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REPRESENTATION OF TYPE A MONOIDS

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A semigroup T consisting of one-one mapping between certain principal left ideals in a type A semigroup S is constructed. T is shown to be a type A semigroup. A representation of S by T is then obtained which is analogous to Vagner-Preston's results on inverse semigroups.

1. INTRODUCTION

Many results are now available in the literature on type A semigroups; some of which are analogous to those on inverse semigroups; see for example Fountain [**6**, **5**], Asibong-Ibe [**2**, **3**, **4**], Amstrong [**1**] and Fountain and Lawson [**7**]. Because of the close relationship which exists between a type A semigroup and an inverse semigroup, each type A being basically a special type of subsemigroup of an inverse semigroup via an embedding, it is natural to ask whether a representation exists for a type A semigroup similar to Vagner-Preston's for inverse semigroup. This paper answers this question.

Let us recall a few definitions. Let S be a semigroup and a, $b \in S$. Then $(a, b) \in \mathcal{L}^*$ if and only if $a\mathcal{L}b$ is an oversemigroup of S. The relation $\mathcal{L}^*$ which properly contains the Green's relation $\mathcal{L}$ on S has the following equivalent characterisation, see [**10**].

LEMMA 1.1. *Let S be a semigroup and a, $b \in S$. The following are equivalent:*

(i) $(a, b) \in \mathcal{L}^*$,
(ii) *for all x, y in S, $ax = ay$ if and only if $bx = by$,*
(iii) *there exists an S-isomorphism $\lambda\colon aS^1 \to bS^1$ such that $a\lambda = b$.*

LEMMA 1.2. *Let S be a semigroup and e an idempotent in S. Then for any a in S, the following are equivalent:*

(i) $(e, a) \in \mathcal{L}^*$,
(ii) $ae = a$, *and for all x, y in S, $ax = ay$ if and only if $ex = ey$.*

$\mathcal{R}^*$ is dual to $\mathcal{L}^*$ and the above definition and properties of $\mathcal{L}^*$ apply in a dual manner to $\mathcal{R}^*$.

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Let S be a semigroup with a semilattice $E(S)$ of idempotents. Then S is said to be an adequate semigroup if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent.

An adequate semigroup S is said to be a type A semigroup if for each a in S and e in $E(S)$, $ea = a(ea)^*$ and $ae = (ae)^+a$, where x^* and x^+ are respectively idempotents in the $\mathcal{L}^*$ and $\mathcal{R}^*$ classes L^*_x and R^*_x. A type A semigroup has been characterised in the following way in [5].

THEOREM 1.3. *Let S be an adequate semigroup. Then for $a \in S$, $e \in E(S)$, the following are equivalent:*

(i) *S is a type A semigroup,*
(ii) *$eS^1 \cap aS^1 = eaS^1$ and $S^1e \cap S^1a = S^1ae$, and*
(iii) *there exist embeddings $\lambda_1 \colon S \to S_1$, and $\lambda_2 \colon S \to S_2$ into inverse semigroups S_1, S_2 such that $a^*\lambda_1 = (a\lambda_1)^{-1}(a\lambda_1)$ and $a^+\lambda_2 = (a\lambda_2)(a\lambda_2)^{-1}$.*

2. TYPE A SEMIGROUP OF MAPPINGS

In this and subsequent sections, the term semigroup S will refer to a type A semigroup S with $E(S)$ as its set of idempotents. Other notation used here agrees with that of [9] and [5].

Let $a \in S$; then a^+, $a^* \in E(S)$, and $aa^* = a^+a = a$. Consider the left principal ideals Sa^+ and Sa^* and let $x_1 \in Sa^+$. Then for some $x \in S$, $x_1 = xa^+ \in Sa^+$ and $x_1a = xa^+a = xa = xaa^* \in Sa^*$. Evidently for every s in S, $saa^* = s(aa^*) = sa \in Sa^*$. Let us define a mapping $\alpha_a \colon Sa^+ \to Sa^*$ by putting for every x in S, $x\alpha_a = xa$, where $a \in S$. Since $aa^* = a$, $Sa = Saa^* \subseteq Sa^*$, so for $x \in S$, $xa = xa^+a = (xa^+)\alpha_a \in (Sa^+)\alpha_a$ so evidently $(Sa^+)\alpha_a = Sa \subseteq Sa^*$. Thus $\operatorname{ran}\alpha_a = Sa$. However, if a is regular then $Sa = Sa^*$, thus in this case $\operatorname{ran}\alpha_a = Sa^*$. Let us show that each α_a, $a \in S$ is a one-to-one mapping.

LEMMA 2.1. *For each $a \in S$, α_a is a one-one mapping from Sa^+ into Sa^*. Also α_a is onto if and only if a is regular.*

PROOF: Consider the mapping $\alpha_a \colon Sa^+ \to Sa^*$, and let $xa = ya$ for x, y in S. Then $(xa^+)\alpha_a = xa = ya = (ya^+)\alpha_a$. But $a\mathcal{R}^*a^+$, so $xa = ya$ if and only if $xa^+ = ya^+$ for all x, y in S. Consequently, α_a is a one-one mapping.

Now if α_a is onto then $(Sa^+)\alpha_a = Sa^*$. Thus $(Sa^+)\alpha_a = Sa^+a = Sa = Sa^*$; consequently $a\mathcal{L}a^*$, and a must be regular. Conversely, if a is regular, $aa^{-1}a = a$, $a^* = a^{-1}a$ and clearly $Sa^* = Sa$, so $\mathcal{L}_a$ is onto. ∎

COROLLARY 2.2. *For each $a \in S$, α_a has inverse $\alpha_{a^{-1}}$ if and only if a is regular.*

PROOF: If $\alpha_a^{-1} = \alpha_{a^{-1}}$ then $xa^+ = (xa)\alpha_a^{-1} = (xa)\alpha_{a^{-1}} = xaa^{-1}$. So $xa^+a =$

$xaa^{-1}a = xa$ and bijectivity of α_a forces $a^+ = aa^{-1}$ so $aa^{-1}a = a$. Conversely if a is regular α_a is bijective so α_a^{-1} exists and obviously $\alpha_a^{-1} = \alpha_{a^{-1}}$. □

Now let a be a non-regular element in S. Let $\lambda\colon Sa \to Sa^+$ be an S-system isomorphism with $a\lambda = a^+$. Thus given $\alpha_a\colon Sa^+ \to Sa^*$ with $\operatorname{ran}\alpha_a = Sa$ we can define $\alpha_a^{-1} \mid Sa \to Sa^+$ by putting $\alpha_a^{-1} = \lambda$ so that $(xa)\alpha_a^{-1} = (xa)\lambda = x(a\lambda) = xa^+$ for $x \in S$. One checks that if $x \in Sa^+$, $x\alpha_a\alpha_a^{-1} = (xa)\alpha_a^{-1} = xa^+ = x$ and for each $y = xa$, we have $y\alpha_a^{-1}\alpha_a = xa^+\alpha_a = xa = y$. Observe that $Sa \neq Sa^*$ because an equality implies regularity of a, which is a contradiction to our assumption.

Now let us consider the subset T of $\mathcal{I}(S)$, the symmetric inverse semigroup where $T = \{\alpha_a \mid a = S, \alpha_a : Sa^+ \to Sa^*\}$ and impose the condition that $\alpha_a^{-1} \in T$ if and only if $\alpha_a^{-1} = \alpha_{a^{-1}}$, that is if and only if a is regular. Thus the domain and codomain of elements of T are respectively the principal left ideals ganerated by a^+ and a^* for any $a \in S$.

An important fact is there is closure in T with respect to the product of its elements. Let us show this as follows. Consider the mappings $\alpha_a\colon Sa^+ \to Sa^*$, $\alpha_b\colon Sb^+ \to Sb^*$. Now $Sa^* \cap Sb^+ = Sa^*b^+$, and $a^*b^+ = (ab^+)^*$. Consequently $ab^+ = aa^*b^+ = a(ab^+)^*$; hence $a(ab^+)^* = ab^+ = (ab^+)^+a$. Since $Sa \subseteq Sa^*$ then $Sa \cap Sb^+ = Sab^+ \subseteq Sa^*b^+$ so that $Sab^+ = S(ab^+)^+a = S(ab)^+a = S(ab)^+\alpha_a$. But $Sab^+ \subseteq Sb^+$, and hence $(Sab^+)\alpha_b \subseteq (Sb^+)\alpha_b$, and $(Sab^+)\alpha_b = S(ab)^+\alpha_a\alpha_b = Sab$. Indeed, since $(Sa^*b^+)\alpha_b = Sa^*b$ and $a^*b = b(a^*b)^* = b(ab)^*$, one checks that $Sa^*b = Sb(ab)^* \subseteq S(ab)^*$. With $Sab \subseteq S(ab)^*$, it is clear that the codomain of $\alpha_a\alpha_b$ is $S(ab)^*$ and its domain is $S(ab)^+$. Evidently, it follows from these facts that $\alpha_a\alpha_b = \alpha_{ab}$, showing closure property in T. It is then clear that T is a semigroup.

Let a, b be regular elements in S. Then (ab) is regular with inverse $(ab)^{-1} \in S$. Also α_a, α_b are regular in T and evidently $\alpha_{ab} = \alpha_a\alpha_b$ is regular in T with inverse $\alpha_{(ab)^{-1}} \in T$, $\alpha_{ab}^{-1} = (\alpha_a\alpha_b)^{-1} = \alpha_b^{-1}\alpha_a^{-1} = \alpha_{b^{-1}}\alpha_{a^{-1}} = \alpha_{b^{-1}a^{-1}} = \alpha_{(ab)^{-1}} \in T$. Let us now show below that T is a type A monoid. □

Theorem 2.3. *For a type A semigroup S, the set $T = \{\alpha_a \mid a \in S, \alpha_a\colon Sa^+ \to Sa^*\}$ such that for each x in S, $x\alpha_a = xa$, is a type A monoid.*

We will prove this fact through the following lemmas.

Lemma 2.4.

(i) $(\alpha_a, \alpha_b) \in \mathcal{L}^*(T)$ *if and only if* $(a, b) \in \mathcal{L}^*(S)$, *and*
(ii) $(\alpha_a, \alpha_b) \in \mathcal{R}^*(T)$ *if and only if* $(a, b) \in \mathcal{R}^*(S)$.

Proof: Let $(\alpha_a, \alpha_b) \in \mathcal{L}^*$ for α_a, α_b in T. Then for all α_c, α_d in T we have that

$$\alpha_a\alpha_c = \alpha_a\alpha_d \text{ if and only if } \alpha_b\alpha_c = \alpha_b\alpha_d.$$

Let $\alpha_a\alpha_c = \alpha_a\alpha_d$. Then $(\operatorname{dom}\alpha_a\alpha_c)\alpha_a = \operatorname{ran}\alpha_a \cap \operatorname{dom}\alpha_c = (\operatorname{ran}\alpha_a \cap \operatorname{dom}\alpha_d) = (\operatorname{dom}\alpha_a\alpha_d)\alpha_a$. Also $(\operatorname{ran}\alpha_a \cap \operatorname{dom}\alpha_c)\alpha_c = (\operatorname{ran}\alpha_a \cap \operatorname{dom}\alpha_d)\alpha_d$. Now if $x\alpha_a \in \operatorname{ran}\alpha_a \cap \operatorname{dom}\alpha_c$, then the equality $\operatorname{ran}\alpha_a \cap \operatorname{dom}\alpha_c = \operatorname{ran}\alpha_a \cap \operatorname{dom}\alpha_d$ implies that for all x in S, $xa\alpha_c = xa\alpha_d$. That is, $xac = xad$ and in particular for $x = a^+$, $ac = a^+\alpha_{ac} = a^+ac = a^+ad = a^+\alpha_{ad} = ad$. Thus if $\alpha_{ac} = \alpha_{ad}$ then $ac = ad$ for any $\alpha_{ac}, \alpha_{ad} \in T$. But $\alpha_{ac} = \alpha_{ad}$ if and only if $\alpha_{bc} = \alpha_{bd}$. It can be shown that whenever this holds then $ac = ad$ if and only if $bc = bd$. Hence $(a, b) \in \mathcal{L}^*(S)$. Conversely, let $ac = ad$. Then $\alpha_{ac} = \alpha_{ad}$ and so $\alpha_a\alpha_c = \alpha_a\alpha_d$. But for all $c, d \in S$, $ac = ad$ implies $bc = bd$ and whenever $ac = ad$, then $\alpha_a\alpha_c = \alpha_a\alpha_d$. Hence for all $c, d \in S$, we can deduce that $\alpha_a\alpha_c = \alpha_a\alpha_d$ implies $\alpha_b\alpha_c = \alpha_b\alpha_d$. Since this is true for all $\alpha_c, \alpha_d \in T$, then $(\alpha_a, \alpha_b) \in \mathcal{L}^*(T)$, which completes the proof of (i). The proof of (ii) is similar, so the lemma is proved. ∎

From the above lemma we have the following.

Corollary 2.5. *Let $\alpha_a, \alpha_b \in T$. Then*

(i) $(\alpha_a, \alpha_b) \in \mathcal{H}^*(T)$ *if and only if* $(a, b) \in \mathcal{H}^*(S)$,

(ii) $(\alpha_a, \alpha_b) \in \mathcal{D}^*(T)$ *if and only if* $(a, b) \in \mathcal{D}^*(S)$.

Proof: (i) If $(\alpha_a, \alpha_b) \in \mathcal{H}^*(T)$, then obviously $(\alpha_a, \alpha_b) \in \mathcal{L}^*(T)$ and $(\alpha_a, \alpha_b) \in \mathcal{R}^*(T)$ and by Lemma 2.4 $(a, b) \in \mathcal{L}^* \cap \mathcal{R}^* = \mathcal{H}^*$. Conversely, if $(a, b) \in \mathcal{H}^*$, then $(\alpha_a, \alpha_b) \in \mathcal{H}^*(T)$ holds from Lemma 2.4.

(ii) For $(\alpha_a, \alpha_b) \in \mathcal{D}^*(T)$, there exist $\alpha_{x_1}\alpha_{x_2}, \dots, \alpha_{x_n} \in T$ such that

$$\alpha_a\mathcal{L}^*\alpha_{x_1}\mathcal{R}^*\alpha_{x_2}\mathcal{L}^* \dots \alpha_{x_n}\mathcal{R}^*\alpha_b.$$

But Lemma 2.4 implies that in S, $a\mathcal{L}^*x_1\mathcal{R}^*x_2\mathcal{L}^*, \dots, x_n\mathcal{R}b$ whence $(a, b) \in \mathcal{D}^*$. The converse can also be shown using Lemma 2.4. ∎

To identify idempotent elements in T, observe that if a in S is an idempotent then $a^+ = a^* = a$. If $x \in Se$, $xe = x$ so that $x\alpha_e = xe$, $\alpha_e = 1_{Se}$.

Lemma 2.6. *An element $\alpha_a \in T$ is an idempotent if and only if a in S is an idempotent. Moreover, $E(T)$ is a semilattice.*

Proof: If α_a is an idempotent then $\alpha_a^2 = \alpha_a$ implies $\operatorname{dom}\alpha_a^2 = (\operatorname{ran}\alpha_a \cap \operatorname{dom}\alpha_a)\alpha_a^{-1} = \operatorname{dom}\alpha_n$, that is, $\operatorname{ran}\alpha_a \cap \operatorname{dom}\alpha_a = \operatorname{ran}\alpha_a$ so that $\operatorname{ran}\alpha_a \subseteq \operatorname{dom}\alpha_a$. Also $\operatorname{ran}\alpha_a^2 = (\operatorname{ran}\alpha_a \cap \operatorname{dom}\alpha_a)\alpha_a = \operatorname{ran}\alpha_a$ hence $\operatorname{dom}\alpha_a, \subseteq \operatorname{ran}\alpha_a$. From both inclusions, $\operatorname{dom}\alpha_a = \operatorname{ran}\alpha_a$. Thus $Sa = Sa^+$ and for $x \in \operatorname{dom}\alpha_a$, $x\alpha_a^2 = x\alpha_a$, that is $xa^2 = xa$, so in particular, for $x = a^+$, $a^2 = a^+a^2 = a^+a = a$. Therefore a is an idempotent in S.

Conversely if a is an idempotent in S then $a^* = a^+$ so that $Sa^+ = Sa^*$ and quite clearly $\operatorname{dom}\alpha_a^2 = \operatorname{dom}\alpha_a = \operatorname{ran}\alpha_a = \operatorname{ran}\alpha_a^2$ and for all $x \in Sa^+$, $xa^2 = xa$. Hence for all $x \in Sa^+$, $x\alpha_a^2 = x\alpha_a$, so $\alpha_a^2 = \alpha_a$.

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Let $\alpha_e, \alpha_f \in E(T)$, the set of idempotents of T. Now $\alpha_e\alpha_f = \alpha_{ef} = \alpha_{fe} = \alpha_f\alpha_e$ and if $e \leqslant f$, $ef = fe = e$, so $\alpha_e\alpha_f = \alpha_f\alpha_e = \alpha_e$. This completes the proof of the lemma. ∎

For $a \in S, a^* \in L_a^*$, $a^+ \in R_a^*$ and $\alpha_a\alpha_{a^*} = \alpha_{aa^*} = \alpha_a$ and $\alpha_{a^+}\alpha_a = \alpha_{a^+a} = \alpha_a$. Evidently $(\alpha_a, \alpha_{a^*}) \in \mathcal{L}^*(T)$ by Lemma 2.4, so we have

LEMMA 2.7. *For each* $\alpha_a \in T$

(i) $(\alpha_a, \alpha_{a^*}) \in \mathcal{L}^*(T)$ *and*

(ii) $(\alpha_a, \alpha_{a^+}) \in \mathcal{R}^*(T)$.

Let $L_{\alpha_a}^*$ and $R_{\alpha_a}^*$ be the $\mathcal{L}^*(T)$ and $\mathcal{R}^*(T)$ classes containing α_a. Let us denote by α_a^* and α_a^+ the unique idempotents in $L_{\alpha_a}^*$ and $R_{\alpha_a}^*$ respectively. Now for $a \in S$, $e \in E(S)$, $ea = a(ea)^*$, $ae = (ae)^+a$, and consequently $\alpha_e\alpha_a = \alpha_{ea} = \alpha_{a(ea)^*} = \alpha_a\alpha_{(ea)^*} = \alpha_a\alpha_{ea}^* = \alpha_a(\alpha_e\alpha_a)^*$ and similarly $\alpha_a\alpha_e = (\alpha_a\alpha_e)^+\alpha_a$. Thus we have proved that

LEMMA 2.8. *For* $\alpha_a, \alpha_e \in T$,

(i) $\alpha_e\alpha_a = \alpha_a(\alpha_e\alpha_a)^*$ *and*

(ii) $\alpha_a\alpha_e = (\alpha_a\alpha_e)^+\alpha_a$.

These last observations together with Lemmas 2.4 to 2.7 complete the proof of Theorem 2.3.

Let $\beta_a \colon a^*S \to a^+S$, $a \in S$ where $x\beta_a = ax$ for $x \in S$; using methods similar to the above, β_a is a one-to-one mapping satisfying Lemmas 2.4 to 2.8 and

COROLLARY 2.9. $T^* = \{\beta_a \mid a \in S\}$ *is a type* A *semigroup.*

3. Representation of type A monoid

We show here that there is a Vagner-Preston type representation from a type A semigroup S into a type A semigroup of mappings on a set X. Let $X = S$, $a \in S$, and let $\varphi \colon S \to T$ be a mapping such that $a\varphi = \alpha_a$, where $T = \{\alpha_a \mid a \in S\}$ is the type A semigroup in Theorem 2.3 above.

THEOREM 3.1. *The mapping* $\varphi \colon S \to T$, *where* $a\varphi = \alpha_a$, *is an isomorphism from* S *onto* T.

PROOF: If $a, b \in S$, then $(ab)\varphi = \alpha_{ab} = \alpha_a\alpha_b = a\varphi . b\varphi$. Also $a\varphi = b\varphi$ implies $\alpha_a = \alpha_b$, which in turn implies that $Sa^+ = Sb^+$, $Sa = Sb$, the domains and ranges of α_a and α_b, respectively, and for all $x \in Sa^+$, $x\alpha_a = x\alpha_b$. Now $Sa^+ = Sb^+$ implies $a^+\mathcal{L}b^+$ and hence $a^+ = b^+$. Similarly $Sa^* = Sb^*$ implies $a^* = b^*$. But $x\alpha_a = x\alpha_b$ implies that $xa = xb$ for all $x \in Sa^+$; hence for $x = a^+$, $a = a^+a = a^+b = b^+b = b$. Thus if $\alpha_a = \alpha_b$ then $a = b$, showing that φ is a one-to-one homomorphism. By definition of T, φ is onto, so the proof is complete. ∎

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From Corollary 2.9, $T' = \{\beta_a \mid a \in S\}$ in type A semigroup and so

COROLLARY 3.2. *Let $\psi\colon S \to T'$ be a mapping given by $a\psi = \beta_a$, for $a \in S$. Then ψ is an isomorphism.*

PROOF: As in Theorem 3.1 above, $(ab)\psi = \beta_{ab} = \beta_a\beta_b = (a\psi)(b\psi)$, so ψ is a one-to-one homomorphism from S onto T'. This completes the proof. ∎

Let S be a left type A monoid and $T = \{\alpha_a \mid a \in S_\gamma \alpha_a\colon Sa^+ \to Sa^*\}$ where $\alpha_a^{-1} \in T$ if and only if $\alpha_a^{-1} = \alpha_{a^{-1}}$, that is, if and only if a is regular.

THEOREM 3.3. *T is a left adequate semigroup.*

PROOF: Consider $\alpha_a\colon Sa^+ \to Sa^*$, $\alpha_b\colon Sb^+ \to Sb^*$ as defined earlier, where $a, b \in S$ are non-regular. Now $\operatorname{ran}\alpha_a = Sa \neq Sa^*$ and $\operatorname{ran}\alpha_a \cap \operatorname{dom}\alpha_b = Sa \cap Sb^+ = Sab^+ = S(ab)^+a = (\operatorname{dom}\alpha_{ab})\alpha_a$. Also $(\operatorname{ran}\alpha_a \cap \operatorname{dom}\alpha_b)\alpha_b = Sab^+b = Sab \subseteq S(ab)^*$, so that $(\operatorname{ran}\alpha_a \cap \operatorname{dom}\alpha_b)\alpha_b = \operatorname{ran}\alpha_{ab}$. Since $(\operatorname{dom}\alpha_{ab})\alpha_a\alpha_b = \operatorname{ran}\alpha_{ab}$, by the previous lemma, T is a semigroup.

The proof of the theorem is complete by noting that the relevant aspects of Lemmas 2.4 – 2.7 above hold for T as well. ∎

In fact T is a left type A semigroup since for α_a, $\alpha_e \in T$, $\alpha_a\alpha_e = (\alpha_a\alpha_e)^+\alpha_a$, which is true by Lemma 2.8 since as S is a left type A monoid, for a in S, $e \in E(S)$, $ae = (ae)^+a$.

Since $ea \neq a(ea)^*$ does not hold in general for a left type A semigroup S with $a \in S$, and e an idempotent, in general the equality $\alpha_e\alpha_a = \alpha_a(\alpha_e\alpha_a)^*$ does not hold. However, we show below an example in which S is left type A and T a type A monoid.

EXAMPLE: Consider the semigroup S with the following multiplication table:

$\cdot$	e	f	z	a	c
e	e	z	z	c	c
f	z	f	z	z	z
z	z	z	z	z	z
a	z	a	z	z	z
S	z	a	z	z	z

The $\mathcal{L}^*$ classes of S are $\{f, a, c\}$, $\{z\}$, $\{e\}$ and the $\mathcal{R}^*$ classes are $\{e, a, c\}$, $\{f\}$, $\{z\}$. It is easy to check that for each idempotent $u \in E(S)$ and each $x \in S$, $xu = (xu)^+x$, and that $c = ea \neq a(ea)^* = af = a$, hence S is left type A but not a right type A monoid.

Now define $\alpha_a\colon Sa^+ \to Sa^*$ as usual. So $T = \{\alpha_e, \alpha_f, \alpha_z, \alpha_a, \alpha_c\}$, with $\mathcal{L}^*$-classes $\{\alpha_a, \alpha_c, \alpha_f\}$, $\{\alpha_e\}$, $\{\alpha_z\}$ and $\mathcal{R}^*$-classes: $\{\alpha_a, \alpha_c, \alpha_e\}$, $\{\alpha_f\}$ and $\{\alpha_z\}$. It is straightforward to verify that α_e, α_f are the only elements with $\alpha_e^{-1} = \alpha_{e^{-1}} = \alpha_e$, $\alpha_f^{-1} = \alpha_{f^{-1}} = \alpha_f$ so $\alpha_{e^{-1}}$, $\alpha_{f^{-1}} \in T$. Now for all $u \in \{e, f\}$, $x \in S$,

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$\alpha_z\alpha_u = (\alpha_z\alpha_u)^+\alpha_z$ but while $ea \neq a(ea)^*$, we have $\alpha_e\alpha_a = \alpha_{ea} = \alpha_c$, and $\alpha_a\alpha_{(ea)^*} = \alpha_a(\alpha_e\alpha_a)^* = \alpha_a\alpha_c^* = \alpha_a\alpha_f = \alpha_a$ and for all $x \in S$, $x\alpha_c = x\alpha_a$, hence $\alpha_c = \alpha_a$, since $Sc^+ = Sa^+$, and $Sc = Sa$. One also finds that $\alpha_e\alpha_c = \alpha_c(\alpha_e\alpha_c)^*$, and in general $\alpha_u\alpha_z = \alpha_z(\alpha_u\alpha_z)^*$ so T is a type A, with $E = \{\alpha_e, \alpha_f, \alpha_z\}$ as a semilattice.

From all the forgoing we have for the left type A semigroup in the table:

THEOREM 3.4. *S is isomorphic to a left type A semigroup of one-to-one mappings on S.*

Let us consider an arbitrary left type A semigroup S and T, the semigroup of one-to-one mappings α_a, $a \in S$. The following result holds.

THEOREM 3.5. *Let S be a left type A semigroup; then T is a left type A semigroup. Moreover S is isomorphic to T.*

To see this clearly, consider an arbitrary left type A monoid S and $T = \{\alpha_a \mid a \in S, \alpha_a : Sa^+ \to Sa^*\}$ where $\alpha_a : Sa^+ \to Sa^*$ is defined by putting

$$x\alpha_a = xa, \text{ for every } x \text{ in } S,$$

and $\alpha_a^{-1} \in T$, $a \in S$ if and only if $\alpha_a^{-1} = \alpha_{a^{-1}}$. Then $Sa \cap Sb^+ = Sab^+$, for $a \in S$, $b^+ \in E(S)$, and if $\alpha_a : Sa^+ \to Sa^*$, $\alpha_b : Sb^+ \to Sb^*$ and $a, b \in S$ have no inverses in S, $\operatorname{ran}\alpha_a = Sa \neq Sa^*$, $\operatorname{ran}\alpha_b = Sb \neq Sb^*$. Also $\operatorname{dom}\alpha_a\alpha_b = S(ab)^+ = \operatorname{dom}\alpha_{ab}$ and $\operatorname{ran}\alpha_a\alpha_b = Sab = \operatorname{ran}\alpha_{ab}$ and T is a semigroup.

$Sa = Sa^*$ if and only if S is regular and in such cases α_a is bijective and $\alpha_a^{-1} = \alpha_{a^{-1}}$.

That T is a left type A semigroup is shown in Theorem 3.3 together with Lemmas 2.4 – 2.6 and the following lemmas.

LEMMA 3.6. $(\alpha_a, \alpha_{a^+}) \in \mathcal{R}^*(T)$ *for all* $a \in S$, $a^+ \in E(S)$.

LEMMA 3.7. $\alpha_a\alpha_e = (\alpha_a\alpha_e)^+\alpha_a$ *for all* $a \in S$, $e \in E(S)$.

PROOF: $\alpha_a\alpha_e = \alpha_{ae} = \alpha_{(ae)^+a} = \alpha_{(ae)^+}\alpha_a = \alpha_{ae}^+\alpha_a = (\alpha_a\alpha_e)^+\alpha_a$, since $ae = (ae)^+a$. ∎

The proof of Theorem 3.5 is complete by noting that if $\psi : S \to T$ is a mapping where ψ is defined by $a\psi = \alpha_a$ for $a \in S$, then for all a, b in S

$$(ab)\psi = (a\psi)(b\psi)$$

and ψ is one-to-one and onto.

If S is an adequate semigroup which is not type A, the above result may not hold. Now for $a, b \in S$ suppose that $z \in Sa \cap Sb^+$. Then $z = sa = tb^+$ for some $s, t \in S$

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and since $z = zb^+ = sab^+ \in Sab^+$ then $Sa \cap Sb^+ \subseteq Sab^+$. To understand the situation clearly, let $S = C \cup D \cup \{1\}$ where $C = \langle a \rangle$ is the free semigroup on a and $D = \langle b \rangle$ the free monoid generated by b, with multiplication in S defined by $a^m b^n = b^{m+n}$, $b^m a^m = a^{m+n}$, for $m > 0$, $n \geqslant 0$, $b^0 = c$, and 1 is the identity in S. The $\mathcal{L}^*$- and $\mathcal{R}^*$-classes of S are respectively $C \cup \{1\}$, D and $\{1\}$, $C \cup D$. For $a, b \in S$, $a^* = 1$, $a^+ = e$, $b^* = b^+ = e$, $Sa \cap Sb^+ = \emptyset$, $Sab^+ = D \setminus \{e\}$, so $Sa \cap Sb^+ \neq Sab^+$. Moreover, $\alpha_a \colon Sa^+ \to Sa^*$ is not one-to-one since for $x = a^t$, $y = b^t$, $x\alpha_a = y\alpha_a$ but $x \neq y$. □

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