the fractional stochastic differential equations

$$dS_1(t) = S_1(t) \diamond (\mu_1 + \sigma_1 W_1^{(H_1)}(t) + \rho_1 W_2^{(H_2)}(t))\,dt, \tag{2.2}$$

$$dS_2(t) = S_2(t) \diamond (\mu_2 + \sigma_2 W_2^{(H_2)}(t) + \rho_2 W_1^{(H_1)}(t))\,dt, \tag{2.3}$$

where $W_i^{(H_i)} = (dB_i^{(H_i)})/(dt)$ is the white noise with respect to $B_i^{(H_i)}$ and $\mu_i, \sigma_i, \rho_i \geq 0$ are constants for $i = 1, 2$.

We now recall the notion of a Wick-self-financing portfolio from Elliott and Van der Hoek [8].

Definition 2.2. A portfolio $\theta = (\theta_{0,t}, \theta_{1,t}, \theta_{2,t})$ is said to be Wick-self-financing if $\theta_{i,t}$ is fWIS integrable with respect to S_i for $i = 1, 2$, and if $X_t(\theta) = \sum_{i=0}^{2} \theta_{i,t} S_i(t)$ for $0 \leq t \leq T$, then

$$X_t(\theta) = X_0(\theta) + \int_0^t \theta_{1,s}\, d^{\diamond} S_1(s) + \int_0^t \theta_{2,s}\, d^{\diamond} S_2(s),$$

where the fWIS integral with respect to S_i is defined by

$$\int_0^t \theta_{i,s}\, d^{\diamond} S_i(s) = \mu_i \int_0^t \theta_{i,s} S_i(s)\, dt + \sigma_i \int_0^t \theta_{i,s} S_i(s)\, dB^{(H_1)} + \rho_i \int_0^t \theta_{i,s} S_i(s)\, dB^{(H_2)}.$$

Remark 2.3. (i) It has been proved that the above model framework is free of arbitrage with the class of Wick-self-financing portfolios [1, Ch. 3].

(ii) In view of the risk preference, in the case of a fractional Brownian market, there is an additional correction to account for the evolution of the past. More precisely, we have historically induced shifts $-\sigma_1^2 t^{2H_1}/2 - \rho_1^2 t^{2H_2}/2$ and $-\sigma_2^2 t^{2H_2}/2 - \rho_2^2 t^{2H_1}/2$ of the distribution, respectively. This means that a positive prediction for the random process of fBm results in a downward correction of the adjusted drift rate. The relationship below these is that the more promising the prediction of S_T due to the observation of the past, the more evident the mispricing of the stock, and, for equilibrium reasons, the stronger the downward adjustment of the deterministic drift rate.

(iii) If we let $\rho_1 = \rho_2 = 0$, then the equations (2.2) and (2.3) become the classical fractional B–S models that account for the main features of Market 1 and Market 2, respectively. The last term $\rho_1 W_2^{(H_2)}$ in (2.2) reflects the effects of Market 2 on the Market 1; and the last term $\rho_2 W_1^{(H_1)}$ in (2.3) reflects the effects of Market 1 on the Market 2. Generally, the effect from another market is limited, and thus we assume that $\sigma_i > \rho_i$.

In this paper, we formulate the following spread options.

Definition 2.4. A spread option with *strike* K written on $S_i(t)$ is an option that at maturity time T, has pay-off function $\{S_2(T) - S_1(T) - K\}^+$, denoted by $V(t, S_1, S_2)$.

https://doi.org/10.1017/S1446181117000220 Published online by Cambridge University Press

3. Main results

This section is devoted to the pricing of spread options within the fWIS integral framework. We need the following lemma and we omit the proof for brevity.

LEMMA 3.1. *Let $f \in C^{1,2,2}(\mathbb{R}^+, \mathbb{R}, \mathbb{R})$, and let the risk assets $S_1(t)$ and $S_2(t)$ satisfy the fractional stochastic differential equations (2.2) and (2.3), respectively. Then, for $0 < t < T$,*

$$f(t, S_1(t), S_2(t)) = f(0, S_1(0), S_2(0)) + \int_0^t \frac{\partial f}{\partial s}\, ds + \int_0^t \frac{\partial f}{\partial S_1}\, d^{\diamond} S_1(s) + \int_0^t \frac{\partial f}{\partial S_2}\, d^{\diamond} S_2(s) + \int_0^t \Big\{ A S_1^2 \frac{\partial^2 f}{\partial S_1^2} + 2B S_1 S_2 \frac{\partial^2 f}{\partial S_2 \partial S_1} + C S_2^2 \frac{\partial^2 f}{\partial S_2^2} \Big\}\, ds \tag{3.1}$$

almost surely, where

$$A = H_1 \sigma_1^2 t^{2H_1 - 1} + H_2 \rho_1^2 t^{2H_2 - 1},$$
$$B = H_1 \sigma_1 \rho_2 t^{2H_1 - 1} + H_2 \sigma_2 \rho_1 t^{2H_2 - 1},$$
$$C = H_1 \rho_2^2 t^{2H_1 - 1} + H_2 \sigma_2^2 t^{2H_2 - 1}.$$

THEOREM 3.2. *Suppose that the risk-free asset $S_0(t)$ satisfies (2.1) and that the underlying $S_1(t)$ and $S_2(t)$ follow equations (2.2) and (2.3), respectively. Then the price of the spread option $V(t, S_1, S_2)$ with exercise price $K = 0$ and expiry date T at time $t \in [0, T]$ can be expressed as*

$$V(t, S_1, S_2) = S_2 \Phi\Big(\frac{\ln S_2 - \ln S_1 + D/2}{\sqrt{D}}\Big) - S_1 \Phi\Big(\frac{\ln S_2 - \ln S_1 - D/2}{\sqrt{D}}\Big), \tag{3.2}$$

where $D = (\sigma_1 - \rho_2)^2 T_1 + (\sigma_2 - \rho_1)^2 T_2$.

PROOF. Using Δ-hedge strategy, we create a portfolio

$$\Pi = \Big(\Big[V(t, S_1, S_2) - S_1(t)\frac{\partial V}{\partial S_1} - S_2(t)\frac{\partial V}{\partial S_2}\Big] \Big/ S_0(t),\ \frac{\partial V}{\partial S_1},\ \frac{\partial V}{\partial S_2}\Big)$$

to replicate the spread option $V(t, S_1, S_2)$. By Lemma 3.1 and equation (2.1),

$$\begin{aligned} & V(t, S_1(t), S_2(t)) - V(0, S_1(0), S_2(0)) \\ &= \int_0^t \Big[\Big(V(s, S_1, S_2) - S_1(s)\frac{\partial V}{\partial S_1} - S_2(s)\frac{\partial V}{\partial S_2}\Big) \Big/ S_0(s)\Big]\, dS_0(s) \\ &\quad + \int_0^t \frac{\partial V}{\partial S_1}\, d^{\diamond} S_1(s) + \int_0^t \frac{\partial V}{\partial S_2}\, d^{\diamond} S_2(s) \\ &\quad + \int_0^t \Big\{ \frac{\partial V}{\partial s} + A S_1^2 \frac{\partial^2 V}{\partial S_1^2} + 2B S_1 S_2 \frac{\partial^2 V}{\partial S_2 \partial S_1} + C S_2^2 \frac{\partial^2 f}{\partial S_2^2} \\ &\quad - \Big(V(s, S_1, S_2) - S_1(s)\frac{\partial V}{\partial S_1} - S_2(s)\frac{\partial V}{\partial S_2}\Big) r \Big\}\, ds. \end{aligned}$$

https://doi.org/10.1017/S1446181117000220 Published online by Cambridge University Press

Comparing equation (3.1) with Definition 2.2, it is easy to see that the portfolio Π will be Wick-self-financing if the last integral vanishes for all $t \in [0, T]$. Thus, letting the integrand of the last term vanish to guarantee the portfolio Π to be Wick-self-financing, and coupling with the boundary condition $V(T, S_1(T), S_2(T)) = (S_2(T) - S_1(T) - K)^+$, we safely obtain

$$\begin{cases} \dfrac{\partial V}{\partial t} + AS_1^2\dfrac{\partial^2 V}{\partial S_1^2} + 2BS_1S_2\dfrac{\partial^2 V}{\partial S_2 \partial S_1} + CS_2^2\dfrac{\partial^2 V}{\partial S_2^2} + rS_1\dfrac{\partial V}{\partial S_1} + rS_2\dfrac{\partial V}{\partial S_2} = rV \\ V(T, S_1, S_2) = (S_2(T) - S_1(T) - K)^+. \end{cases} \tag{3.3}$$

In the case $K = 0$, we let $\eta = S_1/S_2$ and $S_2U(\eta, t) = V(t, S_1, S_2)$; then condition (3.3) is rewritten as

$$\begin{cases} \dfrac{\partial U}{\partial t} + (A - 2B + C)\eta^2\dfrac{\partial^2 U}{\partial \eta^2} = 0 \\ U(\eta, T) = (1 - \eta)^+. \end{cases} \tag{3.4}$$

Putting $\eta = e^s$ and $\tilde{U}(s, t) = U(\eta, t)$, (3.4) yields

$$\begin{cases} \dfrac{\partial \tilde{U}}{\partial t} + (A - 2B + C)\Big(\dfrac{\partial^2 \tilde{U}}{\partial s^2} - \dfrac{\partial \tilde{U}}{\partial s}\Big) = 0 \\ \tilde{U}(s, T) = (1 - e^s)^+ = h(s). \end{cases}$$

Applying the Fourier transform in the space variable s,

$$\begin{cases} \dfrac{d\widehat{\tilde{U}}}{dt} = (A - 2B + C)(4\pi^2\theta^2 + 2\pi i\theta)\widehat{\tilde{U}} \\ \widehat{\tilde{U}}(\theta, T) = \widehat{h}(\theta). \end{cases} \tag{3.5}$$

By a standard calculation, we get the solution to the Cauchy problem (3.5) as

$$\widehat{\tilde{U}}(\theta, t) = \widehat{h}(\theta)\exp\Big\{-\frac{D}{2}(4\pi^2\theta^2 + 2\pi i\theta)\Big\} = \widehat{h}(\theta)\widehat{G_t}(\theta)$$

in distribution sense for the last equation in (3.5), where using the inverse Fourier transform to $\widehat{G_t}(\theta)$ with respect to θ yields

$$G_t(s) = \frac{1}{\sqrt{2\pi D}}\exp\Big\{-\frac{(s - D/2)^2}{2D}\Big\}.$$

Hence, by the inverse Fourier transform, we get the solution of (3.5) as

$$\begin{aligned} \tilde{U}(s, t) = (h * G_t)(s) &= \frac{1}{\sqrt{2\pi D}}\int_{-\infty}^{\infty}(1 - e^{s-y})^+ e^{-(y-D/2)^2/2D}\,dy \\ &= \frac{1}{\sqrt{2\pi D}}\int_s^{\infty} e^{-(y-D/2)^2/2D}\,dy - \frac{e^s}{\sqrt{2\pi D}}\int_s^{\infty} e^{\{-(y-D/2)^2/2D\}-y}\,dy, \end{aligned}$$

which gives the solution of (3.4)

$$U(\eta, t) = \Phi\Big(\frac{-\ln\eta + D/2}{\sqrt{D}}\Big) - \eta\Phi\Big(\frac{-\ln\eta - D/2}{\sqrt{D}}\Big),$$

and thus, after substituting the original variable back in the above, we obtain (3.2).

In the case $K \neq 0$, the dimension of equation (3.3) cannot be reduced by some proper transforms, so the computation becomes more complicated. But the method is similar to the situation when $K = 0$. Now we let $x_1 = \ln S_1$ and $x_2 = \ln S_2$ and denote $U(t, x_1, x_2) = V(t, e^{x_1}, e^{x_2}) = V(t, S_1, S_2)$. Then a direct computation implies that

$$\begin{cases} U_t + AU_{x_1x_1} + 2BU_{x_1x_2} + CU_{x_2x_2} + (r - A)U_{x_1} + (r - C)U_{x_2} = rU \\ U(T, x_1, x_2) = (e^{x_2} - e^{x_1} - K)^+ = h(X), \end{cases} \tag{3.6}$$

where $X = (x_1, x_2)$ is regarded as the two-dimensional space variable. Applying a Fourier transform to (3.6),

$$\widehat{U}(t, \xi) = \widehat{h}(\xi)e^{\int_t^T \varphi(s,\xi)\,ds} = \widehat{h}(\xi)\widehat{G_t}(\xi),$$

where $\xi = (\xi_1, \xi_2)$ and $\widehat{U}(t, \xi)$ is the Fourier transform of $U(t, X)$ with respect to X, and

$$\varphi(t, \xi) = -A(\xi_1^2 + i\xi_1) - 2B\pi^2\xi_1\xi_2 - C(\xi_2^2 + i\xi_2) + i(\xi_1 + \xi_2)r - r.$$

Moreover, the Green function G can be expressed as

$$G(t, y_1, y_2) = \frac{e^{-r(T-t)}}{2\pi\sqrt{D_t}}e^{-g(\cdot)},$$

where

$$g(\cdot) = \frac{A_1\{y_2 - C_1 + r(T - t)\}^2 + C_1\{y_1 - A_1 + r(T - t)\}^2}{D_1} - \frac{2B_1\{y_2 - C_1 + r(T - t)\}\{y_1 - A_1 + r(T - t)\}}{D_1}.$$

Hence, by the convolution theorem, the solution of (3.6) can be written as

$$U(t, x_1, x_2) = (h * G_t)(X) = \int_{\mathbb{R}^2} G(t, y_1, y_2)(e^{x_2-y_2} - e^{x_1-y_1} - K)^+\,dy_1\,dy_2, \tag{3.7}$$

which leads to the pricing formula for the case $K \neq 0$. □

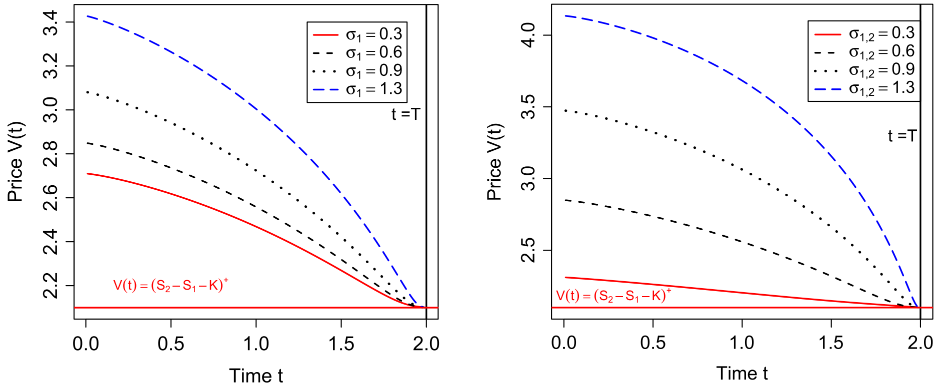
Remark 3.3. (1) To emphasize the fact that equation (3.3) can be solved by a simple and concise method in the case $K = 0$, we divide the proof into two cases.

(2) If we let $H_1 = H_2 = 1/2$ in Lemma 3.1, it becomes the classical Itô's lemma. So the theorem extends the pricing problem of a classical spread option to a more general situation that can more effectively reflect the real market.

4. Behaviour of pricing formula

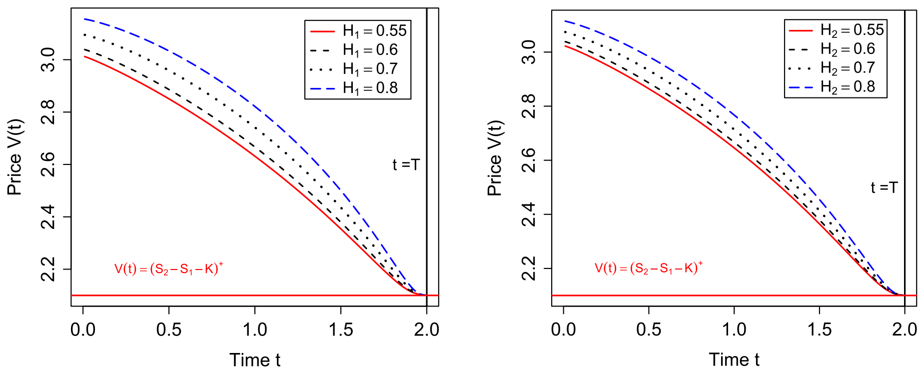
Since the pricing formula (3.7) is complicated, checking its behaviour in a direct way is difficult; but drawing some figures may give us some insight into its behaviour. Firstly, we set the expiration date $T = 2$ and $\rho_1 = \rho_2 = \rho$, the measure of interaction between double markets. Then we calculate the price of the option at given parameter

https://doi.org/10.1017/S1446181117000220 Published online by Cambridge University Press



(a) σ_1 is changing and σ_2 is 0.6 (b) $\sigma_1 = \sigma_2$ is changing

FIGURE 1. The behaviour of price related to time t with different volatility σ, where the other parameters are $\rho = 0.1, r = 0.1, H_1 = 0.6, H_2 = 0.7, T = 2, K = 0.9, S_1(t) = 2, S_2(t) = 5$.

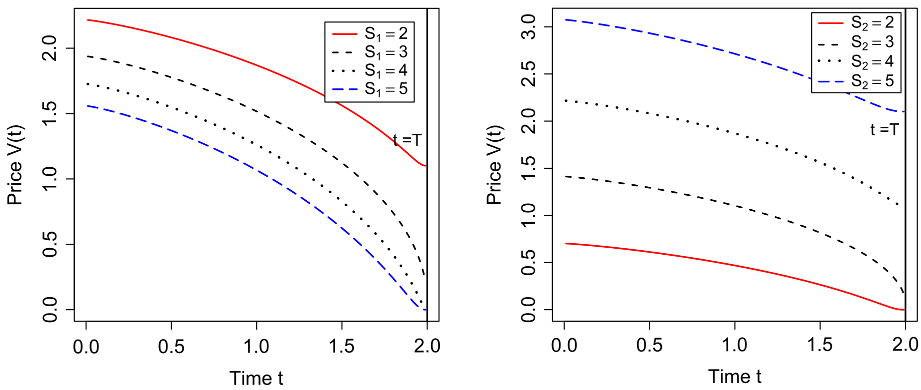


(a) H_1 is changing and H_2 is 0.6 (b) H_2 is changing and H_1 is 0.6

FIGURE 2. The behaviour of price related to time t with different H_i, where the other parameters are $\rho = 0.15, r = 0.1, \sigma_1 = 1, \sigma_2 = 0.6, T = 2, K = 0.9, S_1(t) = 2, S_2(t) = 5$.

values and at every time $t \in [0, T]$. Finally, we plot the change of price with time and attain the term structure of the spread options.

In Figure 1, the relationship between the behaviour of price $V(t, S_1, S_2)$ with respect to t and the volatility σ is examined. It is evident that the volatility affects the option's price dramatically, and the larger volatility σ implies a higher price $V(t, S_1, S_2)$ at any time. This is reasonable, since the larger volatility σ indicates higher risk for the option's writer, requiring a higher return. Figure 2 shows that the larger the Hurst index, the higher the price $V(t, S_1, S_2)$ at any time. Comparing Figures 2(a) and 2(b), we observe that the increase of Hurst index H_1 contributes more to the increase of the price compared with H_2. This is reasonable and compatible with the real market,



(a) S_1 is changing and S_2 is 4 (b) S_2 is changing and S_1 is 2

FIGURE 3. The behaviour of price related to time t with different prices of underlying assets S(t), where the other parameters are $\rho = 0.15, r = 0.1, \sigma_1 = 1, \sigma_2 = 0.6, T = 2, K = 0.9, H_1 = 0.6, H_2 = 0.7$.

since the volatility σ_1 of Market 1 is 1.0, which is higher than the volatility σ_2 of Market 2, which has the value 0.6. On the other hand, the Hurst index measures the memory of the market, that is, a larger Hurst index indicates a more stable market. Thus, increasing the Hurst index in Market 1 contributes to stabilizing the market as well as reducing the risk. This makes intrinsic value play a more important role in option valuation. The graph of $V(t, S_1(t), S_2(t))$ is displayed in Figure 3 for each of the values of $S_1(t)$ and $S_2(t)$. As shown in Figure 3(a), by fixing $S_2(t) = 4$, the price is high for small $S_1(t)$ and low for large $S_1(t)$. This is not unexpected since, when $S_1(t) + K < S_2(t)$, the intrinsic value of the spread option is equal to $S_2(t) - S_1(t) - K$ at any time t, and a large intrinsic value implies a high price. For $S_1(t) = 2$, the intrinsic value goes up with increasing $S_2(t)$ with strike being kept constant. The graph of $V(t, S_1(t), S_2(t))$ is displayed in Figure 4 for each of the values of ρ. It indicates that the price is high for large $|\rho|$ and low for small $|\rho|$. However, it is not affected by the sign of ρ. Another intriguing outcome is that $|\rho|$ affects the option's term structure dramatically when ρ is large. It means that the option premium changes dramatically at different times t when the double markets are closely correlated.

From all the figures, we observe that the price changes mildly at large $(T - t)$ and changes sharply at small $(T - t)$. This is coincident with the nature of the extrinsic value of the spread option, so the behaviour of the pricing formula is compatible with a real market.

5. Conclusion

Inspired by the widespread use of the spread option, we have considered spread option pricing on assets that come from different markets, marked by different Hurst indices. Our market model not only stresses the difference of markets, but also does not neglect the interaction of them. Within an fWIS integral framework, we have obtained

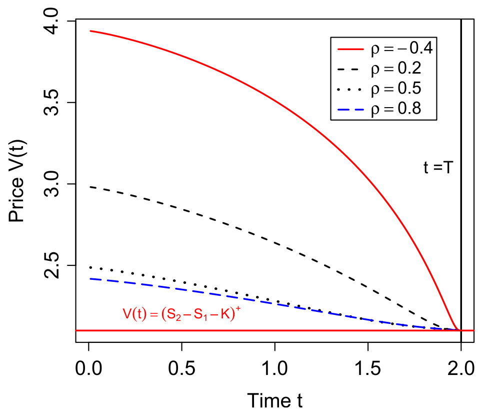


FIGURE 4. The behaviour of price related to time t with different ρ, where the other parameters are $r = 0.1, \sigma_1 = 1, \sigma_2 = 0.6, T = 2, K = 0.9, H_1 = 0.6, H_2 = 0.7, S_1(t) = 2, S_2(t) = 5$.

a pricing formula of the spread option. The figures show that the pricing formula is compatible with the real market.

In summary, we would like to point out that this work is a first attempt at developing a theory of spread option pricing, based on the underlying assets associated with two different fractional markets, whose main characteristics are presented by Hurst parameters H_1 and H_2. We have assumed that these two parameters are constant for the evolution of returns across all timescales. One natural theoretical extension of this work would be to let H_1 and H_2 switch between different values at different times, or be a function of the timescale. Another possible extension to the model would be to include the asymmetric volatility with the underlying asset returns. On the other hand, further work on estimating the parameters such as σ, H_1, H_2, can be done by using econometric methods.

Acknowledgements

This research was supported by the National Nature Science Foundation of China under grant nos. 11171306 and 11571306 and by the project of Zhejiang Provincial Department of Education no. Y201635394. Thanks are also due to the anonymous referees for careful reading and suggestions.

References

[1] F. Biagini, Y. Hu, B. Øksendal and T. Zhang, *Stochastic calculus for fractional Brownian motion and applications* (Springer, London, 2008).

[2] S. Bianchi, A. Pantanella and A. Pianese, “Modeling stock prices by multifractional Brownian motion: an improved estimation of the pointwise regularity”, *Quant. Finance* **11** (2011) 1–14; doi:10.1080/14697688.2011.594080.

[3] R. Caldana and G. Fusai, “A general closed-form spread option pricing formula”, *J. Banking Finance* **37** (2013) 4893–4906; http://dx.doi.org/10.1016/j.jbankfin.2013.08.016.

[4] R. Carmona and V. Durrleman, “Pricing and hedging spread options”, *SIAM Rev.* **45** (2003) 627–685; http://dx.doi.org/10.1137/S0036144503424798.

[5] P. Carr and D. Madan, “Option valuation using the fast Fourier transform”, *J. Comput. Finance* **2** (1999) 61–73; http://dx.doi.org/10.21314/JCF.1999.043.

[6] M. Dempster and S. Hong, “Spread option valuation and the fast Fourier transform”, *Math. Finance Bachelier Congr.* **1** (2000) 203–220; http://dx.doi.org/10.1007/978-3-662-12429-1-10.

[7] S. Deng, M. Li and J. Zhou, “Closed-form approximations for spread option prices and Greeks”, *J. Derivatives* **16** (2008) 58–80.

[8] R. J. Elliott and J. Van der Hoek, “A general fractional white noise theory and applications to finance”, *Math. Finance* **13** (2003) 301–330; http://dx.doi.org/10.1111/1467-9965.00018.

[9] E. F. Fama, “The behaviour of stock market prices”, *J. Bus.* **38** (1965) 34–105; http://www.e-m-h.org/Fama65.pdf.

[10] Y. Hu and B. Øksendal, “Fractional white noise calculus and applications to finance”, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **6** (2003) 1–32; http://dx.doi.org/10.1142/S0219025703001110.

[11] T. R. Hurd and Z. Zhou, “A Fourier transform method for spread option pricing”, *SIAM J. Financial Math.* **1** (2010) 142–157; http://dx.doi.org/10.1137/090750421.

[12] M. Li, J. Zhou and S. J. Deng, “Multi-asset spread option pricing and hedging”, *Quant. Finance* **10** (2010) 305–324; http://dx.doi.org/10.1080/14697680802626323.

[13] A. W. Lo and A. C. MacKinley, “Stock market prices do not follow random walks: Evidence from a simple specification test”, *Rev Financ Stud.* **1** (1988) 41–66; http://dx.doi.org/10.1093/rfs/1.1.41.

[14] C. Necula, “Option pricing in a fractional Brownian motion environment”, *SSRN Electronic J.* **6** (2002) 258–273; http://dx.doi.org/10.2139/ssrn.1286833.

[15] E. Peters, *Fractal market analysis applying chaos theory to investment and economics* (John Wiley & Sons Inc., Berlin, 2002).

[16] K. Ravindran, “Low-fat spreads”, *Risk* **6** (1993) 56–57.

[17] S. Rostek, *Option pricing in fractional Brownian markets* (Springer, New York, 2009).

[18] D. Shimko, “Options on futures spreads: hedging, speculation and valuation”, *J. Futures Mark.* **14** (1994) 183–213; http://dx.doi.org/10.1002/fut.3990140206.

[19] X.-X. Tao and Y.-F. Shi, “Option pricing on fractional correlated stocks”, *Manag. Eng. Appl.* **2010** (2010) 539–545.