

LINEAR GEOMETRIES ON THE MOEBIUS STRIP:  
A THEOREM OF SKORNYAKOV TYPE

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We show that the continuity properties of a stable plane are automatically satisfied if we have a linear space with point set a Moebius strip, provided that the lines are closed subsets homeomorphic to the real line or to the circle. In other words, existence of a unique line joining two distinct points implies continuity of join and intersection. For linear spaces with an open disk as point set, the same result was proved by Skornyakov.

INTRODUCTION

A *flat stable plane*  $(E, \mathcal{L})$  consists of a *point space*  $E$ , which is a surface (topological 2-manifold), and a system  $\mathcal{L}$  of *lines*, which are closed subsets of  $E$ , such that any two points are joined by a unique line and that the operations of join and intersection are continuous. Moreover, it is required that intersection is *stable*, that is, the set of pairs of distinct intersecting lines is open. For a comprehensive introduction to general stable planes (not only flat ones), see [6].

The lines of a flat stable plane are 1-manifolds ([10]), and if they are connected, then the point surface  $E$  is either a topological open disk  $D$ , or the compact nonorientable surface  $P$  of genus 1 (the real projective plane), or an open (that is, boundaryless) Moebius strip  $M$ , see [8, 11]. The flat stable planes with  $E = P$  and lines homeomorphic to the circle are of particular importance and are called *flat projective planes*.

Conversely, if  $(E, \mathcal{L})$  is a linear space (that is, two points are joined by a unique line) and lines are connected 1-manifolds and closed subsets of  $E$ , then it has been proved by Skornyakov for  $E = D$  and by Salzmann for  $E = P$  (see [10, 2.5] and [12, 32.5]), that  $(E, \mathcal{L})$  is a flat stable plane, that is, that join and intersection are automatically continuous. Here we prove a similar result for  $E = M$ :

**THEOREM.** *Let  $M$  be an open Moebius strip and suppose we are given a system  $\mathcal{L}$  of closed subsets  $L \subseteq M$ , called lines, such that lines are connected 1-manifolds, and such that any two points are joined by a unique line.*

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Then one may define a topology on  $\mathcal{L}$  such that  $(M, \mathcal{L})$  becomes a flat stable plane, that is, the operations of join and intersection of point and line pairs, respectively, are continuous and intersection is stable.

Similar characterisations have been derived for circle planes on surfaces in which any three points in ‘general position’ are connected by a unique circle, see [4, 5, 13, 16].

Flat stable planes with point set  $M$  and connected 1-manifolds as lines have been investigated by Betten [2], who completely determined the most homogeneous ones among them. Examples of such flat stable planes arise from flat projective planes ( $E = P$ ) in two ways: either by deleting a single point from  $P$  or by deleting a closed disk which is convex in the sense that it intersects every line in an interval. This disk could be a conic together with its interior points in the real projective plane, yielding the most homogeneous example. The two constructions produce planes with slightly different behaviour. In the first case, every point is incident with a unique noncompact line, and the missing point of the original projective plane can be restituted in a unique way. Such planes were called pointwise coaffine in [7]. In the other case, every point is incident with infinitely many compact and noncompact lines, and it is not clear that all examples arise from projective planes by deletion of disks. The problem of embedding such planes into projective ones amounts to constructing an ideal boundary and pasting a compact disk in the sense of Stroppel [15] to this boundary; this has to be done in a way compatible with the line system. We suspect that this is not always possible.

The proof of our theorem uses the theorem of Skornyakov for the disk. Our strategy is to locate enough subplanes with point set homeomorphic to  $D \approx \mathbb{R}^2$  and lines homeomorphic to  $\mathbb{R}$  (so-called  $(\mathbb{R}^2, \mathbb{R})$ -planes) within the given plane, to apply Skornyakov’s result to them and to derive our result.

We would like to prove a Skornyakov type theorem in full generality: The point set would be an arbitrary surface, and the lines would be closed subsets which are 1-manifolds, connected or not. We roughly got as far as showing that every point is on the boundary of some  $(\mathbb{R}^2, \mathbb{R})$ -subplane, but we would need that every point actually belongs to such a subplane.

#### PROOF OF THE THEOREM

We think of the point set  $M$  as the real projective plane  $P$  minus one point  $\infty$ , even if our geometry is not pointwise coaffine. We depict  $P$  as a circular disk, whose boundary points are identified in antipodal pairs, that is,  $|x| \leq 1$  holds for all points, and  $x = -x$  if  $|x| = 1$ . The point  $\infty$  will always be represented by the pair  $\{(0, 1), (0, -1)\}$ , as in Figure 1.

Since lines are closed subsets  $L \subseteq M$ , their closure  $\bar{L}$  in the one-point compactification  $P$  will always be homeomorphic to a circle. This circle contains the point  $\infty$  if and only if  $L$  is not compact.

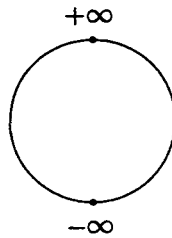


Figure 1: The Moebius strip.

Consider the two-sheeted covering  $\pi : \mathbb{S}_2 \rightarrow P$  and the inverse image  $\pi^{-1}(\bar{L})$  of one of these circles. This set is a covering of  $\bar{L}$ , and there are two possibilities: either  $\pi^{-1}(\bar{L})$  consists of two disjoint circles  $C, -C$  which are interchanged by the antipodal map, or  $\pi^{-1}(\bar{L})$  is a single circle  $C = -C$ , and  $\pi$  restricted to  $C$  is the unique two-fold connected covering of  $\bar{L}$ .

Consider the case  $C = -C$ . Applying the theorem of Schoenflies [9] to each of the two closed disks in  $\mathbb{S}_2$  having  $C = -C$  as their boundary, we obtain a homeomorphism of each disk onto a disk defined by the equator of the sphere. The two homeomorphisms may be adjusted (for example, using polar coordinates) such that they agree on  $C$  and are compatible with the antipodal map, which shows that the pair  $(P, \bar{L})$  is homeomorphic to the pair formed by the real projective plane and one of its lines. In this case, the line  $L$  will be called *essential*. The complement  $P \setminus \bar{L}$  is an open disk.

A similar procedure (needing extra corrections on the annulus defined by  $C$  and  $-C$ ) may be applied in the case of two disjoint circles and results in a proof that  $(P, \bar{L})$  then is homeomorphic to the pair formed by the real projective plane with a conic. In this case, the line  $L$  will be called *inessential*. The complement of  $\bar{L}$  in  $P$  is a disjoint union of an open disk and an open Moebius strip in this case. Our first aim is to show that inessential lines do not occur (Proposition 5).

Each of the two cases splits into subcases depending on the position of  $\infty$  relative to  $\bar{L}$ : whether  $\infty$  is in  $\bar{L}$  or not in  $\bar{L}$ , and in the latter case, which of the complementary components of  $\bar{L}$  contains  $\infty$ . The resulting five possibilities are depicted in Figures 2 through 6.

The figures also show the connected components of the complement of a line relative to  $M$ . Essential lines do not separate the point set  $M$ ; their complement is a punctured disk or an intact disk depending on whether the line is compact or not. An inessential line  $L$  separates  $M$  into two components. If  $L$  is noncompact, then one complementary component is a disk and the other is a Moebius strip. If  $L$  is compact, then one of these components is punctured.

**PROPOSITION 1.** *There are no compact inessential lines.*

**PROOF:** Such a line  $L$  would bound a subset  $D \subseteq M$  homeomorphic to a compact disk or to a compact Moebius strip, and the line  $u \vee v$  joining an interior point  $u \in D$  to a point  $v$  outside  $D$  would intersect the boundary  $\partial D$  twice (remember that the closure of  $u \vee v$  in  $P$  is a topological circle). This is a contradiction.  $\square$

**PROPOSITION 2.** *The geometry induced on the open disk bounded by a noncompact inessential line is an  $(\mathbb{R}^2, \mathbb{R})$ -plane. According to Skornjakov's Theorem, it is a stable plane.*

Remember that our aim is to exclude this particular type of lines. The proposition is merely a step towards this goal, it cannot contribute to our construction of a topology on  $\mathcal{L}$ .

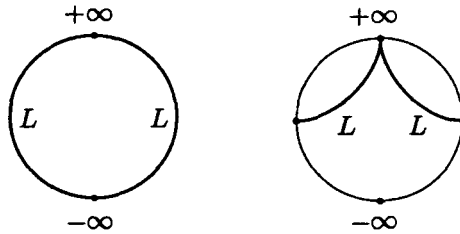


Figure 2: Two noncompact essential lines.

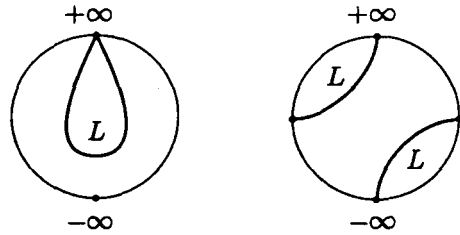


Figure 3: Two noncompact inessential lines.

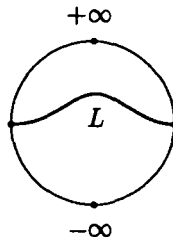


Figure 4: A compact essential line.

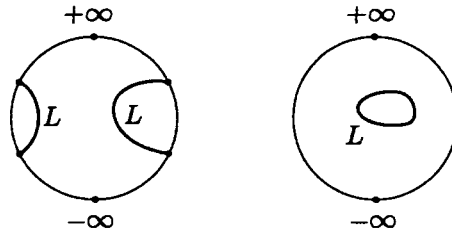


Figure 5: Two compact inessential lines of first type.

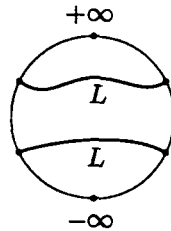


Figure 6: A compact inessential line of second type.

PROOF: Let  $D \approx \mathbb{R}^2$  be this disk and  $L$  the bounding line. We have to show that each line intersects  $D$  either in an open interval or in the empty set. A compact line entirely contained in the closure  $\bar{D}$  would be inessential, contrary to Proposition 1. A compact line intersecting  $\bar{D}$  nontrivially would have to meet the boundary  $L$  of  $D$  twice, again a contradiction.

The only remaining problem concerns a noncompact line,  $K$ , which meets  $L$  (in a single point,  $x$ ) and is entirely contained in the closure  $\bar{D} = D \cup L$ . Such a line would have a disconnected intersection with  $D$ . As indicated in Figure 7, the complement  $D \setminus K$  then has two connected components whose boundaries meet both  $K$  and  $L$ , and whose only common boundary point is  $x$ . The line joining two points  $u, v$ , one in each of these components, has to pass through  $x$ , or else it would meet one of the lines  $K, L$  twice. Therefore, the line  $u \vee x = u \vee v$  has to contain all points of the component containing  $v$ , a contradiction.  $\square$

**PROPOSITION 3.** *If  $x$  is a point in the open disk  $D$  bounded by a noncompact inessential line, then there is an essential line passing through  $x$ .*

PROOF: In view of Proposition 1, we have to exclude the possibility that all lines passing through  $x$  are noncompact inessential. So assume that this is the case. Near  $x$  we are in a stable plane (with point set  $D$ ), and we topologise the pencil  $\mathcal{C}_x^+$  of oriented lines as prescribed by this stable plane. An oriented line  $L$  in this pencil separates  $M$  into a component  $L^+$  which lies to the right of  $L$  and a component  $L^-$ . One of  $L^+, L^-$  is a disk and the other, a Moebius strip. For  $L$  with opposite orientation, the complementary components are switched, hence we have lines where  $L^+$  is a disk and others where  $L^-$  is

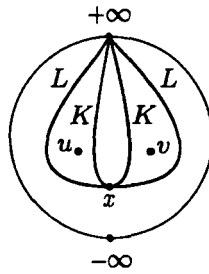


Figure 7:

a disk. Possibly after exchanging ‘right’ and ‘left’, we find a convergent sequence  $L_n \rightarrow L$  in  $\mathcal{L}_x^+$  such that  $L^+$  and all  $L_n^-$  are disks. The intersection  $L^+ \cap L_n^-$  is a half plane in both of the stable planes  $L^+$  and  $L_n^-$ , and it follows that the union  $L^+ \cup L_n^-$  is again a disk. (Observe that lines intersect transversally at  $x$  since we are in a stable plane.) We may assume that the sequence  $L_n$  is monotone in some order of a small neighbourhood of  $L$  in  $\mathcal{L}_x^+$  compatible with the topology.

Figure 8 shows the situation within  $D$ : as  $n$  increases, the union  $L^+ \cup L_n^-$  becomes

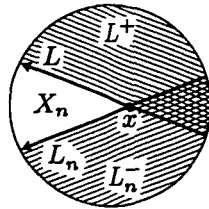


Figure 8:

larger, and its complement  $X_n$  in  $M$  becomes smaller. This remains true even outside the depicted neighbourhood  $D$ , because  $L^+$ , for example, is the union of all half lines with end point  $x$  that meet  $D \cap L^+$ . In fact, the intersection of all  $X_n$  is the positive half line  $X$  of  $L$ . The complement  $M \setminus X$  thus is the union of an increasing sequence of open disks, hence it is an open disk by the following lemma. On the other hand, a look at Figure 3 shows that the complement in  $M$  of a half line of  $L$  is still a Moebius strip. This contradiction proves our claim.  $\square$

**LEMMA 4.** *Let  $\mathcal{D}$  be a set of open 2-disks in some surface  $S$  and suppose that  $\mathcal{D}$  is totally ordered with respect to inclusion. Then the union  $D = \cup \mathcal{D}$  is a 2-disk.*

We remark that there is a far more general result due to Brown, see [3].

**PROOF:** Every loop in  $D$  is covered by finitely many  $X \in \mathcal{D}$ , hence it is contained in the largest one among them and can be contracted therein, hence  $D$  is simply connected. The only simply connected surfaces are  $\mathbb{R}^2$  and the sphere  $\mathbb{S}_2$ , see [1]. If  $D$  were compact, then the empty set  $D \setminus D$  would be the intersection of the nonempty compact complements

$D \setminus X, X \in \mathcal{D}$ , a contradiction because any finite number of those complements has a nonempty intersection.  $\square$

**PROPOSITION 5.** *In a geometry on the Moebius strip, as considered here, all lines are essential.*

**PROOF:** As we have ruled out compact inessential lines (Proposition 1), the existence of an inessential line provides us with an open disk  $D$  on which a stable  $(\mathbb{R}^2, \mathbb{R})$ -plane is induced (Proposition 2). A point  $x \in D$  is incident with some essential line  $L$  by Proposition 3. All lines containing  $x$  are noncompact, because a compact line would have to meet the bounding line of  $D$  twice by Proposition 1. In particular,  $L$  is noncompact essential and can be represented as the boundary of the disk which yields  $M$  upon antipodal identification. We split all lines passing through  $x$  up into half lines with end point  $x$ . Figure 9 represents these half lines.

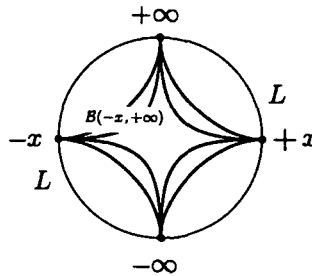


Figure 9:

Each of them connects  $x$  with  $\infty$ . Observe that it makes no difference to us whether these half lines come from essential or inessential lines (this concerns only the way lines are combined from two halves).

In our disk model  $\tilde{P}$  of the projective plane, we have obtained four bundles of half lines; for example, we let  $\mathcal{B}(-x, +\infty)$  denote the bundle of half lines joining the left copy of  $x$  to the upper copy of  $\infty$ ; some of these bundles may be empty, however. Moreover, we let  $\mathcal{B}_{\pm}$  be the set of all half lines with end point  $\pm x$ . On  $\mathcal{B}_{\pm}$  there is a natural order, obtained as follows. Every half line  $H$  separates the disk  $\tilde{P}$  into an upper disk  $H^+$  and a lower disk  $H^-$ , and we say that  $H_1 \leq H_2$  if  $H_1^- \subseteq H_2^-$ .

Since  $x$  is a point of a stable plane, there are convex neighbourhoods  $U$  of  $x$ , see [12, 31.22]. Here, convex means that the intersection of any line with  $U$  is connected. This implies that the set of points covered by  $\mathcal{B}_{\pm}$  is a neighbourhood of  $\pm x$  in the disk  $\tilde{P}$ . It further implies that the order on  $\mathcal{B}_{\pm}$  has no gaps, that is, between any two half lines with end point  $\pm x$  there lies another one having the same end point  $\pm x$ . Finally, we see that the two bundles  $\mathcal{B}_{\pm}$  are both nonempty.

If one of the two bundles  $\mathcal{B}(\pm x, -\infty)$  has a largest element  $H$ , then we replace the disk  $\tilde{P}$  by the upper closed disk  $H^+$ . Similarly, if  $\mathcal{B}(\pm x, +\infty)$  has a smallest element  $H$ ,

we pass to  $H^-$ . Note that at most two times among the four bundles a largest or smallest element can occur, because  $\mathcal{B}_\pm$  has no gaps as we have seen.

The set of points in the interior of the remaining disk is completely covered by the remaining arcs. Because the order of  $\mathcal{B}_\pm$  has no gaps, this set can be described as the total union of at least two totally ordered families of open disks (the interiors of suitable disks  $H^\pm$ ). By Proposition 4, the resulting set is a disjoint union of at least two disjoint open disks. On the other hand, it is itself a connected open disk. The contradiction shows that there can be no inessential lines.  $\square$

Our next aim is to determine the possible positions of pairs of lines, up to homeomorphism of  $M$ . First consider the case where two lines  $K, L$  are given and one of them is compact. As both lines are essential, their closures in  $P$  are covered by two circles  $C_K$  and  $C_L$  in  $\mathbb{S}_2$  which are invariant under the antipodal map. The two circles have at most one antipodal pair of common points  $x, -x$ . None of the circles can be contained in one of the two complementary disks defined by the other, because the antipodal map interchanges those disks. Therefore, the circles must intersect transversally in a pair of antipodal points, and the configuration in  $\mathbb{S}_2$  is homeomorphic to a pair of great circles. The position of  $\pm\infty$  can be on one of the circles or outside. On the Moebius strip, we obtain two possible configurations, represented by Figures 10 and 11.

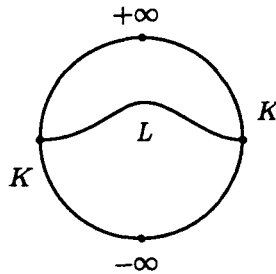


Figure 10: Two intersecting lines, one compact and one noncompact.

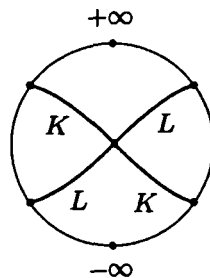


Figure 11: Two compact intersecting lines.

What we have proved is the following.



**PROPOSITION 6.** *Let  $K, L$  be two lines and assume that one of them is compact. Then the lines  $K, L$  intersect in  $M$  transversally, and the configuration is homeomorphic to the one represented in Figure 10 or 11. In particular, two compact lines together bound an open disk in  $M$ .*

Note that the closure of the open disk bounded by two compact lines is a closed disk with two boundary points identified.

We turn to pairs of noncompact lines  $K, L$ . Here the situation is more complicated because the two circles  $C_K, C_L$  have the points  $\infty, -\infty$  in common and may have one more antipodal pair of intersection points. We prefer to look at the picture in  $M$ , representing  $K$  as the boundary circle. The case of disjoint lines is easily dealt with (see Figure 16 below). The case where there is an intersection point  $x \in M$  splits into four subcases depending on whether the intersections at  $x$  and  $\infty$  are transversal or not. If both intersections are of the same kind, then we get the configurations in Figures 12 and 13, and we see that  $L$  is inessential.

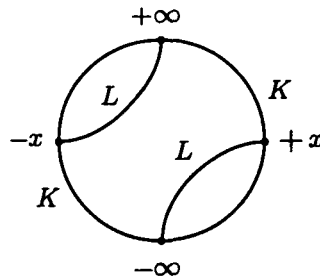


Figure 12: Two noncompact intersecting lines. Both intersections are transversal.

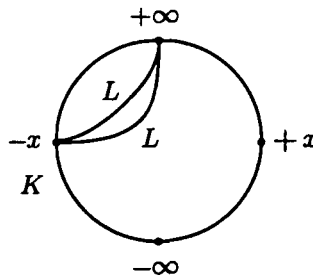


Figure 13: Two noncompact intersecting lines. Both intersections are not transversal.

If the intersection at  $\infty$  is the only transversal one, then the configuration is as in Figure 14.

The line joining the points  $u$  and  $v$  has two ends at  $\pm\infty$  and it must enter the two disks bounded by  $K, L$  and containing  $u$  and  $v$ , respectively. It cannot be shaped like

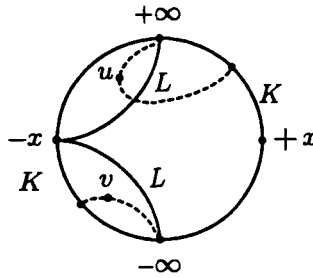


Figure 14: Two noncompact intersecting lines. The intersection at  $\infty$  is transversal, the one at  $x$  is not.

the dotted curve in Figure 14, because this curve is inessential. Hence all lines joining points in the two disks pass through the point  $x$ , and we have a contradiction. The only remaining possibility is that the lines intersect transversally at  $x$  and touch at  $\infty$ , as shown in Figure 15.

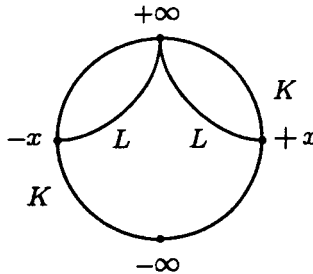


Figure 15: Two noncompact intersecting lines. The intersection at  $x$  is transversal, the one at  $\infty$  is not.

We summarise:

**PROPOSITION 7.** *The configuration of two noncompact lines is as shown in Figure 15 or 16. In particular, if they intersect, they do so transversally.*

**REMARK 8. Ordering a line pencil.** Let  $x$  be a point and  $L$  a noncompact line passing through  $x$ . We want to define an order on the set  $\mathcal{L}_x \setminus \{L\}$  in the same way as we did in the proof of Proposition 5. In the disk model  $\tilde{P}$  with boundary circle  $L$ , every line  $H \in \mathcal{L}_x \setminus \{L\}$  separates  $\tilde{P}$  into an upper part  $H^+$  and a lower part  $H^-$  (the parts may be disconnected). Since we know from Propositions 6 and 7 that lines always intersect transversally, it follows that for two such lines, one of the respective lower parts is entirely contained in the other. We say that  $H_1 \leq H_2$  if  $H_1^- \subseteq H_2^-$ , compare Figure 17.

There are no gaps in this order: if  $H_1 \leq H_2$ , choose any point  $y \in H_2^- \setminus H_1^-$  and join it to  $x$  in order to obtain a line between  $H_1$  and  $H_2$ . We remark that this fact was obtained in a different way in the proof of Proposition 5. There, we used a stable

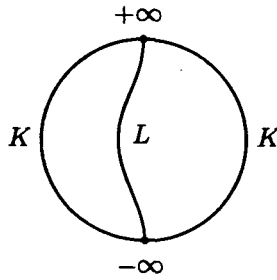


Figure 16: Two noncompact lines that do not intersect in  $M$ .

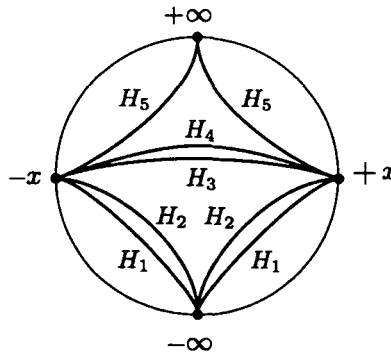


Figure 17:  $H_1 < H_2 < H_3 < H_4 < H_5$

plane containing  $x$ . Here, we used transversality of intersection which we have proved independently.

**PROPOSITION 9.** *For every point  $x$ , the compact lines passing through  $x$  form a nonempty proper subset  $C_x$  of the line pencil  $L_x$ .*

**PROOF:** First we show that there are noncompact lines passing through  $x$ . Assume that this is not the case, and consider three distinct lines in  $C_x$ . By Proposition 6, any two of them together bound an open disk in  $M$ . Looking at the corresponding disks in the sphere covering  $P$ , we see that one of the three disks in  $M$  contains the other two. We keep the two lines bounding that disk and add another line not contained in it. This produces a larger open disk.

Repeating the last argument, we obtain that any two of the disks bounded by pairs of lines in  $C_x$  are contained in another such disk. Inductively, we obtain that any finite number of these disks is contained in a single one. Therefore, we can apply the proof of Lemma 4 to the union  $D$  of all those disks, and  $D$  is a disk. By construction of  $D$ , the complement  $M \setminus D$  is either empty (impossible, since  $M$  is not a disk) or a single compact line. However, the complement of a compact line is an annulus (a punctured disk), and we have again obtained a contradiction.

Now suppose that there are no compact lines passing through  $x$ . In order to reach a contradiction, we proceed as in the proof of Proposition 5. Our starting point differs from Proposition 5 in that we know all lines to be essential, and the nonexistence of compact lines through  $x$  is now an assumption. Since we do not assume the existence of an inessential line, we do not have a stable plane containing  $x$  yielding a convex neighbourhood of  $x$ . We used this in Proposition 5 in order to show that the order considered there had no gaps. This is now obtained from 8, and the proof of Proposition 5 goes through unchanged.  $\square$

**PROPOSITION 10.** *For every point  $x \in M$ , the compact lines passing through  $x$  form a convex subset  $\mathcal{C}_x \neq \emptyset$  of the ordered set  $\mathcal{L}_x \setminus \{L\}$  introduced in 8, and this set has no smallest or largest element. In particular, there are infinitely many compact lines containing  $x$ .*

**PROOF:** Given  $C_1, C_2 \in \mathcal{C}_x, C_1 < C_2$ , we have to show that every line  $C \in \mathcal{L}_x$  such that  $C_1 < C < C_2$  is compact. Now  $C$  is contained in the compact closure of the open disk bounded by  $C_1$  and  $C_2$ , hence it is indeed compact.

If there is a largest compact line  $C \in \mathcal{C}_x$ , for example, then we may repeat the exhaustion arguments indicated in the proofs of Propositions 5 and 9 on the disk  $C^+$ , where we are considering  $M$  in the disk model with boundary  $L$ . Again, this leads to a contradiction.  $\square$

**PROPOSITION 11.** *Let  $K, L \in \mathcal{L}$  be two compact lines. On the open disk  $D = D_{K,L}$  bounded by these lines,  $(M, \mathcal{L})$  induces a (stable)  $(\mathbb{R}^2, \mathbb{R})$ -plane.*

**PROOF:** That two compact lines together bound a disk was shown in Proposition 6. A line  $G$  meeting this disk  $D$  may be contained in its closure. Then  $G$  is compact and intersects  $K$  and  $L$  transversally. This can happen only at the point  $x = K \wedge L$ , and then  $G \cap D = G \setminus \{x\} \approx \mathbb{R}$ . A line meeting  $D$  and passing through  $x$  is entirely contained in the closure of  $D$ , because lines intersect transversally. Therefore, a line  $H$  meeting both the disk  $D$  and the complement of its closure has to meet the boundary  $K \cup L$  at least twice, in points distinct from  $x$ . Since there can be at most one point of intersection with each of the lines  $K, L$ , we have that  $H \cap D \approx \mathbb{R}$ . Thus, the induced structure is an  $(\mathbb{R}^2, \mathbb{R})$ -plane; it is stable by [12, 31.22].  $\square$

**PROPOSITION 12.** *Given a compact proper subset  $A$  of a line  $H$ , we can always find two compact lines  $K, L$  such that  $A$  is contained in the open disk  $D_{K,L}$  bounded by  $K$  and  $L$ .*

**PROOF:** If  $H$  is noncompact, we may simply choose two points  $u, v$  on  $H$  such that  $A$  is contained in the open interval  $I \subseteq L$  between these points, and take arbitrary compact lines  $K$  and  $L$  containing  $u$  and  $v$ , respectively. Then  $I$  must be contained in the bounded complementary component of  $K \cup L$ , which is  $D_{K,L}$ .

If  $H$  is compact, then we choose a point  $x \in H \setminus A$  and a noncompact line  $G$  passing through  $x$ . We order  $\mathcal{L}_x \setminus \{G\}$  according to 8 and choose  $K, L \in \mathcal{C}_x$  such that  $H$  lies

between them in that order; this is possible by Proposition 10. Then  $D_{K,L}$  contains  $H \setminus \{x\}$ .  $\square$

**DEFINITION 13.** We are now in position to define a topology on the line set  $\mathcal{L}$ . Let  $L \in \mathcal{L}$  be a line and let  $H_1, H_2$  be lines meeting  $L$  in distinct points  $x_i = L \wedge H_i$ . Let  $A_i$  be a neighbourhood of  $x_i$  in  $H_i$ ,  $i \in \{1, 2\}$ . If  $A_1 \cap A_2 = \emptyset$ , then the set

$$A_1 \vee A_2 = \{u_1 \vee u_2; u_i \in A_i\} \subseteq \mathcal{L}$$

will be called an  $(H_1, H_2)$ -neighbourhood of  $L$ .

By Propositions 11 and 12, there is an  $(\mathbb{R}^2, \mathbb{R})$ -subplane of  $(M, \mathcal{L})$  containing the points  $x_1, x_2$ , and  $A_1 \vee A_2$  is an open set in the line space of that subplane, provided that  $A_1$  and  $A_2$  are small enough. It follows that the sets  $A_1 \vee A_2$  form the basis of a topology on  $\mathcal{L}$ , which we call the *natural topology*.

**LEMMA 14.** Given two pairs of distinct lines,  $(H_1, H_2)$  and  $(H'_1, H'_2)$ , meeting the line  $L$ , every  $(H_1, H_2)$ -neighbourhood of  $L$  contains an  $(H'_1, H'_2)$ -neighbourhood.

**PROOF:** Using Proposition 12, we can find an  $(\mathbb{R}^2, \mathbb{R})$ -subplane containing the four intersection points of  $L$  with the given lines, and we may also assume that the linear neighbourhoods  $A_i, A'_i$  are all contained in this subplane. Then the assertion is a direct consequence of the continuity of operations in the subplane.  $\square$

**PROPOSITION 15.** If  $(D, \mathcal{D})$  is an  $(\mathbb{R}^2, \mathbb{R})$ -subplane of  $(M, \mathcal{L})$ , then  $\mathcal{D}$  is open with respect to the natural topology on  $\mathcal{L}$ , and the topology induced on  $\mathcal{D}$  by  $\mathcal{L}$  is the topology which makes  $(D, \mathcal{D})$  a stable plane.

**PROOF:** The topology on  $\mathcal{D}$  belonging to the stable plane  $(D, \mathcal{D})$  is generated by all  $(H_1, H_2)$ -neighbourhoods of lines  $L \in \mathcal{D}$  such that the intersection points with  $L$  and the defining sets  $A_i$  are contained in  $D$ . It follows from Proposition 14 that this agrees with the topology induced by the natural topology on  $\mathcal{L}$ .  $\square$

Now it is easy to prove our main result:

**THEOREM 16.** With respect to the natural topology,  $(M, \mathcal{L})$  is a stable plane.

**PROOF:** Since any pair of points and any pair of intersecting lines is contained in some  $(\mathbb{R}^2, \mathbb{R})$ -subplane, which is a stable plane, our assertion is a direct consequence of Proposition 15.  $\square$

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