Canad. Math. Bull.Vol. **36** (4), 1993 pp. 390–397

THE COLLINEATION GROUPS OF FIGUEROA PLANES

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ABSTRACT. We describe the structure of the collineation groups of Figueroa planes, giving examples and explanations that show why the description in Dempwolff [3] is not completely accurate. We also give criteria for when Figueroa planes are isomorphic, and show that certain subplanes are Figueroa or Pappian planes.

1. **Introduction.** This paper is concerned with the projective planes introduced by Figueroa [4] and generalizations of these obtained by Hering-Schaeffer [6] and Dempwolff [2]. First we describe their construction (as given by Grundhöfer [5]) and establish notation which, for the most part, is a combination of that of Dempwolff [2, 3] and Grundhöfer [5].

Throughout, let Π stand for the projective plane $\mathrm{PG}(2,K)$, where K is a field having an automorphism α of order 3. There is an induced collineation on Π which, without danger of confusion, we also call α. The subfield F of K and the subplane Π_0 of Π which are fixed by α satisfy $[K:F]=3$ and $\Pi_0 \cong \mathrm{PG}(2,F)$. It is easy to see that every planar collineation of Π arises in a similar way. Only the ones of order 3 are relevant for the present purposes.

We write $\Pi = (\mathcal{P}, \mathcal{L})$, where the lines (elements of $\mathcal{L}$) are subsets of the point set $\mathcal{P}$. Lower-case letters such as p and q stand for elements of $\mathcal{P}$. The line of Π through distinct points p,q is denoted pq. Distinct lines L,M of Π intersect at a point which, by abuse of notation, is denoted $L\cap M$. The point and line sets of Π can each be partitioned according to the structure of their orbits under α, giving the following subsets:

$$\mathcal{P}_1 = \{p \in \mathcal{P} \mid p = p^\alpha\},$$
$$\mathcal{P}_2 = \{p \in \mathcal{P} \mid p, p^\alpha, p^{\alpha^2} \text{ are distinct and collinear}\},$$
$$\mathcal{P}_3 = \{p \in \mathcal{P} \mid p, p^\alpha, p^{\alpha^2} \text{ are distinct and non-collinear}\},$$
$$\mathcal{L}_1 = \{L \in \mathcal{L} \mid L = L^\alpha\},$$
$$\mathcal{L}_2 = \{L \in \mathcal{L} \mid L, L^\alpha, L^{\alpha^2} \text{ are distinct and concurrent}\},$$
$$\mathcal{L}_3 = \{L \in \mathcal{L} \mid L, L^\alpha, L^{\alpha^2} \text{ are distinct and non-concurrent}\}.$$

For $L \in \mathcal{L}_3$, define $L^* = (L \setminus \mathcal{P}_3) \cup \{M^\mu \mid L^\mu \in M \in \mathcal{L}_3\}$, where μ is the 1–1 correspondence $\mathcal{P}_3 \leftrightarrow \mathcal{L}_3$ given by $p^\mu = p^\alpha p^{\alpha^2}$ and $L^\mu = L^\alpha \cap L^{\alpha^2}$. Set $\mathcal{L}_3^* = \{L^* \mid L \in \mathcal{L}_3\}$ and $\mathcal{L}^* = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3^*$. We write ${}^\alpha\Pi$ for the incidence system $(\mathcal{P}, \mathcal{L}^*)$, and say that

The first author was supported by NSERC grant A3485.

Received by the editors March 11, 1992; revised July 8, 1992.
AMS subject classification: 51A10, 51A35.

https://doi.org/10.4153/CMB-1993-053-0 Published online by Cambridge University Press

it has been obtained from Π by *twisting* via α. All of the above definitions are unaffected if α and its inverse α^2 are interchanged.

In a series of generalizations, several authors proved that ${}^{\alpha}\Pi$ is a projective plane with some interesting properties. In the smallest case, Π has order 8 and ${}^{\alpha}\Pi \cong \Pi$. From now on, we always exclude that case. Then ${}^{\alpha}\Pi$ is a non-Desarguesian plane which is called a *Figueroa plane*. Grundhöfer [5, Theorem 3] shows that if a non-planar collineation of order 3 is used in place of α, then the preceding construction gives a projective plane that is isomorphic to Π. For use in Theorem 4, we also allow the trivial twist by the identity collineation of Π; it has no effect on Π.

Consider the collineation group $G = \mathrm{Aut}({}^{\alpha}\Pi)$, where ${}^{\alpha}\Pi$ (as always) is a Figueroa plane. First suppose that ${}^{\alpha}\Pi$ is finite. Computations in [6] show that $\langle\alpha\rangle \lhd G$ and $G/\langle\alpha\rangle \cong \mathrm{P\Gamma L}(3, F)$. Contrary to claims in [4] and [3], it is now known that this extension is not in general a direct product, or even a semi-direct product. Examples are given in Section 3.

In arbitrary Figueroa planes, there can be further difficulties (explained more fully in Section 3) involving automorphisms of F which do not extend to K. In addition, $\langle\alpha\rangle$ is not always central in G. Dempwolff became aware of these problems only after writing [3]. He has informed us that the statement of his main theorem must consequently be adjusted. We are indebted to Julia Brown for assistance in checking the details of the rather intricate proof, which appears in [3]. The main result that Dempwolff establishes can be stated in the following form:

THEOREM 1 [DEMPWOLFF]. *The orbits of* $\mathrm{Aut}({}^{\alpha}\Pi)$ *on the points and lines of* ${}^{\alpha}\Pi$ *are* $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{L}_1$, $\mathcal{L}_2$, *and* $\mathcal{L}_3^*$. *Moreover,* $\mathrm{Aut}({}^{\alpha}\Pi)$ *acts on* Π_0 *with kernel* $\langle\alpha\rangle$.

The arguments in [3] do not in fact determine the precise structure of $\mathrm{Aut}({}^{\alpha}\Pi)$, but, they go a long way towards doing so. There is a fairly obvious action of $\mathrm{PGL}(3, F)$ on ${}^{\alpha}\Pi$, which is used in the proof of Theorem 1. The only outstanding issue is to determine when collineations of Π_0 arising from field automorphisms can be extended to collineations of ${}^{\alpha}\Pi$. We rely heavily on Dempwolff's theorem in order to compute $\mathrm{Aut}({}^{\alpha}\Pi)$, and to describe when two Figueroa planes are isomorphic. Note that we regard collineations of Π and of ${}^{\alpha}\Pi$ as permutations of the point set $\mathcal{P}$. Our main results are:

THEOREM 2. *Let* α *and* β *be planar order* 3 *collineations of the Pappian planes* Π *and* Π' *respectively. Assume that* ${}^{\alpha}\Pi \cong {}^{\beta}\Pi'$. *Then* $\Pi \cong \Pi'$, *and if* $\Pi = \Pi'$ *then* $\langle\alpha\rangle$ *and* $\langle\beta\rangle$ *are conjugate in* $\mathrm{Aut}(\Pi)$.

THEOREM 3. $\mathrm{Aut}({}^{\alpha}\Pi)$ *consists of all collineations of* Π *which normalize* $\langle\alpha\rangle$. *It has the structure* $\mathrm{PGL}(3, F) \rtimes \{\gamma \in \mathrm{Aut}(K) \mid \gamma(F) = F\}$ *(with the natural group action).*

Our last result is concerned not with collineations, but with subplanes. Let $\mathcal{P}'$ be a subset of the point set $\mathcal{P}$ of Π and ${}^{\alpha}\Pi$. By $\Pi|_{\mathcal{P}'}$ we mean the linear space induced on $\mathcal{P}'$ by Π. We shall find it most convenient to regard $\Pi|_{\mathcal{P}'}$ as the incidence system $(\mathcal{P}', \mathcal{L}', I)$, where $\mathcal{L}'$ is the set of all lines of Π that contain at least two points of $\mathcal{P}'$ and, for all $p \in \mathcal{P}'$, $L \in \mathcal{L}'$, $p\ I\ L \Leftrightarrow p \in L$. Define ${}^{\alpha}\Pi|_{\mathcal{P}'}$ similarly. For example, the subplane Π_0 of Π fixed by α can be written in the form $\Pi|_{\mathcal{P}_1}$, ${}^{\alpha}\Pi|_{\mathcal{P}_1}$, or $(\mathcal{P}_1, \mathcal{L}_1, I)$.

https://doi.org/10.4153/CMB-1993-053-0 Published online by Cambridge University Press

A linear space is *thick* if each of its lines has at least three points. We say that a subset of $\mathcal{P}$ is *α-closed* if it is a union of orbits of $\langle\alpha\rangle$.

THEOREM 4. *Let $\mathcal{P}'$ be a finite α-closed subset of $\mathcal{P}$ such that ${}^{\alpha}\Pi|_{\mathcal{P}'}$ is a projective plane and $\Pi|_{\mathcal{P}'}$ is thick. Then $\Pi|_{\mathcal{P}'}$ is a Pappian projective plane, and ${}^{\alpha}\Pi|_{\mathcal{P}'}$ is the Figueroa or Pappian plane obtained from it by twisting via the restriction of α to $\mathcal{P}'$.*

REMARK. The preceding theorem can be viewed as a partial converse of a result of Julia Brown (personal communication). She has independently proved the existence of proper Figueroa subplanes in Figueroa planes of order p^{3t}, where p is prime and t is not a power of 3, and in many infinite Figueroa planes.

2. **Proofs.** As before, $\mathcal{P}$ is the point set of the projective planes Π and ${}^{\alpha}\Pi$. Any two points p and q determine a unique line of ${}^{\alpha}\Pi$, which is called $\widetilde{pq}$ to distinguish it from the line pq of Π.

From Theorem 1, $\mathrm{Aut}({}^{\alpha}\Pi)$ has three orbits on $\mathcal{P}$ and three on $\mathcal{L}^*$. It is important to know that these orbits can be distinguished from each other by properties intrinsic to ${}^{\alpha}\Pi$, without being given in advance the Pappian plane Π and the collineation α from which ${}^{\alpha}\Pi$ is constructed. Once we know which orbit is $\mathcal{P}_1$, it is easy to describe $\mathcal{L}_1$, then $\mathcal{P}_3$ (the points not on any line in $\mathcal{L}_1$), and then every other orbit. The following lemma gives a criterion for distinguishing $\mathcal{P}_1$ from the other two point orbits.

LEMMA 1. *For $i = 1, 2, 3$, let N_i be the subgroup of* $\mathrm{Aut}({}^{\alpha}\Pi)$ *which fixes every point of $\mathcal{P}_i$. Of these subgroups, only N_1 is nontrivial.*

PROOF. Clearly $\langle\alpha\rangle \leq N_1$; we even have equality. It suffices to prove that N_2 and N_3 are contained in N_1. Given any $p \in \mathcal{P}_2$, N_2 fixes the line pp^{α} in $\mathcal{L}_1$, for it fixes p and p^{α}. Then N_2 fixes all lines in $\mathcal{L}_1$, because $\mathcal{L}_1$ is an orbit of $\mathrm{Aut}({}^{\alpha}\Pi)$ and $N_2 \lhd \mathrm{Aut}({}^{\alpha}\Pi)$. Therefore N_2 fixes all points in $\mathcal{P}_1$. To show that N_3 has the same property, we shall find a line L that contains a point of $\mathcal{P}_1$ and at least two points of $\mathcal{P}_3$. Any such L lies in $\mathcal{L}_2$ and is fixed by N_3. Then N_3 fixes all lines of $\mathcal{L}_2$ and (since $L \cap L^{\alpha} \in \mathcal{P}_1$) all points of $\mathcal{P}_1$. Consequently N_3, as well as N_2, must be trivial.

To find a suitable line L, regard $\mathcal{P}$ as the set of all 1-subspaces of the row space K^3, and use homogeneous coordinates. Choose $a \in K \setminus F$. The field F has more than two elements, so we may also choose $b \in F$, $b \neq 0, 1$. Now let L be the line of ${}^{\alpha}\Pi$ (and of Π) through the points $[1, 0, 0]$, $[1, a, a^{\alpha}]$, and $[b, a, a^{\alpha}]$. The first point is in $\mathcal{P}_1$ and the other two are in $\mathcal{P}_3$. ∎

There is another way to distinguish $\mathcal{P}_1$ from $\mathcal{P}_2$ and $\mathcal{P}_3$, based on the fact that, up to isomorphism, $\mathrm{PGL}(3, F) \leq \mathrm{Aut}({}^{\alpha}\Pi)/\langle\alpha\rangle \leq \mathrm{P\Gamma L}(3, F)$. Then $\langle\alpha\rangle$ is the largest solvable normal subgroup of $\mathrm{Aut}({}^{\alpha}\Pi)$; $\mathcal{P}_1$ is its fixed-point set.

We frequently use the following facts, which follow immediately from the definition (in terms of the 1–1 correspondence μ) of the lines of ${}^{\alpha}\Pi$:

(i) If $pq \in \mathcal{L}_1 \cup \mathcal{L}_2$ then $\widetilde{pq} = pq$.

(ii) For points $p, q \in \mathcal{P}_3$, $q \in \widetilde{pp^{\alpha}}$ iff $p \in \widetilde{qq^{\alpha^2}}$.

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(iii) If $L \in \mathcal{L}_3$, there is a unique point q with $L = qq^{\alpha^2}$. Here $L^* = \widetilde{qq^{\alpha^2}}$ and $q = L \cap L^\alpha = L^* \cap L^{*\alpha} \in \mathcal{P}_3$. This sets up 1–1 correspondences $\mathcal{L}_3 \leftrightarrow \mathcal{P}_3 \leftrightarrow \mathcal{L}_3^*$.

(iv) Let $L^* = \widetilde{qq^{\alpha^2}}$, where $q \in \mathcal{P}_3$. Then $L = (L^* \setminus \mathcal{P}_3) \cup \{x \in \mathcal{P}_3 \mid q \in \widetilde{xx^\alpha}\}$.

The last formula, which is obvious from (ii) and the definition of L^*, is of particular interest. It defines L from L^* in a way which depends only on the internal structure of the Figueroa plane ${}^\alpha\Pi$. In a sense, it shows how to 'untwist' ${}^\alpha\Pi$ to recover Π, once $\langle\alpha\rangle$, $\mathcal{L}_3^*$, and $\mathcal{P}_3$ have been determined. Note that α and α^2 may be interchanged in all of the above formulas, as this has no effect on the definition of the line set $\mathcal{L}^*$.

LEMMA 2. *Let α and β be planar order 3 collineations of the Pappian planes Π and Π' respectively.*

(a) If $\phi: {}^\alpha\Pi \longrightarrow {}^\beta\Pi'$ (regarded as a map between the point sets) is an isomorphism, then so is $\phi: \Pi \longrightarrow \Pi'$.

(b) If $\phi^{-1}\beta\phi \in \langle\alpha\rangle$ and $\phi: \Pi \longrightarrow \Pi'$ is an isomorphism, then so is $\phi: {}^\alpha\Pi \longrightarrow {}^\beta\Pi'$.

PROOF. First assume that $\phi: {}^\alpha\Pi \longrightarrow {}^\beta\Pi'$ is an isomorphism. In view of Dempwolff's theorem and our remarks concerning it, ϕ must carry the sets $\mathcal{P}_i$ ($i = 1, 2, 3$) of ${}^\alpha\Pi$ to the analogous point sets $\mathcal{P}_i'$ of ${}^\beta\Pi'$. A similar result holds for the line orbits. In particular, ϕ maps lines in $\mathcal{L}_1 \cup \mathcal{L}_2$ to lines of Π'. Since $\phi^{-1}\beta\phi$ is a nontrivial collineation of ${}^\alpha\Pi$ fixing $\mathcal{P}_1$ pointwise, it must be α or α^2. We may assume that it is α, by interchanging β and β^2 if necessary.

Let L be an arbitrary line in $\mathcal{L}_3$. Recall that L is of the form qq^{α^2} for some $q \in \mathcal{P}_3$. Then ϕ maps the corresponding line $L^* = \widetilde{qq^{\alpha^2}}$ of ${}^\alpha\Pi$ to the line of ${}^\beta\Pi'$ through $\phi(q)$ and $\phi(q^{\alpha^2})$. The second point is $\phi(q)^{\beta^2}$. From (iv) and our assumption on ϕ, it follows that $\phi(L)$ consists of $\phi(L^* \setminus \mathcal{P}_3)$, together with all points $y \in \mathcal{P}_3'$ such that $\phi(q) \in \widetilde{yy^\beta}$. Therefore $\phi(L)$ is a line of Π'. Then $\phi: \Pi \longrightarrow \Pi'$ is clearly an isomorphism.

A very similar argument, using the definition of L^* from L, establishes (b). ∎

THEOREM 2. *Let α and β be planar order 3 collineations of the Pappian planes Π and Π' respectively. Assume that ${}^\alpha\Pi \cong {}^\beta\Pi'$. Then $\Pi \cong \Pi'$, and if $\Pi = \Pi'$ then $\langle\alpha\rangle$ and $\langle\beta\rangle$ are conjugate in* Aut(Π).

PROOF. Lemma 2 gives the first assertion at once, and from its proof we see that any isomorphism $\phi: {}^\alpha\Pi \longrightarrow {}^\beta\Pi'$ induces an isomorphism $\mathrm{Aut}({}^\alpha\Pi) \longrightarrow \mathrm{Aut}({}^\beta\Pi')$ in which $\langle\alpha\rangle$, the pointwise stabilizer of $\mathcal{P}_1$, must map to the corresponding subgroup $\langle\beta\rangle$. If $\Pi' = \Pi$, this means that $\langle\alpha\rangle$ and $\langle\beta\rangle$ are conjugate via the element ϕ of Aut(Π). ∎

REMARK. If α and β are conjugate elements of order 3 in Aut(Π), then ${}^\alpha\Pi$ and ${}^\beta\Pi$ are clearly isomorphic. Since $\Pi = \mathrm{PG}(2, K)$, there are well-defined automorphisms of K associated with α and β. These are conjugate in Aut(K). In fact (recalling that the case $|K| = 8$ is always excluded), there is a natural 1–1 correspondence between the conjugacy classes of order 3 subgroups of Aut(K) and the isomorphism types of Figueroa planes constructed from PG$(2, K)$.

The easiest way to see this is to start with an arbitrary order 3 collineation γ of Π. Assume that it is planar, for otherwise ${}^\gamma\Pi$ is not a Figueroa plane. Change to a new system

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of homogeneous coordinates over K in which $[1,0,0]$, $[0,1,0]$, $[0,0,1]$, and $[1,1,1]$ all lie in the subplane fixed pointwise by γ. Since γ must be induced by a semilinear transformation, the automorphism $\alpha \in \mathrm{Aut}(K)$ associated with γ has order 3. Recall that there is a corresponding collineation α of Π, acting in the natural way on the canonical coordinate system of $\mathrm{PG}(2,K)$. It acts on these coordinates in exactly the same way that γ acts on the new ones. Therefore γ is conjugate to α in $\mathrm{Aut}(\Pi)$. This means that the desired 1–1 correspondence is now an obvious consequence of a simpler assertion concerning the collineations constructed in a natural way from order 3 automorphisms of K.

THEOREM 3. $\mathrm{Aut}({}^{\alpha}\Pi)$ *consists of all collineations of* Π *which normalize* $\langle\alpha\rangle$. *It has the structure* $\mathrm{PGL}(3,F) \rtimes \{\gamma \in \mathrm{Aut}(K) \mid \gamma(F) = F\}$ *(with the natural group action).*

PROOF. By Theorem 1, $\langle\alpha\rangle \lhd \mathrm{Aut}({}^{\alpha}\Pi)$. In view of this fact, the first part of the theorem is just a special case of Lemma 2. We now compute the structure of the normalizer of $\langle\alpha\rangle$ in $\mathrm{Aut}(\Pi)$. For any $\gamma \in \mathrm{Aut}(K)$, the subfield of K fixed by $\gamma\alpha\gamma^{-1}$ is $\gamma(F)$. By Galois theory (see [7, Theorem 8]), any automorphism of K fixing F is in $\langle\alpha\rangle$, so the condition $\gamma(F) = F$ is equivalent to γ normalizing $\langle\alpha\rangle$. Thus, using the canonical correspondence between automorphisms of K and collineations of Π fixing $[1,0,0]$, $[0,1,0]$, $[0,0,1]$, and $[1,1,1]$ (in the usual coordinate system), we see that $\mathrm{Aut}({}^{\alpha}\Pi)$ has a subgroup which may be identified with $\{\gamma \in \mathrm{Aut}(K) \mid \gamma(F) = F\}$.

Since $\Pi = \mathrm{PG}(2,K)$, $\mathrm{Aut}(\Pi)$ can be identified with $\mathrm{P\Gamma L}(3,K)$. It has a subgroup, canonically isomorphic to $\mathrm{PGL}(3,F)$, which centralizes α. Now consider an arbitrary element of $\mathrm{P\Gamma L}(3,K)$ which normalizes $\langle\alpha\rangle$. It permutes the points of Π_0, as these are the points fixed by $\langle\alpha\rangle$. We can adjust it by multiplying by a suitable element of $\mathrm{PGL}(3,F)$ so that it fixes $[1,0,0]$, $[0,1,0]$, $[0,0,1]$, and $[1,1,1]$. Since this element also normalizes $\langle\alpha\rangle$, it lies in the subgroup of $\mathrm{Aut}({}^{\alpha}\Pi)$ described above. It follows easily that the structure of $\mathrm{Aut}({}^{\alpha}\Pi)$ is as claimed. ∎

COROLLARY. *With notation as before, every collineation of* Π_0 *extends to one of* ${}^{\alpha}\Pi$ *iff every automorphism of* F *extends to one of* K.

PROOF. In its action on Π_0, $\mathrm{Aut}({}^{\alpha}\Pi)$ induces at least the group $\mathrm{PGL}(3,F)$. Any induced collineation of Π_0 corresponding to a field automorphism must arise from the restriction to F of an element in $\{\gamma \in \mathrm{Aut}(K) \mid \gamma(F) = F\}$. Thus $\mathrm{Aut}({}^{\alpha}\Pi)$ induces the full collineation group $\mathrm{P\Gamma L}(3,F)$ on Π_0 precisely when every automorphism of F extends to one of K. ∎

LEMMA 3. *Let* $\mathcal{P}'$ *be an* α*-closed subset of* $\mathcal{P}$ *such that* ${}^{\alpha}\Pi|_{\mathcal{P}'}$ *is a projective plane and* $\Pi|_{\mathcal{P}'}$ *is thick. Then*

(a) for each line L *of* $\Pi|_{\mathcal{P}'}$, $L \cap L^{\alpha}$ *contains a point of* $\mathcal{P}'$, *and*

(b) there exists a 1–1 correspondence between lines of $\Pi|_{\mathcal{P}'}$ *and lines of* ${}^{\alpha}\Pi|_{\mathcal{P}'}$, *given by* $L \leftrightarrow L$ *for* $L \in \mathcal{L}_1 \cup \mathcal{L}_2$, *and* $xx^{\alpha^2} \leftrightarrow \widetilde{xx^{\alpha^2}}$ *for* $x \in \mathcal{P}' \cap \mathcal{P}_3$.

PROOF. (a) This is trivial if $L \in \mathcal{L}_1$. If $L \in \mathcal{L}_2$, then L and L^{α} are distinct lines of the projective plane ${}^{\alpha}\Pi|_{\mathcal{P}'}$, so they intersect at a point in $\mathcal{P}'$. Suppose $L \in \mathcal{L}_3$, and let

$q = L \cap L^\alpha$. We must show that $q \in \mathcal{P}'$. Since $|L \cap \mathcal{P}'| \geq 3$, we may write $L = uv$, where $u, v \in \mathcal{P}' \cap \mathcal{P}_2$ or $u, v \in \mathcal{P}' \cap \mathcal{P}_3$.

In the first case, $\widetilde{uv} = \widetilde{qq^{\alpha^2}}$ and $\widetilde{u^\alpha v^\alpha} = \widetilde{qq^\alpha}$. These are distinct lines of ${}^\alpha\Pi|_{\mathcal{P}'}$ which intersect at q. If instead we have $u, v \in \mathcal{P}_3$, then the lines $\widetilde{uu^\alpha}$ and $\widetilde{vv^\alpha}$ are distinct. By (ii), they intersect at q. Therefore, in either case, q lies in $\mathcal{P}'$.

(b) By (iii), the lines in $\mathcal{L}_3$ are of the form xx^{α^2}, $x \in \mathcal{P}_3$. Those which are lines of $\Pi|_{\mathcal{P}'}$ are the ones satisfying $x \in \mathcal{P}'$, by (a). As ${}^\alpha\Pi|_{\mathcal{P}'}$ is a projective plane, its lines in $\mathcal{L}_3^*$ are of the form $\widetilde{xx^{\alpha^2}}$, where $x \in \mathcal{P}' \cap \mathcal{P}_3$. This gives the desired 1–1 correspondence between the lines of $\Pi|_{\mathcal{P}'}$ and those of ${}^\alpha\Pi|_{\mathcal{P}'}$. ∎

THEOREM 4. *Let $\mathcal{P}'$ be a finite α-closed subset of $\mathcal{P}$ such that ${}^\alpha\Pi|_{\mathcal{P}'}$ is a projective plane and $\Pi|_{\mathcal{P}'}$ is thick. Then $\Pi|_{\mathcal{P}'}$ is a Pappian projective plane, and ${}^\alpha\Pi|_{\mathcal{P}'}$ is the Figueroa or Pappian plane obtained from it by twisting via the restriction of α to $\mathcal{P}'$.*

PROOF. The linear space $\Pi|_{\mathcal{P}'}$ and the projective plane ${}^\alpha\Pi|_{\mathcal{P}'}$ are finite and have the same point set. By the 1–1 correspondence of Lemma 3, they have the same number of lines. Thus $\Pi|_{\mathcal{P}'}$ is a thick linear space with equally many points and lines. By a result of de Bruijn and Erdős [1], $\Pi|_{\mathcal{P}'}$ is a projective plane. This plane is Pappian, as it is a subplane of $\Pi = \mathrm{PG}(2, K)$. Moreover, the plane obtained by twisting it must have the same lines (regarded as subsets of $\mathcal{P}'$) as ${}^\alpha\Pi|_{\mathcal{P}'}$. ∎

REMARK. We are unable to determine whether or not there are analogous results with $\mathcal{P}'$ infinite or $\Pi|_{\mathcal{P}'}$ not thick. Of course the methods of the preceding proof cannot be adapted to those cases.

3. **Examples.** As always, ${}^\alpha\Pi$ stands for a Figueroa plane constructed from $\Pi = \mathrm{PG}(2, K)$ and $\alpha \in \mathrm{Aut}(K)$, while F denotes the fixed subfield. The nature of the field K has some influence on properties of the collineation group of ${}^\alpha\Pi$. We provide several examples which illustrate this fact. Since the theory of fields, rather than geometry, comes to the fore in this section, we now let p denote a prime number.

First let K be a finite field of order p^{3n}. Automorphisms of K induce collineations of Π and (by Theorem 3) of ${}^\alpha\Pi$, where α arises from a field automorphism of order 3. If n is not a multiple of 3 then $\mathrm{Aut}(K) \cong Z_{3n} \cong Z_3 \times Z_n$, and it is easy to conclude from Theorem 3 that $\mathrm{Aut}({}^\alpha\Pi) \cong Z_3 \times \mathrm{P\Gamma L}(3, F)$. Assume instead that n is a multiple of 3. Then there is a collineation β of ${}^\alpha\Pi$ with $\beta^3 = \alpha$. This makes it clear that $\mathrm{Aut}({}^\alpha\Pi)$ is now a non-split extension of $\langle\alpha\rangle$ by $\mathrm{P\Gamma L}(3, F)$.

Next we examine the situation in the Corollary for various choices of fields K. Define H to be the subgroup $N_{\mathrm{Aut}(K)}(\langle\alpha\rangle)$ of $\mathrm{Aut}(K)$. As shown in the proof of Theorem 3, it is the group of all automorphisms of K which stabilize F. The pointwise stabilizer of F is just $\langle\alpha\rangle$, so the group of all automorphisms of F which extend to K is isomorphic to $H/\langle\alpha\rangle$.

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We shall construct examples for which $H/\langle\alpha\rangle \not\cong \mathrm{Aut}(F)$. Thus, in the corresponding Figueroa plane ${}^{\alpha}\Pi$, there are collineations of the distinguished subplane Π_0 which do not extend to collineations of ${}^{\alpha}\Pi$. Any such Figueroa plane must be infinite. We can even say a little more: the field K cannot be an algebraic extension of $\mathbf{F}_p$, for in that case it is not hard to show that automorphisms of subfields always extend to automorphisms of the whole field.

Let $K = \mathbf{F}_p(x)$ be a transcendental extension of a field with p elements. Every automorphism γ of K is determined by the image $\gamma(x)$ of x. We need two basic results which can be found in [8, §10.2]. First, $\gamma(x)$ must be of the form $(ax+b)/(cx+d)$, where $a,b,c,d \in \mathbf{F}_p$ and $ad-bc \neq 0$. Thus $\mathrm{Aut}(K)$ is isomorphic to the finite group $\mathrm{PGL}(2,p)$. Secondly, Lüroth's theorem implies that, for any subfield L of K other than $\mathbf{F}_p$, L is isomorphic to K and hence $\mathrm{Aut}(L) \cong \mathrm{PGL}(2,p)$.

In particular, consider the order 3 automorphism α of K satisfying $\alpha(x) = -1-x^{-1}$. The automorphism of K which moves x to x^{-1} normalizes but does not centralize $\langle\alpha\rangle$. We have $[K : F] = 3$, where F is the subfield of K fixed pointwise by α. There is a corresponding Figueroa plane ${}^{\alpha}\Pi$. Because $\mathrm{Aut}(F)$ and $\mathrm{Aut}(K)$ have the same finite order and $H/\langle\alpha\rangle$ has a smaller order, it follows that $\mathrm{Aut}({}^{\alpha}\Pi)/\langle\alpha\rangle \not\cong \mathrm{P\Gamma L}(3,F)$, where α is regarded as a collineation of ${}^{\alpha}\Pi$. Moreover, α is not central in $\mathrm{Aut}({}^{\alpha}\Pi)$.

In characteristic 0, there is no need to use transcendental extensions. It is widely conjectured that every finite group G can be realized as the Galois group of a normal extension field M of $\mathbb{Q}$. We shall be content with examining one small example in detail.

Let M be such an extension with $\mathrm{Aut}(M) \cong Z_3 \times S_3$. We use the Galois correspondence between subgroups of $G = \mathrm{Aut}(M)$ and subfields of M. Let H be one of the non-normal subgroups of G which has order 3. Then $N_G(H)$ has order 9. Let the subfields K and F of M correspond to H and $N_G(H)$ respectively. Then $[K : F] = 3$ and $[F : \mathbb{Q}] = 2$. The group of automorphisms of M which leave K invariant is $N_G(H)$. Moreover (see Example 9 of [7, §3]), every automorphism of K extends to one of M. Therefore $\mathrm{Aut}(K) \cong N_G(H)/H \cong Z_3$. The subfield fixed by a generator α of $\mathrm{Aut}(K)$ is F. Thus, from $\Pi = \mathrm{PG}(2,K)$ and α, we can construct a Figueroa plane ${}^{\alpha}\Pi$ whose subplane Π_0 is isomorphic to $\mathrm{PG}(2,F)$. However F has an automorphism of order 2; it does not extend to K. The corresponding collineation of Π_0 does not extend to ${}^{\alpha}\Pi$. In this example, $\mathrm{Aut}({}^{\alpha}\Pi) \cong Z_3 \times \mathrm{PGL}(3,F)$.

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