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SOME CHARACTERIZATIONS OF HARDY SPACES ASSOCIATED WITH TWISTED CONVOLUTION

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Abstract

In this paper, we shall give some characterizations of the Hardy space associated with twisted convolution, including Lusin area integral, Littlewood–Paley g-function and heat maximal function.

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1. Introduction

On $\mathbb{C}^n$ consider the $2n$ linear differential operators

$$Z_j = \frac{\partial}{\partial_{z_j}} + \frac{1}{4}\overline{z_j}, \quad \overline{Z_j} = \frac{\partial}{\partial_{\overline{z_j}}} - \frac{1}{4}z_j, \quad j = 1, 2, \ldots, n. \tag{1}$$

Together with the identity they generate a Lie algebra h^n which is isomorphic to the $2n+1$-dimensional Heisenberg algebra. The only nontrivial commutation relations are

$$[Z_j, \overline{Z_j}] = -\tfrac{1}{2}I, \quad j = 1, 2, \ldots, n. \tag{2}$$

The operator L defined by

$$L = -\frac{1}{2}\sum_{j=1}^{n}\left(Z_j\overline{Z_j} + \overline{Z_j}Z_j\right)$$

is nonnegative, self-adjoint and elliptic. Therefore it generates a diffusion semigroup $\{T_t^L\}_{t>0} = \{e^{-tL}\}_{t>0}$. The operators in (1) generate a family of ‘twisted translations’

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τ_w on $\mathbb{C}^n$ defined on measurable functions by

$$(\tau_w f)(z) = \exp\left(\frac{1}{2}\sum_{j=1}^{n}(w_j z_j + \overline{w_j z_j})\right) f(z)$$
$$= f(z+w)\exp\left(\frac{i}{2}\mathrm{Im}(z\cdot\overline{w})\right).$$

The 'twisted convolution' of two functions f and g on $\mathbb{C}^n$ can now be defined as

$$(f \times g)(z) = \int_{\mathbb{C}^n} f(w)\tau_{-w} g(z)\,dw$$
$$= \int_{\mathbb{C}^n} f(z-w)g(w)\overline{\omega}(z,w)\,dw,$$

where $\omega(z, w) = \exp\big(i\mathrm{Im}(z\cdot\overline{w})/2\big)$. More about twisted convolution can be found in [1, 7, 9].

In [8] the authors defined the Hardy space $H^1_L(\mathbb{C}^n)$ associated with twisted convolution. They gave several characterizations of $H^1_L(\mathbb{C}^n)$, such as maximal function, atomic decomposition and Riesz transform. The purpose of this paper is to consider other characterizations, including Lusin area integral and Littlewood–Paley g-function. In order to prove our result, we also give a heat maximal functon characterization for $H^1_L(\mathbb{C}^n)$.

We first give some basic notation for $H^1_L(\mathbb{C}^n)$. Let $\mathcal{B}$ denote the class of C^∞-functions φ on $\mathbb{C}^n$, supported on the ball $B(0, 1)$ such that $\|\varphi\|_\infty \le 1$ and $\|\nabla\varphi\|_\infty \le 2$. For $t > 0$, let $\varphi_t(z) = t^{-2n}\varphi(z/t)$. Given $\sigma > 0$, $0 < \sigma \le +\infty$ and a tempered distribution f, define the grand maximal function

$$M_\sigma f(z) = \sup_{\varphi\in\mathcal{B}}\ \sup_{0<t<\sigma} |\varphi_t \times f(z)|.$$

Then the Hardy space $H^1_L(\mathbb{C}^n)$ can be defined by

$$H^1_L(\mathbb{C}^n) = \{f \in L^1(\mathbb{C}^n) \mid M_\infty f \in L^1(\mathbb{C}^n)\}.$$

We define atoms for $H^1_L(\mathbb{C}^n)$ as follows. A function $a(z)$ is an atom for the Hardy space $H^1_L(\mathbb{C}^n)$ associated to a ball $B(z_0, r)$ if the following properties hold.

(1) supp $a \subset B(z_0, r)$.
(2) $\|a\|_\infty \le |B(z_0, r)|^{-1}$.
(3) $\int a(w)\overline{\omega}(z_0, w)\,dw = 0$.

The atomic quasi-norm in $H^1_L(\mathbb{C}^n)$ is defined by

$$\|f\|_{L\text{-atom}} = \inf\{\Sigma|\lambda_j|\},$$

where the infimum is taken over all decompositions $f = \sum\lambda_j a_j$ and a_j are atoms.

The following result has been proved in [8].

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PROPOSITION 1. *For a tempered distribution f on $\mathbb{C}^n$ the following properties are equivalent.*

(i) $M_\infty f \in L^1(\mathbb{C}^n)$.

(ii) *For some* σ, $0 < \sigma < +\infty$, $M_\sigma f \in L^1(\mathbb{C}^n)$.

(iii) *For some radial* $\varphi \in \mathcal{S}$, *such that* $\int \varphi(z)\,dz \neq 0$, *we have*

$$\sup_{0<t<1} |\varphi_t \times f(z)| \in L^1(\mathbb{C}^n).$$

(iv) *The distribution f can be decomposed as* $f = \sum \lambda_j a_j$, *where a_j are atoms and* $\sum |\lambda_j| < +\infty$.

It is well known that there are many equivalent characterizations of the classical Hardy spaces (see [4]), we shall consider other characterizations for $H_L^1(\mathbb{C}^n)$ in this paper.

We define the Lusin area integral operator by

$$(S_L^\alpha f)(z) = \left(\int_0^{+\infty} \int_{|z-w|<\alpha t} |Q_t^L f(w)|^2 \frac{dw\,dt}{t^{2n+1}} \right)^{1/2},$$

where $Q_t^L f(x) = t^2 (\partial_s T_s^L|_{s=t^2} f)(z)$.

REMARK 2. It is easy to see that the definition of area integral operator is independent of α in the sense of $\|(S_L^\alpha f)\|_{L^p} \sim \|(S_L^\beta f)\|_{L^p}$, for $0 < \alpha < \beta < \infty$ and $0 < p < \infty$ (see [3]). In the following we use S_L to denote S_L^1.

We can characterize $H_L^1(\mathbb{C}^n)$ as follows.

THEOREM 3. *A function $f \in H_L^1(\mathbb{C}^n)$ if and only if its area integral $S_L f \in L^1(\mathbb{C}^n)$ and $f \in L^1(\mathbb{C}^n)$. Moreover,*

$$\|f\|_{H_L^1} \sim \|S_L f\|_{L^1}.$$

The Littlewood–Paley g-function is defined by

$$\mathcal{G}_L(f)(z) = \left(\int_0^\infty |Q_t^L f(z)|^2 \frac{dt}{t} \right)^{1/2}.$$

The Hardy space $H_L^1(\mathbb{C}^n)$ can also be characterized by $\mathcal{G}_L$ as in the following theorem.

THEOREM 4. *A function $f \in H_L^1(\mathbb{C}^n)$ if and only if $\mathcal{G}_L f \in L^1(\mathbb{C}^n)$ and $f \in L^1(\mathbb{C}^n)$. Moreover,*

$$\|f\|_{H_L^1} \sim \|\mathcal{G}_L f\|_{L^1}.$$

We also need some basic propositions about the tent space (see [3]).

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Let $0 < p < \infty$, and $1 \le q \le \infty$, then the tent space T_q^p is defined as the space of functions f on $\mathbb{C}^n \times \mathbb{R}^+$, so that

$$\left(\int_{\Gamma(z)} |f(w,t)|^q \frac{dw\,dt}{t^{2n+1}}\right)^{1/q} \in L^p(\mathbb{C}^n) \quad \text{when } 1 \le q < \infty,$$

and

$$\sup_{(w,t)\in\Gamma(z)} |f(w,t)| \in L^p(\mathbb{C}^n) \quad \text{when } q = \infty,$$

where $\Gamma(z)$ is the standard cone whose vertex is $z \in \mathbb{C}^n$, that is,

$$\Gamma(z) = \{(w,t) : |w - z| < t\}.$$

Assume $B(z_0, r)$ is a ball in $\mathbb{C}^n$, its tent $\widehat{B}$ is defined by

$$\widehat{B} = \{(w,t) : |w - z_0| \le r - t\}.$$

A function $a(z,t)$ that supported in a tent $\widehat{B}$, B is a ball in $\mathbb{C}^n$, is said to be an atom in the tent space T_q^p if and only if it satisfies

$$\left(\int_{\widehat{B}} |a(z,t)|^2 \frac{dz\,dt}{t}\right)^{1/2} \le |B|^{1/2-1/p}.$$

The atomic decomposition of T_q^p is stated as in the following proposition.

Proposition 5. *When $0 < p \le 1$, then any $f \in T_2^p$ can be written as $f = \sum \lambda_k a_k$, where a_k are atoms and $\sum |\lambda_k|^p \le C\|f\|_{T_2^p}^p$.*

The paper is organized as follows: in Section 2, we give some estimates of the kernels; in Section 3, we prove the main results of this paper.

Throughout the article, we shall use A and C to denote the positive constants, which are independent of main parameters and may be different at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $1/C \le B_1/B_2 \le C$.

2. Preliminaries

In this section, we give some estimates of the kernel of Q_t^L that we shall use subsequently.

Let $\{T_t^L\}_{t>0}$ be the heat semigroup generated by the operator L, then, for $f \in L^2(\mathbb{C}^n)$, the function $e^{-tL} f$ has the special Hermite expansion (see [11])

$$e^{-tL} f(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-(2k+n)t} f \times \varphi_k(z),$$

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where φ_k is the Laguerre function. Therefore $e^{-tL}f$ is given by twisted convolution with the kernel

$$K_t(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-(2k+n)t} \varphi_k(z). \tag{3}$$

Let

$$L_k^\alpha(x) = \sum_{j=0}^{k} \frac{\Gamma(k+\alpha+1)}{\Gamma(k-j+1)\Gamma(j+\alpha+1)} \frac{(-x)^j}{j!}$$

be the Laguerre polynomials of type α and degree k, then we have the following generating function for Laguerre polynomials:

$$\sum_{k=0}^{\infty} L_k^\alpha(x) r^k = (1-r)^{-\alpha-1} e^{-(r/(1-r))x}. \tag{4}$$

From (4) we obtain

$$K_t(z) = (4\pi)^{-n} (\sinh t)^{-n} e^{-\frac{1}{4}|z|^2(\coth t)}. \tag{5}$$

It is easy to prove that the heat kernel $K_t(z)$ has the following estimates (see the proof of Lemma 7).

LEMMA 6. *There exists a positive constant $C > 0$ such that the following inequalities hold.*

(i) $\left|K_t(z)\right| \le C t^{-n} e^{-C|z|^2/t}$.

(ii) $\left|\nabla K_t(z)\right| \le C t^{-n-\frac{1}{2}} e^{-C|z|^2/t}$.

Let $Q_t^L(z)$ be the twisted convolution kernel of Q_t^L, then

$$Q_t^L(z) = t^2 \partial_s K_s(z)|_{s=t^2}.$$

LEMMA 7. *There exists a constant $C > 0$ such that the following inequalities hold.*

(i) $\left|Q_t^L(z)\right| \le C t^{-2n} e^{-C\, t^{-2}|z|^2}$.

(ii) $\left|\nabla Q_t^L(z)\right| \le C t^{-2n-1} e^{-C\, t^{-2}|z|^2}$.

PROOF. It is easy to see that

$$\begin{aligned}\partial_t K_t(z) = {} & (4\pi)^{-n}(-n)(\sinh t)^{-n-1}(\cosh t) e^{-\frac{1}{4}|z|^2(\coth t)} \\ & + (4\pi)^{-n}(\sinh t)^{-n-2}\left(-\tfrac{1}{4}|z|^2\right) e^{-\frac{1}{4}|z|^2(\coth t)}.\end{aligned}$$

Therefore

$$\begin{aligned}|\partial_t K_t(z)| \le {} & C\big((\sinh t)^{-n-1}(\cosh t) e^{-\frac{1}{4}|z|^2(\coth t)} \\ & + (\sinh t)^{-n-2}|z|^2 e^{-\frac{1}{4}|z|^2(\coth t)}\big).\end{aligned}$$

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Noting that there exists $C_1 > 0$ such that for all $t > 0$ we have

$$\sinh t \geq C_1 t, \qquad \coth t \geq C_1 t^{-1} \tag{6}$$

and

$$(\sinh t)^{-n-1}\cosh t \leq \begin{cases} C_1(\sinh t)^{-n-1} \leq C_1 t^{-n-1} & \text{if } 0 < t \leq 1, \\ C_1(\sinh t)^{-n}\coth t \leq C_1 t^{-n-1} & \text{if } 1 < t < \infty. \end{cases} \tag{7}$$

From (6) and (7) we have

$$\begin{aligned} |\partial_t K_t(z)| &\leq C(t^{-n-1}e^{-\frac{1}{4}t^{-1}|z|^2} + t^{-n-2}|z|^2 e^{-\frac{1}{4}t^{-1}|z|^2}) \\ &\leq Ct^{-n-1}e^{-Ct^{-1}|z|^2}. \end{aligned}$$

So, we have

$$|Q_t^L(z)| \leq Ct^{-2n}e^{-C\,t^{-2}|z|^2}.$$

This proves part (i).

To prove part (ii), it is sufficient to prove

$$|\partial_{z_j}\partial_t K_t(z)| \leq Ct^{-n-\frac{3}{2}}e^{-Ct^{-1}|z|^2}, \qquad j = 1, 2, \ldots, n. \tag{8}$$

It is easy to calculate

$$\begin{aligned} \partial_{z_j}\partial_t K_t(z) &= (4\pi)^{-n}(-n)(\sinh t)^{-n-1}(\cosh t)2z_j\big(-\tfrac{1}{4}\coth t\big)e^{-\frac{1}{4}|z|^2(\coth t)} \\ &\quad + (4\pi)^{-n}(\sinh t)^{-n-2}\big(-\tfrac{1}{4}\cosh t\big)2z_j\big(-\tfrac{1}{4}\coth t\big) \\ &\qquad \times \big(-\tfrac{1}{4}|z|^2\big)e^{-\frac{1}{4}|z|^2(\coth t)} \\ &\quad + (4\pi)^{-n}(\sinh t)^{-n-2}\big(-\tfrac{1}{2}z_j\big)e^{-\frac{1}{4}|z|^2(\coth t)}. \end{aligned}$$

By (6) and (7),

$$\begin{aligned} |\partial_{z_j}\partial_t K_t(z)| &\leq C\bigg(t^{-n-\frac{3}{2}}\frac{|z|}{\sqrt{t}}e^{-\frac{1}{4}t^{-1}|z|^2} + t^{-n-\frac{3}{2}}\bigg(\frac{|z|}{\sqrt{t}}\bigg)^3 e^{-\frac{1}{4}t^{-1}|z|^2} \\ &\qquad + t^{-n-\frac{3}{2}}\frac{|z|}{\sqrt{t}}e^{-\frac{1}{4}t^{-1}|z|^2}\bigg) \\ &\leq Ct^{-n-\frac{3}{2}}e^{-Ct^{-1}|z|^2}. \end{aligned}$$

This completes the proof of (8) and so part (ii) is proved. □

We can also consider the following operator $Q_t^2 = t^4\partial_s^2 T_s^L|_{s=t^2}$. If we use $Q_t^2(z)$ to denote the twisted convolution kernel of Q_t^2, then similarly as Lemma 7, we can prove the following result.

LEMMA 8. *There exists a constant $C > 0$ such that the following inequalities hold.*

(i) $|Q_t^2(z)| \leq Ct^{-2n}e^{-C\,t^{-2}|z|^2}$.

(ii) $|\nabla Q_t^2(z)| \leq Ct^{-2n-1}e^{-C\,t^{-2}|z|^2}$.

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3. The proofs of the main results

In this section, we shall give the proofs of the main results of this paper.

We first give the maximal function characterization for $H^1_L(\mathbb{C}^n)$. By Proposition 1, we have the following lemma (see [8, p. 281]).

LEMMA 9. *$f \in H^1_L(\mathbb{C}^n)$ if and only if*

$$\widetilde{M} f(z) = \sup_{0<t<1} |K_t \times f(z)| \in L^1(\mathbb{C}^n)$$

and $f \in L^1(\mathbb{C}^n)$.

Let

$$Mf(z) = \sup_{t>0} |K_t \times f(z)| \in L^1(\mathbb{C}^n),$$

then we can characterize $H^1_L(\mathbb{C}^n)$ by the maximal function Mf as follows.

THEOREM 10. *$f \in H^1_L(\mathbb{C}^n)$ if and only if $Mf \in L^1(\mathbb{C}^n)$ and $f \in L^1(\mathbb{C}^n)$.*

PROOF. By Lemma 9, we know $f \in H^1_L(\mathbb{C}^n)$ if $Mf \in L^1(\mathbb{C}^n)$ and $f \in L^1(\mathbb{C}^n)$.

For the reverse, we just need to prove that there exists $C > 0$ such that for any atom $a(z)$ of $H^1_L(\mathbb{C}^n)$,

$$\|Ma\|_{L^1} \leq C.$$

Let $a(z)$ be an atom of $H^1_L(\mathbb{C}^n)$. By twisted translation, we can assume that $\operatorname{supp} a \subset B(0, r)$. Then, we have $\int a(w)\, dw = 0$.

By Lemma 6, we have

$$\int_{|w|\leq 2r} |Ma(w)|\, dw \leq \|M_{H-L} a\|_{L^2} (2r)^n \leq C,$$

where M_{H-L} is the Hardy–Littlewood maximal function.

For $|z| > 2r$, we get

$$\begin{aligned} K_t \times a(z) &= \int K_t(z-w) a(w) \overline{\omega}(z, w)\, dw \\ &= \int (K_t(z-w) - K_t(z)) a(w) \overline{\omega}(z, w)\, dw + K_t(z) \widehat{a}\Big(-\frac{i}{2} z\Big) \\ &= I_1 + I_2. \end{aligned}$$

By Lemma 6 again, we can prove

$$I_1 \leq C \int t^{-n-\frac{1}{2}} e^{-Ct^{-1}|z-w|^2} |w| |a(w)|\, dw \leq Cr|z|^{-2n-1}.$$

Therefore

$$\int_{|z|>2r} |I_1|\, dz \leq C.$$

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By Lemma 6 (i) and Hardy's inequality (see [6, p. 341, Theorem 7.22]) , we get

$$\int_{|z|>2r} |I_2|\,dz \le C \int_{\mathbb{C}^n} \frac{|\widehat{a}(-(i/2)z)|}{|z|^{2n}}\,dz \le C.$$

Therefore we have $\|Ma\|_{L^1} \le C$ and Theorem 10 is proved. □

In order to get our results, we need the following lemma.

LEMMA 11. *The operators S_L and $\mathcal{G}_L$ are isometries on $L^2(\mathbb{C}^n)$ up to constant factors. Exactly,*

$$\|\mathcal{G}_L f\|_{L^2} = \frac{1}{2}\,\|f\|_{L^2}, \quad \|S_L f\|_{L^2} = \frac{\sqrt{c_n}}{2}\,\|f\|_{L^2}.$$

PROOF. The L^2 equality for $\mathcal{G}_L$ is established in [12, Proposition 3.1]. As a consequence we have

$$\begin{aligned}\|S_L f\|_{L^2}^2 &= \int_{\mathbb{C}^n}\int_{\mathbb{C}^n\times\mathbb{R}^+} |Q_t^L f(w)|^2\,\chi_{\Gamma(z)}(w,t)\frac{dw\,dt}{t^{2n+1}}\,dz\\ &= c_n \int_{\mathbb{C}^n\times\mathbb{R}^+} |Q_t^L f(w)|^2\,\frac{dw\,dt}{t} = c_n\,\|\mathcal{G}_L f\|_{L^2}^2\\ &= \frac{c_n}{4}\,\|f\|_{L^2}^2.\end{aligned}$$

□

PROOF OF THEOREM 3. As in the proof of Theorem 10, by Lemma 7, we can prove that there exists a constant $C > 0$ such that for any atom $a(z)$ of $H_L^1(\mathbb{C}^n)$ we have

$$\|S_L a\|_{L^1} \le C. \tag{9}$$

Now we prove that $f \in H_L^1(\mathbb{C}^n)$ when $S_L f \in L^1(\mathbb{C}^n)$ and $f \in L^1(\mathbb{C}^n)$.

We first assume that $f \in L^1(\mathbb{C}^n) \cap L^2(\mathbb{C}^n)$. When $S_L f \in L^1(\mathbb{C}^n)$, we know that $Q_t^L f \in T_2^1$. By Proposition 5 we get

$$Q_t^L f(z) = \sum_j \lambda_j a_j(z,t), \tag{10}$$

where $a_j(z,t)$ are atoms of T_2^1 and $\sum_j |\lambda_j| < \infty$. By the spectral theorem (see [10]), we can prove

$$f(z) = 4\int_0^\infty Q_t^L(Q_t^L f(z))\frac{dt}{t}. \tag{11}$$

By (10) and (11), we get

$$f(z) = 4\int_0^{+\infty} Q_t^L\Big(\sum_j \lambda_j a_j(z,t)\Big)\frac{dt}{t} = C\sum_j \lambda_j \int_0^{+\infty} Q_t^L a_j(z,t)\frac{dt}{t}.$$

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Therefore it is sufficient to prove that the functions

$$\alpha_j = \int_0^{+\infty} Q_t^L a_j(z, t) \frac{dt}{t}, \quad i = 1, 2, \ldots,$$

are bounded in $H_L^1(\mathbb{C}^n)$ uniformly; that is, there exists a constant $C > 0$ such that for any atom $a(z, t)$ in T_2^1,

$$\|\alpha\|_{H_L^1} = \left\| \int_0^{+\infty} Q_t^L a(z, t) \frac{dt}{t} \right\|_{H_L^1} \le C.$$

We assume that $a(z, t)$ is supported in $\widehat{B}(z_0, r)$, where $\widehat{B}(z_0, r)$ denotes the tent of the ball $B(z_0, r)$, then

$$\begin{aligned}\left\| \sup_{t>0} |e^{-tL}\alpha(z)| \right\|_{L^1} &\le \left\| \left(\sup_{t>0} |e^{-tL}\alpha(z)| \right) \chi_{B^*} \right\|_{L^1} + \left\| \left(\sup_{t>0} |e^{-tL}\alpha(z)| \right) \chi_{(B^*)^c} \right\|_{L^1} \\ &= I_1 + I_2,\end{aligned}$$

where $B^* = B(z_0, 2r)$.

By the Hölder inequality, we get

$$I_1 \le |B^*|^{1/2} \left(\int_{\mathbb{C}^n} \left(\sup_{t>0} |e^{-tL}\alpha(z)| \right)^2 dz \right)^{1/2} \le |B^*|^{1/2} \|\alpha\|_{L^2}.$$

By the self-adjointness of Q_t^L and Lemma 11 we have

$$\begin{aligned}\|\alpha\|_{L^2} &= \sup_{\|\beta\|_{L^2} \le 1} \int_{\mathbb{C}^n} \alpha(z) \overline{\beta}(z)\, dz \\ &= \sup_{\|\beta\|_{L^2} \le 1} \int_{\mathbb{C}^n} \left(\int_0^{+\infty} Q_t^L a(z, t) \frac{dt}{t} \right) \overline{\beta}(z)\, dz \\ &= \sup_{\|\beta\|_{L^2} \le 1} \int_0^{+\infty} \int_{\mathbb{C}^n} Q_t^L a(z, t) \overline{\beta}(z)\, dz \frac{dt}{t} \\ &= \sup_{\|\beta\|_{L^2} \le 1} \int_0^{+\infty} \int_{\mathbb{C}^n} a(z, t) Q_t^L \overline{\beta}(z)\, dz \frac{dt}{t} \\ &\le \sup_{\|\beta\|_{L^2} \le 1} \left(\int_{\mathbb{C}^n} \int_0^{+\infty} |a(z, t)|^2 \frac{dz\, dt}{t} \right)^{1/2} \\ &\qquad \times \left(\int_{\mathbb{C}^n} \int_0^{+\infty} |Q_t^L \overline{\beta}(z)|^2 \frac{dz\, dt}{t} \right)^{1/2} \\ &\le |B|^{-1/2} \|\beta\|_{L^2} \le |B|^{-1/2}.\end{aligned}$$

This proves $I_1 \le C$.

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By Lemma 7 we can prove

$$
\begin{aligned}
&\sup_{s>0}\left|e^{-sL}\int_0^{+\infty} Q_t^L a(z,t)\frac{dt}{t}\right| \\
&\quad= \sup_{s>0}\left|e^{-sL}\int_0^{+\infty}(-L)te^{-tL}a(z,t)\frac{dt}{t}\right| \\
&\quad= \sup_{s>0}\left|\int_0^{+\infty}(-tL)e^{-(s+t)L}a(z,t)\frac{dt}{t}\right| \\
&\quad= \sup_{s>0}\left|\int_0^{+\infty}\left(\frac{t}{s+t}\right)((s+t)L)e^{-(s+t)L}a(z,t)\frac{dt}{t}\right| \\
&\quad= \sup_{s>0}\left|\int_0^{+\infty}\left(\frac{t}{s+t}\right)\int_{\mathbb{C}^n} Q_{s+t}^L(z,w)a(w,t)\frac{dw\,dt}{t}\right| \\
&\quad\le \sup_{s>0}\int_0^{+\infty}\frac{t}{s+t}\int_{\mathbb{C}^n}(s+t)^{-2n}\exp\left(-\frac{|z-w|^2}{(s+t)^2}\right)|a(w,t)|\frac{dw\,dt}{t} \\
&\quad\le \sup_{s>0}\int_0^{+\infty}\frac{t}{s+t}\int_{\mathbb{C}^n}(s+t)^{-2n}\left(1+\frac{|z-w|^2}{(s+t)^2}\right)^{-(n+1)}|a(w,t)|\frac{dw\,dt}{t} \\
&\quad\le \sup_{s>0}\left(\int_0^r\int_B(s+t)^{-4n}\left(1+\frac{|z-w|^2}{(s+t)^2}\right)^{-(2n+1)}\left(\frac{t}{s+t}\right)^2\frac{dw\,dt}{t}\right)^{1/2} \\
&\qquad\times\left(\int_0^r\int_B|a(w,t)|^2\frac{dw\,dt}{t}\right)^{1/2} \\
&\quad\le |B|^{-1/2}|z-z_0|^{2n+1}\left(\int_0^r\int_B t\,dw\,dt\right)^{1/2} \\
&\quad\le Cr|z-z_0|^{-(2n+1)}.
\end{aligned}
$$

Then we have

$$I_2 \le Cr\int_{(B^*)^c}|z-z_0|^{-(2n+1)}\,dz \le C.$$

When $f \in L^1(\mathbb{C}^n)$ we can prove the result as in [2, Proposition 14]. In fact, we let $f_k = T_{2^{-k}}^L f,\ \ k \ge 0$. Then, by $f \in L^1(\mathbb{C}^n)$ and Lemma 6, we know $f_k \in L^2(\mathbb{C}^n)$ and $\|S_L f_k\|_1 \le \|S_L f\|$. By the case of $f \in L^1 \cap L^2$, we get

$$\|f_k\|_{H_L^1(\mathbb{C}^n)} \lesssim \|S_L f_k\|_{L^1} \le \|S_L f\|_{L^1}.$$

By the monotone theorem, we have

$$\|f_k - f_n\|_{H_L^1} \le \|S_k^L(f_k - f_n)\|_{L^1} \to 0 \quad \text{when } k, n \to +\infty.$$

Therefore $\{f_k\}$ is a Cauchy sequence in $H_L^1(\mathbb{R}^d)$ and there exists $g \in H_L^1(\mathbb{R}^d)$ such that

$$\lim_{m\to+\infty} f_k = g \quad \text{in } H_L^1(\mathbb{R}^d).$$

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As

$$\lim_{k\to+\infty} f_k = f \quad \text{in } (BMO_L)^*,$$

we know $f = g \in H^1_L(\mathbb{R}^d)$ and $\|f\|_{H^1_L(\mathbb{C}^n)} \lesssim \|S_L f\|_{L^1}$.

This gives the proof of Theorem 3. □

We define $\widetilde{S^\alpha_L} f(z)$ by

$$\widetilde{S_L}(f)(z) = \left(\int_0^{+\infty}\int_{|z-w|<\alpha t} |Q_t^2 f(w)|^2 \frac{dw\,dt}{t^{2n+1}}\right)^{1/2},$$

where $\alpha > 0$. Then in the same way as the proof of Theorem 3, we can prove the following result.

COROLLARY 12. *A function $f \in H^1_L(\mathbb{C}^n)$ if and only if its area integral $\widetilde{S^\alpha_L} f \in L^1(\mathbb{C}^n)$ and $f \in L^1(\mathbb{C}^n)$. Moreover,*

$$\|f\|_{H^1_L} \sim \|\widetilde{S^\alpha_L} f\|_{L^1}.$$

Now we can give the proof of Theorem 4.

PROOF OF THEOREM 4. Firstly, by Lemma 7, we can prove that there exists a positive constant C such that for any atom $a(z)$ of $H^1_L(\mathbb{C}^n)$, we have

$$\|\mathcal{G}_L a\|_{L^1} \le C.$$

For the converse, by Corollary 12, it is sufficient to prove

$$\|\widetilde{S^1_L} f\|_{L^1} \le C\|\mathcal{G}_L f\|_{L^1}. \tag{12}$$

Our proof is motivated by [5]. Let

$$F(z)(t) = (\partial_t e^{-tL} f)(z), \qquad V(z,s) = e^{-sL}F(z),$$

then

$$V(z,s)(t) = e^{-sL}(\partial_t e^{-tL} f)(z) = (\partial_t e^{-(s+t)L} f)(z).$$

Therefore

$$\begin{aligned}\int_0^{+\infty} |V(z,s)(t)|^2 t\,dt &= \int_0^{+\infty} |(\partial_t e^{-(s+t)L} f)(z)|^2 t\,dt\\ &= \int_s^{+\infty} |(\partial_t e^{-tL} f)(z)|^2 (t-s)\,dt.\end{aligned}$$

Hence

$$\sup_{s>0}\int_0^{+\infty} |V(z,s)(t)|^2 t\,dt \le \int_0^{+\infty} |(\partial_t e^{-tL} t f)(z)|^2 t\,dt = (\mathcal{G}_L f(z))^2.$$

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Let $\mathbf{X} = L^2((0, \infty), t\, dt)$, then

$$\sup_{s>0} \|e^{-sL} F(z)\|_{\mathbf{X}} = \mathcal{G}_L f(z) \in L^1(\mathbb{C}^n).$$

Therefore $F \in H^1_{\mathbf{X}}(\mathbb{C}^n)$ and here $H^1_{\mathbf{X}}(\mathbb{C}^n)$ can be seen as a vector-valued Hardy space. This shows that $S_1^2 F(z) \in L^1(\mathbb{C}^n)$, where

$$S_1^2 F(z) = \left(\int_0^{+\infty} \int_{|z-w|<2t} \|Q_t^L F(w)\|^2_{\mathbf{X}} \frac{dw\, dt}{t^{2n+1}} \right)^{1/2}.$$

By

$$\begin{aligned}
(S_1^2 F(z))^2 &= \int_0^{+\infty} \int_{|z-w|<2t} \|Q_t^L F(z)\|^2_{\mathbf{X}} \frac{dw\, dt}{t^{2n+1}} \\
&= \int_0^{+\infty} \int_{|z-w|<2t} \int_0^{+\infty} |t(-L)e^{-tL} F(w)(s)|^2 s\, ds \frac{dw\, dt}{t^{2n+1}} \\
&= \int_0^{+\infty} \int_0^{+\infty} \int_{|z-w|<2t} |(-L)^2 e^{-(s+t)L} f(w)|^2 t^{1-2n} s\, dw\, dt\, ds \\
&= \int_0^{+\infty} \int_s^{+\infty} \int_{|z-w|<2(t-s)} |(-L)^2 e^{-tL} f(w)|^2 (t-s)^{1-2n} s\, dw\, dt\, ds \\
&= \int_0^{+\infty} \int_0^{t} \int_{|z-w|<2(t-s)} |(-L)^2 e^{-tL} f(w)|^2 (t-s)^{1-2n} s\, dw\, ds\, dt \\
&\geq \int_0^{+\infty} \int_0^{t/2} \int_{|z-w|<2(t-s)} |(-L)^2 e^{-tL} f(w)|^2 (t-s)^{1-2n} s\, dw\, ds\, dt \\
&\geq \int_0^{+\infty} \int_0^{t/2} \int_{|z-w|<t} |(-L)^2 e^{-tL} f(w)|^2 t^{1-2n} s\, dw\, ds\, dt \\
&= \frac{1}{8} \int_0^{+\infty} \int_{|z-w|<t} |(-L)^2 e^{-tL} f(w)|^2 t^{3-2n}\, dw\, dt \\
&= \frac{1}{8} \int_0^{+\infty} \int_{|z-w|<t} |Q_t^2 f(w)|^2 \frac{dw\, dt}{t^{2n+1}} = \frac{1}{8} (\widetilde{S_L^1} f(z))^2,
\end{aligned}$$

we get $\widetilde{S_L^1} f \in L^1(\mathbb{C}^n)$. Then $f \in H^1_L(\mathbb{C}^n)$ follows from Corollary 12.

This completes the proof of Theorem 4. □

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