HOMOLOGY OF DELETED PRODUCTS OF CONTRACTIBLE 2-DIMENSIONAL POLYHEDRA. II

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1. Introduction. The deleted product space X^* of a space X is $X \times X - \Delta$. In (**4**), I computed the homology groups of the deleted product of a polyhedron in a subcollection $\mathfrak{B}$ (see §2 of this paper for the definition of $\mathfrak{B}$) of the finite, contractible, 2-dimensional polyhedra. In the present paper, I show that there is an infinite subcollection $\mathfrak{C}$ of $\mathfrak{B}$ such that the deleted product of each member of $\mathfrak{C}$ has the homotopy type of the 2-sphere. One of these, call it C, can be embedded in the others, and we show that C can be embedded in a member X of $\mathfrak{B}$ if and only if $H_2(X^*) \neq 0$. Using this, I show that such a polyhedron X can be embedded in the plane if and only if $H_2(X^*) = 0$. It follows from my work in (**4**) that if X is a member of $\mathfrak{B}$, then $H_4(X^*) = 0$ and X^* does not have the homotopy type of a 3-sphere. However, here I show that there is a member CC of $\mathfrak{B}$ which can be embedded in X if and only if $H_3(X^*) \neq 0$.

The homology groups used throughout this paper will be the reduced homology groups with integral coefficients, and the customary tilde over the H has been omitted. If X is a finite polyhedron, let

$$P(X^*) = \bigcup \{\sigma \times \tau |\ \sigma \text{ and } \tau \text{ are simplexes of } X \text{ and } \sigma \cap \tau = \emptyset\}.$$

Hu (**1**) has proved that X^* and $P(X^*)$ are homotopically equivalent.

2. Relation between $H_2(X^*)$ and embeddings. In (**3**), I defined a c-point as follows. A point x in a finite, contractible, 2-dimensional polyhedron X is called a *c-point* of X if there exist 2-simplexes, $\tau_1, \tau_2, \ldots, \tau_n$, of X and a simplex τ of X such that:

(a) τ is not a face of τ_i for any i,

(b) x is a vertex of τ and of τ_i for each i,

(c) $\tau_n \cap \tau_1$ is a 1-simplex,

(d) for each $i = 1, 2, \ldots, n - 1$, $\tau_i \cap \tau_{i+1}$ is a 1-simplex, and

(e) $\tau_i \cap \tau_j = \{x\}$ unless i and j satisfy the conditions of either (c) or (d).

In (**4**), I observed that if X is a finite, contractible, 2-dimensional polyhedron and A is a 2-simplex, then a homeomorph of X can be constructed out of A by appending n-simplexes ($n = 1, 2$). The construction may be factored

$$A = X_1 \to X_2 \to \ldots \to X_p = X$$

so that X_i is obtained from X_{i-1} by

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(a) adding a 1-simplex which meets X_{i-1} in just one of its vertices,
(b) adding a 2-simplex which meets X_{i-1} in just one of its vertices,
(c) adding a 2-simplex which meets X_{i-1} in just one of its 1-faces, or
(d) adding a 2-simplex which meets X_{i-1} in exactly two of its 1-faces.

We may choose the order in which we add simplexes so that if τ is a 2-simplex such that $X_i = X_{i-1} \cup \tau$ and $X_{i-1} \cap \tau = s_1 \cup s_2$, where s_1 and s_2 are 1-simplexes of X_{i-1} and τ, $s_1 \cap s_2 = \{u_3\}$, and u_i is the vertex of s_i different from u_3, then there is a sequence $r_1, r_2, \ldots, r_n$ of 1-simplexes in $\partial(\mathrm{St}(u_3, X_{i-1}))$ such that u_1 is a vertex of r_1, u_2 is a vertex of r_n, $r_j \cap r_{j+1}$ is a vertex, and $r_j \cap r_k = \emptyset$ if $|j - k| > 1$.

Let $\mathfrak{B}$ be the subcollection of the finite, contractible, 2-dimensional polyhedra consisting of those X which can be constructed so that if τ is a 2-simplex such that $X_i = X_{i-1} \cup \tau$ and $X_{i-1} \cap \tau = s_1 \cup s_2$, where s_1 and s_2 are 1-simplexes of X_{i-1} and τ, $s_1 \cap s_2 = \{u_3\}$, and u_i is the vertex of s_i different from u_3, and S is a simple closed curve in $\partial(\mathrm{St}(u_3, X_{i-1}))$ such that u_1 and u_2 are not in S, then the sequence $r_1, r_2, \ldots, r_n$ can be chosen so that $r_j \cap S = \emptyset$ for each j.

For each $i = 1, 2, 3$, let σ_i be a 2-simplex, and let r be a 1-simplex. Throughout this paper, let C denote the polyhedron, consisting of these simplexes and their faces, which satisfies the following conditions:
(a) r is not a face of σ_i for any i,
(b) there is a vertex c_0 which is a vertex of r and of σ_i for each i,
(c) for each $i < j$, $\sigma_i \cap \sigma_j$ is a 1-simplex r_{ij}, and
(d) $r_{ij} \neq r_{km}$ unless $i = k$ and $j = m$.

THEOREM 1. *If $X \in \mathfrak{B}$, then $H_2(X^*) \neq 0$ if and only if C can be embedded in X.*

Proof. Suppose C can be embedded in X. By Theorem 9 of (**3**), either X has a vertex which is a c-point or X has a 1-simplex which is a face of at least three 2-simplexes. If X has a vertex v which is a c-point, let K be the subpolyhedron of X consisting of a collection of simplexes, $\tau_1, \tau_2, \ldots, \tau_n, \tau$, such that v and $\tau_1, \tau_2, \ldots, \tau_n, \tau$ satisfy the definition of c-point. If X does not have a vertex which is a c-point, let s be a 1-simplex which is a face of at least three simplexes, and let K be the subpolyhedron of X consisting of these three 2-simplexes. By Theorems 6 and 7 of (**3**), $H_2(K^*) \neq 0$. It follows immediately from my work (**4**) that $H_2(X^*) \neq 0$.

Suppose $H_2(X^*) \neq 0$. In the construction of X,

$$A = X_1 \rightarrow X_2 \rightarrow \ldots \rightarrow X_p = X,$$

since $H_2(A^*) = 0$, there is an i such that $H_2(X_i^*) \neq 0$ but $H_2(X_{i-1}^*) = 0$. It is sufficient to show that C can be embedded in X_i. Suppose X_i is obtained from X_{i-1} by addition of an n-simplex ($n = 1, 2$) at an m-simplex σ ($m = 0, 1$). Then, by Theorems 5 to 10 of (**4**), $H_1(\partial(\mathrm{St}(\sigma, X_{i-1}))) \neq 0$. Therefore X_{i-1} contains a disk with centre at the barycentre v of σ. Hence v is either a c-point of X_i or σ is a 1-simplex which is a face of at least three 2-simplexes of X_i. In either case C can be embedded in X_i. Suppose X_i is obtained

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from X_{i-1} by addition of a 2-simplex at two 1-simplexes. Let B be the 2-simplex such that $X_i = X_{i-1} \cup B$, and let $r_1, r_2, \ldots, r_n$ be a sequence of 1-simplexes in $\partial(\mathrm{St}(u_3, X_{i-1}))$ such that u_1 is a vertex of r_1, u_2 is a vertex of r_n, $r_j \cap r_{j+1}$ is a vertex, and $r_j \cap r_k = \emptyset$ if $|j-k| > 1$. For each j, let σ_j be the 2-simplex which has u_3 as a vertex and r_j as a face. Then

$$\left(\bigcup_{j=1}^{n} \sigma_j\right) \cup B$$

is a disk with centre at u_3. By Theorem 14 of (**4**),

$$\partial(\mathrm{St}(u_3, X_{i-1})) - \bigcup_{k=1}^{2} \mathrm{St}(u_k, X_{i-1})$$

is not connected. Therefore there is a vertex w in $\partial(\mathrm{St}(u_3, X_{i-1}))$ such that

$$w \notin \bigcup_{j=1}^{n} r_j.$$

Hence u_3 is a c-point of X_i, and C can be embedded in X_i.

THEOREM 2. *An element X of $\mathfrak{B}$ can be embedded in the plane if and only if $H_2(X^*) = 0$.*

Proof. Suppose $H_2(X^*) \neq 0$. Then, by Theorem 1, C can be embedded in X. It is obvious that C cannot be embedded in the plane, and therefore X cannot be embedded in the plane.

Now suppose X cannot be embedded in the plane. Define an equivalence relation on the collection of 2-simplexes of X by $\sigma_1 \sim \sigma_2$ if and only if there is a sequence $\tau_1, \tau_2, \ldots, \tau_n$ of 2-simplexes such that $\tau_1 = \sigma_1$, $\tau_n = \sigma_2$, and $\tau_i \cap \tau_{i+1}$ is a 1-simplex for each i. If R is an equivalence class, let $K_R = \bigcup \{\sigma | \sigma \in R\}$. Let $K_1, K_2, \ldots, K_n$ denote the subpolyhedra of X obtained in this manner. If, for some i, K_i has a 1-simplex which is a face of at least three 2-simplexes, then C can be embedded in K_i and hence in X. Thus $H_2(X^*) \neq 0$ by Theorem 1. Suppose that, for each i, K_i does not have such a 1-simplex. Then each K_i is homeomorphic to a disk. If there exist i and j ($i \neq j$) such that $K_i \cap K_j$ is an interior point of the disk K_i, then C can be embedded in $K_i \cup K_j$ and hence in X. Again, by Theorem 1, this means that $H_2(X^*) \neq 0$. Suppose that for each i and j, $K_i \cap K_j$ is either empty or a boundary point of each. Then, since X is contractible, $\bigcup_{i=1}^{n} K_i$ can be embedded in the plane. Let $s_1, s_2, \ldots, s_m$ denote the 1-simplexes of X which are not faces of 2-simplexes, let $L_1, L_2, \ldots, L_p$ denote the components of $\bigcup_{i=1}^{n} K_i$, and let $T_1, T_2, \ldots, T_q$ denote the components of $\bigcup_{j=1}^{m} s_j$. Now, for each i and j, $L_i \cap T_j$ is either empty or a single point. If, for some i and j, $L_i \cap T_j$ is an interior point of L_i, then C can be embedded in $L_i \cup T_j$ and therefore $H_2(X^*) \neq 0$. Suppose that for each i and j, $L_i \cap T_j$ is either empty or a boundary point of L_i. Then, since X is contractible, it can be embedded in the plane.

For each $i = 1, 2, 3$, let σ_i be a 2-simplex, and suppose there is a 1-simplex r such that $\sigma_i \cap \sigma_j = r$ for all $i \neq j$. Throughout this paper, let D denote the polyhedron consisting of σ_1, σ_2, σ_3, and all their faces.

By Theorems 6 and 7 of (**3**), C^* and D^* have the homotopy type of the 2-sphere. By Theorems 13 and 16 of (**3**), there are two isotopy classes of embeddings of C in C, and, by Theorems 9 and 21 of (**3**), there are six isotopy classes of embeddings of C in D.

For each $i = 1, 2, 3$, let σ_i be a 2-simplex and r_i a 1-simplex. Let X_1 denote the polyhedron, consisting of these simplexes and their faces, which satisfies the following conditions:

(a) r_i is not a face of σ_j for any i and j,

(b) there is a vertex c_0 which is a vertex of σ_i and r_i for each i,

(c) for each $i < j$, $\sigma_i \cap \sigma_j$ is a 1-simplex, r_{ij},

(d) $r_{ij} \neq r_{km}$ unless $i = k$ and $j = m$, and

(e) $r_i \cap r_j = \{c_0\}$ for all $i \neq j$.

By Theorems 9, 11, 13, and 16 of (**3**), the number of isotopy classes of embeddings of C in X_1 is six. Therefore the number of isotopy classes of embeddings of C in X_1 is the same as the number of isotopy classes of embeddings of C in D. However, by Theorem 6 of (**4**), $H_2(X_1^*)$ is the free abelian group on five generators and $H_1(X_1^*)$ is the free abelian group on six generators.

The above examples show that if X is a finite, contractible, 2-dimensional polyhedron, then a combination of the homology groups of X^* and the number of isotopy classes of embeddings of C in X gives us more information about X than either one separately. However, as the following example shows, a combination of these two things does not distinguish finite, contractible, 2-dimensional polyhedra.

For each $i = 1, 2, 3$, let σ_i be a 2-simplex, and, for each $j = 1, 2$, let r_j be a 1-simplex. Let X_2 denote the polyhedron, consisting of these simplexes and their faces, which satisfies the following conditions:

(a) r_i is not a face of σ_j for any i and j,

(b) there is a vertex c_0 which is a vertex of σ_i and r_j for each i and j,

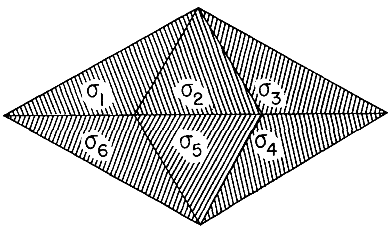
(c) for each $i < j$, $\sigma_i \cap \sigma_j$ is a 1-simplex, r_{ij},

(d) $r_{ij} \neq r_{km}$ unless $i = k$ and $j = m$, and

(e) $r_1 \cap r_2 = \{c_0\}$.

For each $i = 1, 2, \ldots, 7$, let σ_i be a 2-simplex, and let r be a 1-simplex. Let X_3 denote the polyhedron, consisting of these simplexes and their faces, which satisfies the following conditions.

(a) $\bigcup_{i=1}^{6} \sigma_i$ is a disk as indicated below.



(b) If $v_1 = \sigma_1 \cap \sigma_5$ and $v_2 = \sigma_2 \cap \sigma_4$, then

$$\sigma_7 \cap \bigcup_{i=1}^{6} \sigma_i = \{v_2\} \quad \text{and} \quad r \cap \bigcup_{i=1}^{7} \sigma_i = \{v_1\}.$$

By Theorems 9, 11, 13, and 16 of (**3**), for each $i = 2, 3$, the number of isotopy classes of embeddings of C in X_i is four. By Theorem 6 of (**4**), $H_2(X_2^*)$ is the free abelian group on three generators, $H_1(X_2^*)$ is the free abelian group on two generators, and $H_k(X_2^*) = 0$ if $1 \neq k \neq 2$. By Theorem 8 of (**4**), $H_2(X_3^*)$ is the free abelian group on three generators, $H_1(X_3^*)$ is the free abelian group on two generators, and $H_k(X_3^*) = 0$ if $1 \neq k \neq 2$.

For the sake of completeness, we observe that essentially the same thing happens for trees (finite, contractible, 1-dimensional polyhedra). It follows from Theorems 2.2 and 3.1 of (**2**) that if X is a tree, then $H_1(X^*) \neq 0$ if and only if the triod can be embedded in X. Let X_4 be the tree that has five vertices of order three and all other vertices of order one, and let X_5 be the tree that has one vertex of order four, one of order three, and the remainder of order one. Then, by Theorem 4 of (**3**), for each $i = 4, 5$, the number of isotopy classes of embeddings of the triod in X_i is 30. However, by Theorem 5 in (**3**), $H_1(X_4^*)$ is the free abelian group on nine generators and $H_1(X_5^*)$ is the free abelian group on seven generators.

Let X_6 be the tree that has four vertices of order three and all other vertices of order one. Then $H_1(X_6^*)$ is the free abelian group on seven generators and hence $H_1(X_6^*)$ is isomorphic to $H_1(X_5^*)$. However, the number of isotopy classes of embeddings of the triod in X_6 is 24.

The following example shows that if X is a tree, then a combination of the homology groups of X^* and the number of isotopy classes of embeddings of the triod in X does not give as much information as counting the orders of vertices. Let X_7 be a tree that has 60 vertices of order three, 10 vertices of order five, and all other vertices of order one. Let X_8 be a tree that has 40 vertices of order four and all other vertices of order one. Then, for each $i = 7, 8$, by Theorem 5 of (**3**), $H_1(X_i^*)$ is the free abelian group on 239 generators, and, by Theorem 6 of (**3**), the number of isotopy classes of embeddings of the triod in X_i is 960.

3. Homotopy type of the 2-sphere. In (**4**), I defined pronged and the simple 2-dimensional deleted product number as follows.

If X is a finite, contractible, 2-dimensional polyhedron and v is a vertex of X, then X is *pronged* at v provided $\partial(\mathrm{St}(v, X))$ contains a simple closed curve and if $\partial(\mathrm{St}(v, X))$ is a simple closed curve S, then there is a simple closed curve S' in the 1-skeleton of $X - \mathrm{St}(v, X)$, a 2-chain

$$c = \sum_{j=1}^{n} a_j \sigma_j$$

($a_j \neq 0$ for each $j = 1, 2, \ldots, n$) in $X - \mathrm{St}(v, X)$, and either a 1-simplex $r \in X - \mathrm{St}(v, X)$ such that $\partial c = z_S - z_{S'}$, $r \cap S' = \emptyset$, and

$$r \cap \bigcup_{j=1}^{n} \sigma_j$$

is a vertex, or a 2-simplex $\tau \in X - \mathrm{St}(v, X)$ and a 1-face μ of τ such that if L denotes the line segment in τ from the barycentre of τ to the barycentre of μ, then $\partial c = z_S - z_{S'}$, $L \cap S' = \emptyset$, and

$$L \cap \bigcup_{j=1}^{n} \sigma_j$$

is a vertex. If s is a 1-simplex of X, then X is *pronged* at s provided the first barycentric subdivision of X is pronged at the barycentre of s.

If X is a finite, contractible, 2-dimensional polyhedron, u_3 is a vertex of X, and u_1 and u_2 are vertices in a component of $\partial(\mathrm{St}(u_3, X))$, let $K = \bigcup\{\sigma|$ σ is a 2-simplex and there is a sequence $\sigma_1, \sigma_2, \ldots, \sigma_n$ of 2-simplexes in X with the property that $\sigma = \sigma_1$, u_1 is a vertex of σ_n, and $\sigma_j \cap \sigma_{j+1}$ is a 1-simplex for each $j\}$. If

$$H_0\left(\partial(\mathrm{St}(u_3, X)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, X)\right) = 0,$$

there is a vertex w in K such that $\partial(\mathrm{St}(w, K))$ contains a simple closed curve and w is a c-point of X, or there is a 1-simplex in K which is a face of at least three 2-simplexes, then the *simple* 2-*dimensional deleted product number* is 0. Otherwise, it is 1.

Let $\mathfrak{A}$ be the collection consisting of the polyhedra C and D and all finite, contractible, 2-dimensional polyhedra A such that a homeomorph of A can be constructed out of D by appending 2-simplexes in such a way that if the construction is factored

$$D = X_1 \to X_2 \to \ldots \to X_p = A,$$

then X_i is obtained from X_{i-1} by adding a 2-simplex τ such that

$$X_{i-1} \cap \tau = s_1 \cup s_2,$$

where s_1 and s_2 are distinct 1-simplexes of X_{i-1} and τ, and, if $s_1 \cap s_2 = \{u_3\}$ and u_j is the vertex of s_j different from u_3, then

$$\partial(\mathrm{St}(u_3, X_{i-1})) - \bigcup_{j=1}^{2} \mathrm{St}(u_j, X_{i-1})$$

is contractible. (Of course, one may take a finite subdivision of X_{i-1} before adding τ.)

If $A \in \mathfrak{A}$, let $E_A = \{x|\ x$ is in a 2-simplex of A and x is not the centre of a disk which is contained in $A\}$.

THEOREM 3. *If $X \in \mathfrak{B}$, then X^* has the homotopy type of the 2-sphere if and only if there is a member A of $\mathfrak{A}$ and a non-negative integer m such that a homeomorph of X can be constructed out of A by appending m 1-simplexes at m distinct points of E_A.*

Proof. We have already observed that C^* and D^* have the homotopy type of the 2-sphere. Let $A \in \mathfrak{A}$ such that $C \neq A \neq D$. Since a homeomorph of A can be constructed out of D in the manner described above, in order to show that A^* has the homotopy type of the 2-sphere, it is sufficient to show that if X_{i-1} is a finite, contractible, 2-dimensional polyhedron such that X_{i-1}^* has the homotopy type of the 2-sphere and τ is a 2-simplex such that $X_i = X_{i-1} \cup \tau$ and $X_{i-1} \cap \tau = s_1 \cup s_2$, where s_1 and s_2 are distinct 1-simplexes of X_{i-1} and τ, and, if $s_1 \cap s_2 = \{u_3\}$ and u_j is the vertex of s_j different from u_3, then

$$\partial(\mathrm{St}(u_3, X_{i-1})) - \bigcup_{j=1}^{2} \mathrm{St}(u_j, X_{i-1})$$

is contractible, then X_i^* has the homotopy type of the 2-sphere. Let s denote the 1-face of τ which is not in X_{i-1}. Then

$$\begin{aligned} P(X_i^*) = P(X_{i-1}^*) &\cup \left(\left[X_{i-1} - \bigcup_{j=1}^{3} \mathrm{St}(u_j, X_{i-1})\right] \times \tau\right) \\ &\cup \left(\left[\overline{\mathrm{St}(u_3, X_{i-1})} - \bigcup_{j=1}^{2} \mathrm{St}(u_j, X_{i-1})\right] \times s\right) \\ &\cup \left(\tau \times \left[X_{i-1} - \bigcup_{j=1}^{3} \mathrm{St}(u_j, X_{i-1})\right]\right) \\ &\cup \left(s \times \left[\overline{\mathrm{St}(u_3, X_{i-1})} - \bigcup_{j=1}^{2} \mathrm{St}(u_j, X_{i-1})\right]\right). \end{aligned}$$

Since

$$\begin{aligned} P(X_{i-1}^*) \cap \left(\left[X_{i-1} - \bigcup_{j=1}^{3} \mathrm{St}(u_j, X_{i-1})\right] \times \tau\right) = \\ \left[X_{i-1} - \bigcup_{j=1}^{3} \mathrm{St}(u_j, X_{i-1})\right] \times (s_1 \cup s_2), \end{aligned}$$

then

$$P(X_{i-1}^*) \cup \left(\left[X_{i-1} - \bigcup_{j=1}^{3} \mathrm{St}(u_j, X_{i-1})\right] \times \tau\right)$$

is homotopically equivalent to $P(X_{i-1}^*)$. Now

$$\begin{aligned} \left[P(X_{i-1}^*) \cup \left(\left[X_{i-1} - \bigcup_{j=1}^{3} \mathrm{St}(u_j, X_{i-1})\right] \times \tau\right)\right] \\ \cap \left(\left[\overline{\mathrm{St}(u_3, X_{i-1})} - \bigcup_{j=1}^{2} \mathrm{St}(u_j, X_{i-1})\right] \times s\right) \\ = \left(\left[\overline{\mathrm{St}(u_3, X_{i-1})} - \bigcup_{j=1}^{2} \mathrm{St}(u_j, X_{i-1})\right] \times \{u_1\}\right) \\ \cup \left(\left[\overline{\mathrm{St}(u_3, X_{i-1})} - \bigcup_{j=1}^{2} \mathrm{St}(u_j, X_{i-1})\right] \times \{u_2\}\right) \\ \cup \left(\left[\partial(\mathrm{St}(u_3, X_{i-1})) - \bigcup_{j=1}^{2} \mathrm{St}(u_j, X_{i-1})\right] \times s\right). \end{aligned}$$

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Since

$$\left[P(X_{i-1}^*) \cup \left(\left[X_{i-1} - \bigcup_{j=1}^{3} \operatorname{St}(u_j, X_{i-1})\right] \times \tau\right)\right] \cap \left(\left[\overline{\operatorname{St}(u_3, X_{i-1})} - \bigcup_{j=1}^{2} \operatorname{St}(u_j, X_{i-1})\right] \times s\right)$$

is a deformation retract of

$$\left[\overline{\operatorname{St}(u_3, X_{i-1})} - \bigcup_{j=1}^{2} \operatorname{St}(u_j, X_{i-1})\right] \times s,$$

then

$$P(X_{i-1}^*) \cup \left(\left[X_{i-1} - \bigcup_{j=1}^{3} \operatorname{St}(u_j, X_{i-1})\right] \times \tau\right) \cup \left(\left[\overline{\operatorname{St}(u_3, X_{i-1})} - \bigcup_{j=1}^{2} \operatorname{St}(u_j, X_{i-1})\right] \times s\right)$$

is homotopically equivalent to $P(X_{i-1}^*)$. Continuing,

$$\begin{aligned}\Bigg[P(X_{i-1}^*) \cup \left(\left[X_{i-1} - \bigcup_{j=1}^{3} \operatorname{St}(u_j, X_{i-1})\right] \times \tau\right) \\ \cup \left(\left[\overline{\operatorname{St}(u_3, X_{i-1})} - \bigcup_{j=1}^{2} \operatorname{St}(u_j, X_{i-1})\right] \times s\right)\Bigg] \\ \cap \left(\tau \times \left[X_{i-1} - \bigcup_{j=1}^{3} \operatorname{St}(u_j, X_{i-1})\right]\right) \\ = (s_1 \cup s_2) \times \left[X_{i-1} - \bigcup_{j=1}^{3} \operatorname{St}(u_j, X_{i-1})\right],\end{aligned}$$

and therefore

$$\begin{aligned}P(X_{i-1}^*) \cup \left(\left[X_{i-1} - \bigcup_{j=1}^{3} \operatorname{St}(u_j, X_{i-1})\right] \times \tau\right) \\ \cup \left(\left[\overline{\operatorname{St}(u_3, X_{i-1})} - \bigcup_{j=1}^{2} \operatorname{St}(u_j, X_{i-1})\right] \times s\right) \\ \cup \left(\tau \times \left[X_{i-1} - \bigcup_{j=1}^{3} \operatorname{St}(u_j, X_{i-1})\right]\right)\end{aligned}$$

is homotopically equivalent to $P(X_{i-1}^*)$. Also

$$\begin{aligned}\Bigg[P(X_{i-1}^*) \cup \left(\left[X_{i-1} - \bigcup_{j=1}^{3} \operatorname{St}(u_j, X_{i-1})\right] \times \tau\right) \\ \cup \left(\left[\overline{\operatorname{St}(u_3, X_{i-1})} - \bigcup_{j=1}^{2} \operatorname{St}(u_j, X_{i-1})\right] \times s\right) \\ \cup \left(\tau \times \left[X_{i-1} - \bigcup_{j=1}^{3} \operatorname{St}(u_j, X_{i-1})\right]\right)\Bigg] \\ \cap \left(s \times \left[\overline{\operatorname{St}(u_3, X_{i-1})} - \bigcup_{j=1}^{2} \operatorname{St}(u_j, X_{i-1})\right]\right)\end{aligned}$$

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$$= \left(\{u_1\} \times \left[\overline{\mathrm{St}(u_3, X_{i-1})} - \bigcup_{j=1}^{2} \mathrm{St}(u_j, X_{i-1})\right]\right)$$
$$\cup \left(\{u_2\} \times \left[\overline{\mathrm{St}(u_3, X_{i-1})} - \bigcup_{j=1}^{2} \mathrm{St}(u_j, X_{i-1})\right]\right)$$
$$\cup \left(s \times \left[\partial(\mathrm{St}(u_3, X_{i-1})) - \bigcup_{j=1}^{2} \mathrm{St}(u_j, X_{i-1})\right]\right),$$

and hence, for the same reason as above, $P(X_i^*)$ is homotopically equivalent to $P(X_{i-1}^*)$.

Now suppose $A \in \mathfrak{A}$, m is a positive integer, and a homeomorph of X can be constructed out of A by appending m 1-simplexes at m distinct points of E_A. The construction may be factored

$$A = X_1 \to X_2 \to \ldots \to X_{m+1} = X$$

so that X_i is obtained from X_{i-1} by adding a 1-simplex at a vertex of E_A. Thus, in order to show that X^* has the homotopy type of the 2-sphere, it is sufficient to show that if X_{i-1}^* has the homotopy type of the 2-sphere, then so does X_i^*. Let s be the 1-simplex such that $X_i = X_{i-1} \cup s$, let $v = X_{i-1} \cap s$, and let u be the vertex of s which is not in X_{i-1}. Then

$$P(X_i^*) = P(X_{i-1}^*) \cup ((X_{i-1} - \mathrm{St}(v, X_{i-1})) \times s) \cup (\overline{\mathrm{St}(v, X_{i-1})} \times \{u\})$$
$$\cup (s \times (X_{i-1} - \mathrm{St}(v, X_{i-1}))) \cup (\{u\} \times \overline{\mathrm{St}(v, X_{i-1})}).$$

Now $P(X_{i-1}^*) \cap ((X_{i-1} - \mathrm{St}(v, X_{i-1})) \times s) = (X_{i-1} - \mathrm{St}(v, X_{i-1})) \times \{v\}$, and hence $P(X_{i-1}^*) \cap ((X_{i-1} - \mathrm{St}(v, X_{i-1})) \times s)$ is homotopically equivalent to $P(X_{i-1}^*)$. Also

$$[P(X_{i-1}^*) \cup ((X_{i-1} - \mathrm{St}(v, X_{i-1})) \times s)] \cap (\overline{\mathrm{St}(v, X_{i-1})} \times \{u\}) =$$
$$\partial(\mathrm{St}(v, X_{i-1})) \times \{u\}.$$

Since v is a point of E_A and every simplex of X_{i-1} which has v as a vertex is a simplex of A, then $\partial(\mathrm{St}(v, X_{i-1}))$ is contractible. Therefore

$$P(X_{i-1}^*) \cup ((X_{i-1} - \mathrm{St}(v, X_{i-1})) \times s) \cup (\overline{\mathrm{St}(v, X_{i-1})} \times \{u\})$$

is homotopically equivalent to $P(X_{i-1}^*)$. Continuing,

$$[P(X_{i-1}^*) \cup ((X_{i-1} - \mathrm{St}(v, X_{i-1})) \times s) \cup (\overline{\mathrm{St}(v, X_{i-1})} \times \{u\})]$$
$$\cap (s \times (X_{i-1} - \mathrm{St}(v, X_{i-1}))) = \{v\} \times (X_{i-1} - \mathrm{St}(v, X_{i-1})),$$

and hence

$$P(X_{i-1}^*) \cup ((X_{i-1} - \mathrm{St}(v, X_{i-1})) \times s)$$
$$\cup (\overline{\mathrm{St}(v, X_{i-1})} \times \{u\}) \cup (s \times (X_{i-1} - \mathrm{St}(v, X_{i-1})))$$

is homotopically equivalent to $P(X_{i-1}^*)$. Finally,

$$[P(X_{i-1}^*) \cup ((X_{i-1} - \text{St}(v, X_{i-1})) \times s) \cup \overline{(\text{St}(v, X_{i-1})} \times \{u\}) \cup (s \times (X_{i-1} - \text{St}(v, X_{i-1})))] \cap (\{u\} \times \overline{\text{St}(v, X_{i-1})}) = \{u\} \times \partial(\text{St}(v, X_{i-1})).$$

Therefore, for the same reason as above, $P(X_i^*)$ is homotopically equivalent to $P(X_{i-1}^*)$.

Now suppose $X \in \mathfrak{B}$ and X^* has the homotopy type of the 2-sphere. A homeomorph of X can be constructed out of a 2-simplex B, and the construction may be factored

$$B = X_1 \to X_2 \to \ldots \to X_q = X$$

so that X_i is obtained from X_{i-1} by one of the four types of additions described in the second paragraph of §2. We may assume, without loss of generality, that, for $i > 1$, X_i is not homeomorphic to a disk. Since X^* has the homotopy type of the 2-sphere, $H_3(X_i^*) = 0$ for all i by my work in (**4**). Let n be the smallest integer such that $H_2(X_n^*) \neq 0$. Again, it follows from the theorems of (**4**) that $H_2(X_i^*)$ is isomorphic to the group of integers and $H_1(X_i^*) = 0$ for $n \leqslant i \leqslant q$ and, for $i > n$, X_i is obtained from X_{i-1} by

(1) adding a 1-simplex at a vertex v, where $H_k(\partial(\text{St}(v, X_{i-1}))) = 0$ for all k,

(2) adding a 2-simplex at a 1-simplex s, where $H_1(\partial(\text{St}(s, X_{i-1}))) = 0$, or

(3) adding a 2-simplex τ such that $X_{i-1} \cap \tau = s_1 \cup s_2$, where s_1 and s_2 are distinct 1-simplexes of X_{i-1} and τ, and, if $s_1 \cap s_2 = \{u_3\}$ and u_j is the vertex of s_j different from u_3, then

$$H_k\left(\partial(\text{St}(u_3, X_{i-1})) - \bigcup_{j=1}^{2} \text{St}(u_j, X_{i-1})\right) = 0 \quad \text{for all } k.$$

If X_i is obtained from X_{i-1} by (2), then X_i is homeomorphic to X_{i-1}, and hence we may assume that X_i is obtained from X_{i-1} by either (1) or (3).

Now it follows also from the theorems of (**4**) that X_n is obtained from X_{n-1} by

(4) adding a 1-simplex at a vertex v, where X_{n-1} is not pronged at v and $H_1(\partial(\text{St}(v, X_{n-1})))$ is isomorphic to the group of integers,

(5) adding a 2-simplex at a 1-simplex s, where X_{n-1} is not pronged at s and $H_1(\partial(\text{St}(s, X_{n-1})))$ is isomorphic to the group of integers, or

(6) adding a 2-simplex τ such that $X_{n-1} \cap \tau = s_1 \cup s_2$, where s_1 and s_2 are distinct 1-simplexes of X_{n-1} and τ, and, if $s_1 \cap s_2 = \{u_3\}$ and u_j is the vertex of s_j different from u_3, then

$$H_1\left(\partial(\text{St}(u_3, X_{n-1})) - \bigcup_{j=1}^{2} \text{St}(u_j, X_{n-1})\right) = 0,$$

$$H_0\left(\partial(\text{St}(u_3, X_{n-1})) - \bigcup_{j=1}^{2} \text{St}(u_j, X_{n-1})\right)$$

is isomorphic to the group of integers, and the simple 2-dimensional deleted product number of X_{n-1} is 1.

By my work in (**4**), $H_1(X_i^*)$ is isomorphic to the group of integers for $1 \leqslant i \leqslant n-1$ and, for $1 < i \leqslant n-1$, X_i is obtained from X_{i-1} by

(7) adding a 1-simplex at a vertex v, where $H_k(\partial(\mathrm{St}(v, X_{i-1}))) = 0$ for all k,

(8) adding a 2-simplex at a 1-simplex s, where $H_1(\partial(\mathrm{St}(s, X_{i-1}))) = 0$, or

(9) adding a 2-simplex τ such that $X_{i-1} \cap \tau = s_1 \cup s_2$, where s_1 and s_2 are distinct 1-simplexes of X_{i-1} and τ, and, if $s_1 \cap s_2 = \{u_3\}$ and u_j is the vertex of s_j different from u_3, then

$$H_k\left(\partial(\mathrm{St}(u_3, X_{i-1})) - \bigcup_{j=1}^{2} \mathrm{St}(u_j, X_{i-1})\right) = 0 \quad \text{for all } k.$$

If X_i is obtained from X_{i-1} by either (8) or (9), then X_i is homeomorphic to X_{i-1} and hence we may assume that X_i is obtained from X_{i-1} by (7). Therefore, there is a non-negative integer α such that X_{n-1} is homeomorphic to a disk with α 1-simplexes attached to the disk at α distinct points of the boundary. If X_n is obtained from X_{n-1} by (4), then a homeomorph of X_n can be constructed out of C by appending α 1-simplexes at α distinct points of E_C. Therefore, there is a non-negative integer β such that a homeomorph of X can be constructed out of C by appending β 1-simplexes at β distinct points of E_C. If X_n is obtained from X_{n-1} by either (5) or (6), then there is a member A_1 of $\mathfrak{A}(A_1 \neq C)$ and a non-negative integer α_1 such that a homeomorph of X_n can be constructed out of A_1 by appending α_1 1-simplexes at α_1 distinct points of E_{A_1}. Therefore, there is a member A_2 of $\mathfrak{A}(A_2 \neq C)$ and a non-negative integer α_2 such that a homeomorph of X can be constructed out of A_2 by appending α_2 1-simplexes at α_2 distinct points of E_{A_2}.

4. Relation between $H_3(X^*)$ and embeddings. For each $i = 1, 2, 3$, let σ_i and τ_i be 2-simplexes, and let CC denote the polyhedron, consisting of these simplexes and their faces, which satisfies the following conditions:

(1) There is a vertex c_0 which is a vertex of σ_i and of τ_i for each i.

(2) $\sigma_i \cap \tau_j = \{c_0\}$ for each i and j.

(3) For each $i < j$, $\sigma_i \cap \sigma_j$ is a 1-simplex r_{ij} and $\tau_i \cap \tau_j$ is a 1-simplex s_{ij}.

(4) If either $i \neq k$ or $j \neq m$, then $r_{ij} \neq r_{km}$ and $s_{ij} \neq s_{km}$.

Throughout this section, we let CC, σ_i, τ_i, and c_0 denote the specific objects described above.

Definition 1. Let X be a finite, contractible, 2-dimensional polyhedron. A point $x \in X$ is called a *double c-point* of X if there exist 2-simplexes $\lambda_1, \lambda_2, \ldots, \lambda_m$ and $\xi_1, \xi_2, \ldots, \xi_n$ of X such that

(1) x is a vertex of λ_i and of ξ_j for each i and j,

(2) $\lambda_i \cap \xi_j = \{x\}$ for each i and j,

(3) $\lambda_m \cap \lambda_1$ is a 1-simplex,

(4) for each $i = 1, 2, \ldots, m-1$, $\lambda_i \cap \lambda_{i+1}$ is a 1-simplex,

(5) $\lambda_i \cap \lambda_k = \{x\}$ unless i and k satisfy the conditions of either (3) or (4),

(6) $\xi_n \cap \xi_1$ is a 1-simplex,
(7) for each $j = 1, 2, \ldots, n-1$, $\xi_j \cap \xi_{j+1}$ is a 1-simplex, and
(8) $\xi_j \cap \xi_k = \{x\}$ unless j and k satisfy the conditions of either (6) or (7).

THEOREM 4. *If X is a finite, contractible, 2-dimensional polyhedron and $f: CC \rightarrow X$ is an embedding, then $f(c_0)$ is a double c-point of X.*

Proof. It is easy to see that $f(c_0)$ is not an interior point of a 2-simplex. Suppose $f(c_0)$ is an interior point of a 1-simplex u. Now $f(c_0)$ is an interior point of

$$f\left(\bigcup_{i=1}^{3} \sigma_i\right).$$

Therefore there exist two 2-simplexes μ_1 and μ_2, which have u as a face, and a disk D_1 such that

$$f(c_0) \in D_1^0 \subset D_1 \subset (\mu_1 \cup \mu_2) \cap f\left(\bigcup_{i=1}^{3} \sigma_i\right).$$

Likewise, there exist two 2-simplexes ν_1 and ν_2 which have u as a face, and a disk D_2 such that

$$f(c_0) \in D_2^0 \subset D_2 \subset (\nu_1 \cup \nu_2) \cap f\left(\bigcup_{i=1}^{3} \tau_i\right).$$

Therefore $D_1 \cap D_2$ contains a non-degenerate closed interval and f is not an embedding. Hence $f(c_0)$ is a vertex, and, using an argument similar to the one above, it is easy to see that $f(c_0)$ is a double c-point.

THEOREM 5. *If $X \in \mathfrak{B}$, then $H_3(X^*) \neq 0$ if and only if CC can be embedded in X.*

Proof. Suppose CC can be embedded in X. Then, by Theorem 4, X has a vertex v which is a double c-point. Let K be the subpolyhedron of X consisting of a collection of 2-simplexes, $\lambda_1, \lambda_2, \ldots, \lambda_m$ and $\xi_1, \xi_2, \ldots, \xi_n$ such that $\lambda_1, \lambda_2, \ldots, \lambda_m, \xi_1, \xi_2, \ldots, \xi_n$ and v satisfy the definition of double c-point. By Theorem 14 of (**4**), $H_3(K^*) \neq 0$. It also follows immediately from my work in (**4**) that $H_3(X^*) \neq 0$.

Suppose $H_3(X^*) \neq 0$. In the construction of X out of a 2-simplex,

$$A = X_1 \rightarrow X_2 \rightarrow \ldots \rightarrow X_p = X,$$

there is an i such that $H_3(X_i^*) \neq 0$ but $H_3(X_{i-1}^*) = 0$. It is sufficient to show that CC can be embedded in X_i. By my work in (**4**), X_i is obtained from X_{i-1} by adding a 2-simplex at two 1-simplexes. Let B be the 2-simplex such that $X_i = X_{i-1} \cup B$, and suppose $X_{i-1} \cap B = s_1 \cup s_2$, where $s_1 \cap s_2 = \{u_3\}$. For each j, let u_j be the vertex of s_j different from u_3. Then, by Theorem 14 of (**4**),

$$H_1\left(\partial(\mathrm{St}(u_3, X_{i-1})) - \bigcup_{j=1}^{2} \mathrm{St}(u_j, X_{i-1})\right) \neq 0.$$

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Let S be a simple closed curve in

$$\partial(\mathrm{St}(u_3, X_{i-1})) - \bigcup_{j=1}^{2} \mathrm{St}(u_j, X_{i-1}),$$

and let $r_1, r_2, \ldots, r_n$ be a sequence of 1-simplexes in $\partial(\mathrm{St}(u_3, X_{i-1}))$ such that u_1 is a vertex of r_1, u_2 is a vertex of r_n, $r_\alpha \cap r_{\alpha+1}$ is a vertex, $r_\alpha \cap r_\beta = \emptyset$ if $|\alpha - \beta| > 1$, and $r_\alpha \cap S = \emptyset$ for each α. For each α, let σ_α be the 2-simplex which has u_3 as a vertex and r_α as a face. Then

$$\left(\bigcup_{\alpha=1}^{n} r_\alpha\right) \cup B$$

is a disk with centre at u_3. Let $s_1, s_2, \ldots, s_m$ be the 1-simplexes of S, and, for each γ, let τ_γ be the 2-simplex which has u_3 as a vertex and s_γ as a face. Then

$$\bigcup_{\gamma=1}^{m} \tau_\gamma$$

is a disk with centre at u_3, and

$$\left[\left(\bigcup_{\alpha=1}^{n} r_\alpha\right) \cup B\right] \cap \left[\bigcup_{\gamma=1}^{m} \tau_\gamma\right] = \{u_3\}.$$

Therefore CC can be embedded in X.

References

1. S. T. Hu, *Isotopy invariants of topological spaces*, Proc. Roy. Soc. (London), Ser. A *255* (1960) 331–366.
2. C. W. Patty, *The fundamental group of certain deleted product spaces*, Trans. Amer. Math. Soc. *105* (1962), 314–321.
3. ——— *Isotopy classes of imbeddings* Trans. Amer. Math. Soc. *128* (1967), 232–247.
4. ——— *Homology of deleted products of contractible 2-dimensional polyhedra*. I, Can. J. Math. *20* (1968), 416–441.

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