WORKSHOP

CREDIT RISK AND PREPAYMENT OPTION[1]

BY PHILIPPE ARTZNER

Université Louis Pasteur

AND FREDDY DELBAEN

Vrije Universiteit Brussel

ABSTRACT

The paper examines a type of insurance contract for which secondary markets do exist: default risk insurance is implicit in corporate bonds and other risky debts. It applies risk neutral martingale measure pricing to evaluate the option for a borrower with default risk, to prepay a fixed rate loan. A simple "matchbox" example is presented with a spreadsheet treatment.

KEYWORDS

C-1 risk; C-3 risk; credit risk insurance; default risk; interest rate risk; financial reserve; insurance reserve; market values; martingale; optimal stopping; premium process; risk process; swap; swaption.

1. INTRODUCTION

This paper attempts to illustrate, in a discrete time setting with a spreadsheet construction, the use of a martingale measure for the pricing of traded securities subject to both interest rate risk and default risk. We shall encounter several notions pertaining to the field of insurance, and strong similarities with classical notions in this field. Notice nevertheless that, when "premiums" and "reserves" are mentioned, the reader is invited to think in terms of prices rather than present values.

We shall emphasize the distinction between *defining* or describing securities and *evaluating* them. For this second task, we shall *start* from the *risk-neutral* probability of the financial economics litterature (see HARRISON and KREPS (1979), DUFFIE (1988) and HUANG and LITZENBERGER (1988)). This probability, implied by *market* considerations such as the absence of arbitrage opportunities between marketed securities, reflects the market's attitude towards risk and, in the simple model presented, is *supposed* to be known: its determination is left for other research work.

[1] An earlier version of this work has been presented at the ASTIN Colloquium, Montreux 1990, and at the Erasmus University Conference on Insurance, Solvency and Finance, Rotterdam 1991, under the title "Credit Insurance with Prepayment Option".

Fixed rate default-free loans introduce the notion of financial reserve (Section 2.3), while risky loans introduce both an insurance process and a insurance premium payments proces, with the difference of the prices of these two securities being the proper concept of insurance reserve (Section 3.4). Prepaying a risky loan is a call on the sum of the negative of the two reserves (Section 4.2).

Notation: $1\{A\}$ denotes the indicator function of the event A, F_t (or F_t in the spreadsheet) denotes the set of events known to occur or not by date t. The three types of numbers: 1, **7** and *9* refer to columns in respectively each of the three parts "Quantities independent of default risk" "**Effect of default risk**" and "*Value of prepayment option*" of the spreadsheet presentation.

2. THE SHORT TERM INTEREST RATE PROCESS

2.1. The discount factor process

There are three trading dates $t = 0, 1, 2$, for borrowing or lending money on a default-free or default-prone basis. In columns 1, 11, and 25 of the first spreadsheet: Quantities independent of default risk, is given the *short term spot rate process* $r(t)$, $t = 0, 1, 2$, where $r(t)$ is the market rate of interest for borrowing or lending from the date t to te next trading date $t+1$ on a default-free basis. The conditional one-period risk-neutral probabilities are all supposed to be 1/2 (see also ARTZNER and DELBAEN (1990a), ARTZNER and SHIU (1989), HARRISON and KREPS (1979) and HUANG and LITZENBERGER (1988)).

The *(generalized) discout factor process* D is defined in a recurrent way by the equalities:

$$D(0) = 1 \qquad \text{and, for } t = 0, 1, 2: \qquad D(t+1) = (1+r(t))^{-1} \cdot D(t).$$

It is therefore a *predictable* process (see ARTZNER and DELBAEN (1990a), DACUNHA-CASTELLE and DUFLO (1986)), that is the actual value of $D(t+1)$ is part of the information available at date t, which shows up in the setting of $D(1)$, $D(2)$ and $D(3)$ in columns 2, 12 and 26 which, respectively, relate to dates $t = 0$, $t = 1$ and $t = 2$. One unit invested at date zero at the short term rate, and rolled over at each trading date, becomes at date t, $1/D(t)$ units, but it must be already pointed out that in this model of stochastic interest rates, capitalisation just described as rolling over, is *not* the inverse from actualisation, the second operation being defined tentatively, as "valuing at date s, one unit available at future date t" (see below). This could be a reason for the qualifying *generalized* in definition of D (we omit it in the spreadsheet).

2.2. Description and pricing of the zero-coupon bond price process

It is assumed that, for each "maturity date" t, the *zero-coupon bond* which provides, without risk of default, one unit at date t, is traded on the market. In the absence of arbitrage opportunities between these bonds and the capitalisa-

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tion process of Section 2.1, the zero-coupon bond maturing at date t, has a ("current price") price $B(s;t)$ at date $s \leq t$, in date s units, given by:

$$B(s;t) = E[D(t)/D(s)|F_s], \text{ or, for each } s' \text{ in } [s,t]$$
$$D(s)\cdot B(s;t) = E[D(s')\cdot B(s';t)|F_s],$$

that is, the (discounted) price is a martingale (see GERBER (1979), p. 34 and the conclusion of TILLEY (1989)) *with respect to* the information structure *and* the assumed risk neutral probability measure (see ARTZNER and DELBAEN (1990b) 3.2, ARTZNER and SHIU (1989), DACUNHA-CASTELLE and DUFLO (1986), 2.2.1, HARRISON and KREPS (1979) and HUANG and LITZENBERGER (1988), 8.4.8); this fact appears in the columns 14, 2 and 3 for, respectively, $[s = 1, s' = 2, t = 3]$ and $[s = 0, s' = 1, t = 2$ and $t = 3]$. The name "martingale measure" is often given to the risk-neutral measure being used, because of this property of discounted prices under it. Notice that in columns 13 and 27, the bond prices $B(1;2)$ and $B(2;3)$ are simply the reciprocals of $1+r(1)$ and $1+r(2)$, respectively.

The price $B(0;t)$ *could* also be considered a candidate for being the "discount factor" from date t tot date 0, and the choice of terminology is an open question.

2.3. Financial reserve in a default free, fixed rate, loan

The general principle of mathematically describing a security by what *has been* collected out of possession of the security (see ARTZNER and DELBAEN (1990b), 2.1, DUFFIE (1988), 16, p. 148, HUANG and LITZENBERGER (1988), 8.5, p. 229), leads to consider an *adapted* stochastic process S (see HUANG and LITZENBERGER (1988), 7.7, p. 189), where $S(t)$ stands for the sum of all payments, discounted back to date 0, received from date 0 to date t included, out of possession of the security.

This mathematical description may be contrasted with a *legal* description as in SHARPE and ALEXANDER (1990), p. 3, where a security is defined as "*the legal representation of the right to receive prospective* future *benefits under stated conditions*". It is *very* important to *carefully* distinguish between the security described by the process X, and the pricing process PX, of the security.

The mathematical approach applies of course to the capitalisation process of Section 2.1 as well as to zero-coupon bonds (see ARTZNER and DELBAEN (1990a)), where it can be considered as a good modelisation exercise. We shall start applying it to simple, default-free loans.

A default-free loan of one unit, from date 0 to date 3, which is being paid for by payment of fixed interest β at dates 1, 2 and 3, together with repayment of principal at date 3, is essentially a *default-free bond* with maturity date 3; it is the building block of more usual loans which contain an amortization component. For simplicity we consider only the first type of loan, the exchange of the two securities:

1 paid at date 0, and

β, β and $1+\beta$ paid at dates 1, 2 and 3 respectively.

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The equality of the market prices, *at date* 0 of these securities, exchanged at date 0, by the lender and the borrower respectively, is a natural condition in a frictionless market. It requires that:

$$1 = \beta \cdot B(0;1) + \beta B(0;2) + (1+\beta)\, B(0;3), \text{ or, by Section 2.2. with } s = 0:$$

$$1 = E[\beta \cdot (D(1) + D(2) + D(3)) + D(3)],$$

which determines the market rate β for *this* loan (see rows 23 to 25, columns 1 to 4).

At a later date s, and depending on the evolution of the short term rate of interest, the two securities may have different values: the *financial reserve process FR* is the difference between the *market value* of the commitments (liabilities) of the lender which is 1 in current units (since the lender can recover the money lent at this current price on the market (see also ARTZNER and DELBAEN (1990a), 2.2. and ARTZNER and DELBAEN (1990b), 3.3 and Section 4.1 below), and the *market value* of the commitments of the borrower. This reserve appears in columns 20 and 33.

3. A RISKY, FIXED RATE, LOAN

3.1. Description of the default by a stopping time

An agent with default risk borrows one unit from date 0 to date 3, and may stop fulfilling his obligations from some random date ∂ onwards. The technical requirement on ∂ is to be a *stopping time* (see ARTZNER and DELBAEN (1990a), 3.1, ARTZNER and DELBAEN (1990b), 2.4, DACUNHA-CASTELLE and DUFLO (1986), 2.3.1), meaning that, by date t, the event $\{\partial = t\}$ is known to be true or not. For the interpretation, we notice that, in the worked example, the random variable ∂ is dependent on the spot interest rate process r, but, for ease of the spreadsheet use, not in the utmost generality: if default has not occured at date t, the (risk-neutral) conditional probability that it occurs at date $t+1$ depends on the current value $r(t)$ but not on $r(t+1)$: see columns **7**, **17** and **30** of the second spreadsheet: **Effect of default risk**. It is also important to realize that the model does not make ∂ a decision variable, and does not cover moral hazard phenomena.

As in Section 2.3, the type of loan just considered is the building block for the study of risky loans with an amortization component. The type we study is in fact a risky bond which will have level interest rate payments that we analyse as the sum of the level interest rate β determined in Section 2.3 and a level default risk insurance premium π to be determined.

3.2. Description and pricing of the loan's insurance viewed as a security

The mathematical approach to defining a security is applied to the one which is *implicit* in the risky loan, namely the right granted to the borrower at date of default: to be dispensed of due interest and principal payments (in this simple model the lender has no recourse). This *defines* a security I, called the (cumulative, discounted) *insurance process* or *risk process*:

$I(0) = 0$ (see column **5**) and, for $t = 0, 1, 2$ (see columns **15**, **28**, **38**):

$$I(t+1) = I(t)+1\{\partial = t+1\}\cdot E[(\beta\cdot(D(t+1)+\ \dots\ +D(3))+D(3))|\boldsymbol{F}_t].$$

Notice that we had to *price* the loan in order to *define* I. Risky bonds, as well as risky loans in the case of credit securitization, being traded on the market, we want to apply to the (discounted) *price process PI* of the implicit security I, that is to the *insurance prices process*, the general rule of financial economics: "discounted dividends so far collected plus discounted current price form a martingale" (see ARTZNER and DELBAEN (1990a), 2.1 and 3, ARTZNER and SHIU (1989), HARRISON and KREPS (1979), HUANG and LITZENBERGER (1988) 8.5 and 8.6). This provides the relations in columns **31**, **18** and **8** (in this order), out of the following equality, used for $t = 2$, 1 and 0 successively:

$$PI(t)+I(t) = E[PI(t+1)+I(t+1)|\boldsymbol{F}_t].$$

3.3. Description and pricing of the insurance premium payments

The level "insurance premium" π asked to the default prone borrower, is of the "post-numerando" type, since it is paid together with the interest β due for the past time period. This is a difference with, for example, classical life insurance, and accounts for the non predictability (see ARTZNER and DELBAEN (1990a), 3.1, as well as ARTZNER and DELBAEN (1990b), 4.3) of the *premium payments process* Π, another security, described in columns **6**, **16**, **29** and **39** by the relations:

$$\Pi(0) = 0 \quad \text{and} \quad \text{for } t = 0, 1 \text{ and } 2:$$
$$\Pi(t+1) = \Pi(t)+\pi\cdot D(t+1)\cdot 1\{\partial > t+1\}.$$

The level premium π has yet to be defined: this has to do with the price of the security Π. Since, at date 0, the security Π is given by the borrower, who will pay the premiums, in exchange for the security I, given by the lender, who provides the insurance coverage, a frictionless market will require the equality of the prices at date 0, of Π and I, the first price being obtained as the expectation (at date 0) of all "future benefits" π at date 1, π at date 2 and π at date 3, under "stated conditions" namely these dates being smaller than default date ∂. This gives the relation:

$$PI(0) = \pi\cdot E[D(1)\cdot 1\{\partial > 1\}+D(2)\cdot 1\{\partial > 2\}+D(3)\cdot 1\{\partial > 3\}],$$

as it appears in rows **31** and **32**, columns **31** to **39**. This reminds us of the "equivalence principle" of classical life insurance, but we have to notice the randomness of discounting factors and the risk-neutral character of the probability measure used.

3.4. The insurance reserve process

After the initial date 0, there is no further reason for the difference of the *price processes* of the insurance process I and the premium payments process Π, to

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be zero. This difference is called the *insurance reserve process* and is denoted V (see columns **19** and **32**): from its very definition we see that, as in classical actuarial mathematics (BOWERS et al. (1986), Ch. 7) it has to do with conditional expectations of the difference of the cumulative, discounted, commitments of the "insurer" (the lender, who has no recourse in case of default) and of the "insured" (the borrower, who has no more debt at date ∂). Notice two differences:

(i) in order to define the reserve as a stochastic process, a function of *all* dates and states of nature, we do not restrict ourselves to the stochastic time interval $[[0, \partial[[$, as actuarial mathematics does (see BOWERS et al. (1986), p. 192: "*for an insured* surviving *at the end of* t *years*");

(ii) we use a risk-neutral probability, in order to consider *market* values: reserves have to do with valuation.

4. THE PREPAYMENT OPTION

4.1. Prepayment of a default-free, fixed rate loan, as swap from fixed to variable rate

We shall examine in this Section the value of the option which a fixed rate default-free borrower may ask to the lender: prepaying the loan. We could have included this value into the net level payments due by the borrower but chose to separate it, for expository purpose.

If a default-free borrower prepays his loan at date t by paying the fair interest β plus the principal, one current unit, he is in fact exchanging the following securities:

commitment of fixed rate payments of β at date $t+1, \ldots, 1+\beta$ at date 3
commitment of variable rate payments of $r(t)$ at date $t+1, \ldots, 1+r(2)$ at date 3.

The second security is indeed worth $\Delta(t)$ where Δ fulfills the following equalities:

$$\Delta(t) = r(t) \cdot D(t+1) + E[\Delta(t+1) | F_t] \quad \text{for} \quad t+1 < 3,$$
$$\Delta(t) = (1+r(t)) \cdot D(t+1) \quad \text{for} \quad t+1 = 3, \quad \text{and} \quad \Delta(3) = 0,$$

which allows us to conclude, by backwards induction, that $\Delta(t) = D(t)$ for $t < 3$ (see also ARTZNER and DELBAEN (1990a), 2.2 and ARTZNER and DELBAEN (1990b), 3.3). The prepayment of the principal by the borrower at date t is therefore equal to the current value, namely 1, of the second security.

This exchange is called an interest rate *swap* (ARTZNER and DELBAEN (1990a), ARTZNER and DELBAEN (1990b), 2.3, DUFFIE (1989), p. 269, TURNBULL (1987)). The financial reserve at date t, as defined in 2.3, is precisely the negative of the market value (in date 0 units) of this swap at date t. When the financial reserve is negative, it is in the borrower's interest to prepay.

4.2. Prepayment of a risky, fixed rate loan and the two reserves

The case of prepayment at date t by a default-prone borrower is more complex: it involves paying at date t the amount $\beta+\pi+1$ (i.e. interest, insurance premium and principal) and, from then on, stopping any interest *and* insurance premium payment. This implies that, next to performing the swap transaction described in Section 4.1, the prepayer also engages in the following:

> receiving from date t onwards the security Π^t consisting of the level premium payments π at the various dates $t+1, \ldots \partial-1$

and

> giving up from date t onwards the security I^t, the coverage $I(\partial)$ at date ∂.

This second transaction has the (discounted) market value given by a price difference, namely $P\Pi^t(t)-PI^t(t)$ which is equal to $P\Pi(t)-PI(t)$, that is to the negative of the insurance reserve.

A borrower prepaying at date t receives therefore the negative of the *sum* of the financial and insurance reserves, that is the quantity $-FR(t)-V(t)$.

4.3. American prepayment option as an optimal stopping problem

A rational borrower, allowed to prepay at some *fixed* date t, $t = 1$ or $t = 2$, would do so only if, at *this* date, the quantity $-FR(t)-V(t)$ is positive, receiving the (discounted) "exercise gain"

$$G(t) = 1\{\partial > t\}\cdot \max\{0, -FR(t)-V(t)\},$$

(see columns *21* and *34* of the third spreadsheet: *Value of prepayment option*). Notice that we do not speak of "prepayment risk" in this case of rational exercise; see ARTZNER (1990) for other cases.

We now *define* a new security S_t by $S_t(s) = 0$ if $s < t$, $S_t(s) = G(t)$ if $s \geq t$, for which the (ex dividend) price process would be $PS_t(s) = E[S_t(t)|\boldsymbol{F}_s]$ if $s < t$ and 0 otherwise.

If this borrower can prepay at *any* one of the two dates, he faces the problem of choosing the stopping time τ maximizing the expectation of $G(\tau)$: this is an optimal stopping problem arising from the *American* type of the swap option he has been granted (see ARTZNER and DELBAEN (1990a), 2.3, DACUNHA-CASTELLE and DUFLO (1986), 5.1.3).

The solution is described by computing the value $PP(t)$ of the prepayment option at date t, columns *9*, *22* and *34*, and by the condition of equality between exercise gain and option value (no "time value") for rational exercice, columns *23* and *35*.

5. CONCLUSION

Stochastic processes are necessary to *describe* complex contracts, in particular when payments involved have several sources of randomness, as for example in credit risk insurance.

For the *pricing* or *evaluation* of such contracts, a risk-neutral probability is the tool allowing averaging discounted payments to be consistent with prices of related marketed contracts. Research has to done for specifying such a probability out of some observed market prices.

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PHILIPPE ARTZNER
Institut de Recherches Mathématiques Avancées et Laboratoire de Recherches en Gestion, 7 rue René Descartes, F 67084 Strasbourg Cedex, France, A18604@FRCCSC21.

FREDDY DELBAEN
Department of Mathematics and Information Sciences, V.U.B., Institute of Actuarial Sciences, Pleinlaan 2, B 1050 Brussels, Belgium, fdelbaen @tena2.vub.ac.be.

Artzner/Delbaen, Credit Risk

Quantities independent of default risk

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	DATE	Discount	Bond	Bond							DATE	Discount	Current	Current						Financial
2	t = 0	factor	price	price							t = 1	factor	bond	bond						reserve
3	short rate	D(1) =	B(0;2) =	B(0;3) =							short rate	D(2) =	price	price						process
4	r(0)	1/(1+r(0))	D(1) * E [B(1;2)]	D(1) * E [B(1;3)]							r(1)	D(1) /	B(1;2) =	B(1;3) =						FR(1) =
5	(*)										(*)	(1+r(1))	1 / (1+r(1))	(D(2)/D(1))						D(1)
6														* E [B(2;3)						- ß * D(2)
7														/ F 1]						- (1+ß) *
8																				B(1;3) *
9																				D(1)
10																				
11																				
12																				
13											7.00%	0.890076	0.934579	0.862943						0.032834
14																				
15																				
16																				
17																				
18																				
19	5.00%	0.952381	0.9007	0.8460																
20																				
21																				
22																				
23	Fixed rate ß	for default																		
24	free loan	= (1 - E [D(	3)]) / E [D(1) +	D(2) + D(3)]																
25			ß =	5.7068%							4.50%	0.9114	0.9569	0.9136						-0.019370
26																				
27																				
28																				
29																				
30																				
31											E [D(2)] =	0.9007								
32																				

(*) all one-period conditional risk neutral probabilites are 0.5

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Artzner/Delbaen, Credit Risk

Quantities independent of default risk

	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
1					DATE	Discount	Current						Financial				DATE		
2					t = 2	factor	bond price						reserve				t = 3		
3					short rate	D(3) =	B(2;3) =						process						
4					r(2)	D(2) / (1+r(2))	1/(1+r(2))						FR(2) =						
5													D(2) - (1+ß)*						
6													D(3)						
7																			
8																			
9																			
10					11.20%	0.8004	0.8993						0.043969						
11																			
12																			
13																			
14																			
15																			
16					5.55%	0.8433	0.9474						-0.001322						
17																			
18																			
19																			
20																			
21																			
22					5.50%	0.8639	0.9479						-0.001787						
23																			
24																			
25																			
26																			
27																			
28					4.00%	0.8763	0.9615						-0.014957						
29																			
30																			
31					E [D(3)] =	0.8460													
32																			

(*) all one-period conditional risk neutral probabilites are 0.5

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Artzner/Delbaen, Credit Risk

Effect of default risk

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	DATE	Discount	Bond	Bond	Cumulative	Cumulative	Risk	Insurance			DATE	Discount	Current	Current
2	t = 0	factor	price	price	discounted	discounted	neutral	price			t = 1	factor	bond	bond
3	short rate	D(1) =	B(0;2) =	B(0;3) =	insurance	premium	probability	process			short rate	D(2) =	price	price
4	r(0)	1/(1+r(0))	D(1) *	D(1) *	process	process	of default	PI(0) =			r(1)	D(1) /	B(1;2) =	B(1;3) =
5	(*)		E [B(1;2)]	E[B(1;3)]	I(0)	Π(0)	at the	E[I(1) + PI(1)]			(*)	(1+r(1))	1 / (1+r(1))	(D(2)/D(1)) *
6							next date							E [B(2;3) / F 1]
7							(**)							
8														
9														
10														
11														
12														
13											7.00%	0.890076	0.934579	0.862943
14											7.00%	and	payments	default (**) :
15														
16														
17														
18														
19	5.00%	0.952381	0.9007	0.8460	0	0	0.165	0.263012						
20														
21														
22														
23														
24	Fixed rate ß	for default free	loan = (1 - E [	D(3)]) /	E [D(1) +	D(2) + D(3)	]							
25			ß =	5.7068%							4.50%	0.9114	0.9569	0.9136
26											4.50%	and	payments	default (**) :
27														
28														
29														
30														
31											E [D(2)] =	0.9007		
32														

(*) all one-period conditional risk neutral probabilites are 0.5 (**) default at the end of the period independent of following spot short rates

Artzner/Delbaen, Credit Risk

Effect of default risk

	15	16	17	18	19	20	21	22	23	24	25	26	27
1	Cumulative	Cumulative	**Conditional**	Insurance	Insurance	Financial					**DATE**	Discount	Current
2	discounted	discounted	**risk**	price	reserve	reserve					**t = 2**	factor	bond price
3	insurance	premium	**neutral**	process	process	process					**short rate**	D(3) =	B(2;3) =
4	process	process	**probability**	PI(1) = E [	V(1) = PI(1)	FR(1) =					**r(2)**	D(2) / (1+r(2))	1/(1+r(2))
5	I(1) = I(0) +	∏(1) = ∏(0)	**of default**	I(2)-I(1)	- E [∏(3)	D(1)					**(*)**		
6	1{∂=1} *	+ π * D(1) *	**at the**	+ PI(2)	- ∏(1)	- ß * D(2)							
7	(ß * D(1)	1{∂>1}	**next date**	/ F 1]	/ F 1]	- (1+ß) *							
8	+ß * D(2)		**(**)**			B(1;3) *							
9	+ (1+ß)*					D(1)							
10	* B(1;3) *										**11.20%**	0.8004	0.8993
11	D(1))										**11.20%**	and payments	default (**) :
12													
13	0	0.119438	**0.120**	0.148372	-0.036046	0.032834							
14	0.973898	0		0	0								
15													
16											**5.55%**	0.8433	0.9474
17											**5.55%**	and payments	default (**) :
18													
19													
20													
21													
22											**5.50%**	0.8639	0.9479
23											**5.50%**	and payments	default (**) :
24													
25	0	0.119438	**0.060**	0.086388	-0.120288	-0.019370							
26	1.026102	0		0	0								
27													
28											**4.00%**	0.8763	0.9615
29											**4.00%**	and payments	default (**) :
30													
31											E [D(3)] =	0.8460	
32													

(*) all one-period conditional risk neutral probabilites are 0.5 (**) default at the end of the period independent of following spot short rates

Artzner/Delbaen, Credit Risk

Effect of default risk

	28	29	30	31	32	33	34	35	36	37	38	39
1	Cumulative	Cumulative	**Conditional**	Insurance	Insurance	Financial				DATE	Cumulative	Cumulative
2	discounted	discounted	**risk**	price	reserve	reserve				t = 3	discounted	discounted
3	insurance	premium	**neutral**	process	process	process					insurance process	premium
4	process	process	**probability**	PI(2) =	V(2) =	FR(2) =					I(3) = I(2) + 1{∂=3} *	process
5	I(2) = I(1) +	∏(2) = ∏(1)	**of default**	E [I(3) - I(2)	PI(2) - E [	D(2) - (1+ß)*					(1+ß)*D(3)	∏(3) = ∏(2)
6	1{∂=2} *	+ π * D(2)	**at the**	/ F 2]	∏(3) - ∏(2)	D(3)						+
7	(ß * D(2) +	* 1{∂>2}	**next date**		/ F 2]							π * D(3) * 1{∂>3}
8	(1 + ß) * D(3))		**(**)**									
9												
10	0	0.231062	**0.060**	0.050766	-0.043592	0.043969				no default :	0	0.331443
11	0.896902	0.119438		0	0					default :	0.846107	0.231062
12												
13												
14												
15												
16	0	0.231062	**0.040**	0.035656	-0.065869	-0.001322				no default :	0	0.336817
17	0.942193	0.119438		0	0					default :	0.891398	0.231062
18												
19												
20												
21												
22	0	0.233732	**0.035**	0.031960	-0.072584	-0.001787				no default :	0	0.342068
23	0.965166	0.119438		0	0					default :	0.913156	0.233732
24												
25												
26												
27												
28	0	0.233732	**0.030**	0.027790	-0.078812	-0.014957				no default :	0	0.343631
29	0.978337	0.119438		0	0					default :	0.926326	0.233732
30												
31				level	default risk	premium	:	π	=	PI(0) / E [	1{∂>1} * D(1) + 1{∂>2} * D(2) +	1{∂>3} * D(3)]
32										π =	0.125410	

(*) all one-period conditional risk neutral probabilites are 0.5 (**) default at the end of the period independent of following spot short rates

Artzner/Delbaen, Credit Risk

Value of prepayment option

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	DATE	Discount	Bond	Bond	Cumulative	Cumulative	Risk	Insurance	Value		DATE	Discount	Current	Current
2	t = 0	factor	price	price	discounted	discounted	neutral	price	of the		t = 1	factor	bond	bond
3	short	D(1) =	B(0;2) =	B(0;3) =	insurance	premium	probability	process	prepayment		short	D(2) =	price	price
4	rate	1/(1+r(0))	D(1) *	D(1) *	process	process	of default	PI(0) =	option		rate	D(1) /	B(1;2) =	B(1;3) =
5	r(0)		E [B(1;2)]	E [B(1;3)]	I(0)	Π(0)	at the	E [I(1) + PI(1)]	PP(0) =		r(1)	(1+r(1))	1 / (1+r(1))	(D(2)/D(1)) *
6	(*)						next date		E [PP(1)]		(*)			E [B(2;3) / F 1]
7							(**)							
8														
9														
10														
11														
12														
13											7.00%	0.890076	0.934579	0.862943
14											7.00%	and	payments	default (**) :
15														
16														
17														
18														
19	5.00%	0.952381	0.900722	0.845969	0.000000	0	0.165	0.263012	0.070650					
20														
21														
22														
23														
24	Fixed rate ß	for default	free	loan	= (1 - E [	D(3)]) /	E [D(1) +	D(2) + D(3)	]					
25			ß =	5.7068%							4.50%	0.911369	0.956938	0.913591
26											4.50%	and	payments	default (**) :
27														
28														
29														
30														
31											E [D(2)] =	0.9007		
32														

(*) all one-period conditional risk neutral probabilites are 0.5 (**) default at the end of the period independent of following spot short rates

Artzner/Delbaen, Credit Risk

Value of prepayment option

	15	16	17	18	19	20	21	22	23	24	25	26	27
1	Cumulative	Cumulative	**Conditional**	Insurance	Insurance	Financial	Exercise	Value of	Should		**DATE**	Discount	Current
2	discounted	discounted	**risk**	price	reserve	reserve	gain	the	you		**t = 2**	factor	bond price
3	insurance	premium	**neutral**	process	process	process	G(1) =	prepayment	prepay		**short**	D(3) =	B(2;3) =
4	process	process	**probability**	PI(1) = E [	V(1) = PI(1)	FR(1) =	1{∂>1} *	option	the		**rate**	D(2) / (1+r(2))	1/(1+r(2))
5	I(1) = I(0) +	Π(1) = Π(0)	**of default**	I(2)-I(1)	- E [Π(3)	D(1)	Max	PP(1) =	loan?		**r(2)**		
6	1{∂=1} *	+ π * D(1) *	**at the**	+ PI(2)	- Π(1)	- β * D(2)	{ 0,	Max{ G(1),			**(*)**		
7	(β * D(1)	1{∂>1}	**next date**	/ F 1]	/ F 1]	- (1+β) *	- V(1)	E [PP(2) /					
8	+β * D(2)		**(**)**			B(1;3) *	-FR(1)}	F 1] }					
9	+ (1+β)*					D(1)							
10	B(1;3) *										**11.20%**	0.800428	0.899281
11	D(1)										**11.20%**	and payments	default (**) :
12													
13	0	0.1194	**0.120**	0.148372	-0.036046	0.032834	0.003213	0.029564	no				
14	0.973898	0.0000		0	0		0	0					
15													
16											**5.55%**	0.843274	0.947418
17											**5.55%**	and payments	default (**) :
18													
19													
20													
21													
22											**5.50%**	0.863857	0.947867
23											**5.50%**	and payments	default (**) :
24													
25	0	0.119438	**0.060**	0.086388	-0.120288	-0.019370	0.139658	0.139658	yes				
26	1.026102	0		0	0		0	0					
27													
28											**4.00%**	0.876317	0.961538
29											**4.00%**	and payments	default (**) :
30													
31											E [D(3)] =	0.845969	
32													

(*) all one-period conditional risk neutral probabilites are 0.5 (**) default at the end of the period independent of following spot short rates

Artzner/Delbaen, Credit Risk

Value of prepayment option

	28	29	30	31	32	33	34	35	36	37	38	39
1	Cumulative	Cumulative	**Conditional**	Insurance	Insurance	Financial	Value of the	Should		DATE	Cumulative	Cumulative
2	discounted	discounted	**risk-neutral**	price	reserve	reserve	prepayment	you		t = 3	discounted	discounted
3	insurance	premium	**probability**	process	process	process	option	prepay			insurance	premium
4	process	process	**of default**	PI(2) =	V(2) =	FR(2) =	PP(2) =	the			process	process
5	I(2) = I(1) +	Π(2) = Π(1)	**at the next**	E [I(3) - I(2)	PI(2) - E [	D(2) - (1+ß)*	G(2) =	loan?			I(3) = I(2) +	Π(3) = Π(2)
6	1{∂=2} *	+	**date**	/ F 2]	Π(3) - Π(2)	D(3)	1{∂>2} *				1{∂=3} *	+
7	(ß * D(2) +	π * D(2)	**(**)**		/ F 2]		Max{0; -V(2)				(1+ß) *	π * D(3)
8	(1 + ß) * D(3))	* 1{∂>2}					- FR(2) }				D(3)	* 1{∂>3}
9												
10	0	0.231062	**0.060**	0.050766	-0.043592	0.043969	0.000000	no		no default :	0	0.331443
11	0.896902	0.119438		0	0		0			default :	0.846107	0.231062
12												
13												
14												
15												
16	0	0.231062	**0.040**	0.035656	-0.065869	-0.001322	0.067191	yes		no default :	0	0.336817
17	0.942193	0.119438		0	0		0			default :	0.891398	0.231062
18												
19												
20												
21												
22	0	0.233732	**0.035**	0.031960	-0.072584	-0.001787	0.074370	yes		no default :	0	0.342068
23	0.965166	0.119438		0	0		0			default :	0.913156	0.233732
24												
25												
26												
27												
28	0	0.233732	**0.030**	0.027790	-0.078812	-0.014957	0.093769	yes		no default :	0	0.343631
29	0.978337	0.119438		0	0		0			default :	0.926326	0.233732
30												
31		level	default risk	premium	π = PI(0) /	E [1{∂>1} *	D(1) + 1{∂>2}	* D(2)	+	1{∂>3} *	D(3)]	
32										π =	0.125410	

(*) all one-period conditional risk neutral probabilites are 0.5 (**) default at the end of the period independent of following spot short rates

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