THE PROPAGATION OF A PLANE SHOCK INTO A QUIET ATMOSPHERE

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Introduction. In this note the problem of determining the propagation of a plane shock moving through a polytropic gas into an undisturbed body of the gas, leaving a non-isentropic disturbance behind it, is reduced to the solution of the problem of Cauchy for a Monge-Ampère partial differential equation of special type. The application of this result to specific examples is being studied with a view to later publication.

1. First principles. In one-dimensional unsteady flow of a gas the velocity u, density ρ, and pressure p must satisfy [**3**, p. 508] the underdetermined system

$$\rho(u_x u + u_t) + p_x = 0, \quad (\rho u)_x + \rho_t = 0, \tag{1}$$

of partial differential equations implied by the physical hypotheses of conservation of momentum and of mass. By multiplying the second equation by u and adding the result to the first, we arrive at the equivalent system

$$(p + \rho u^2)_x + (\rho u)_t = 0, \quad (\rho u)_x + \rho_t = 0.$$

If u, ρ, p are functions of x, t which satisfy this system, we can infer the existence of two functions $\bar{\xi}$, ψ of x, t defined by

$$d\bar{\xi} = \rho u\, dx - (p + \rho u^2)\, dt, \quad d\psi = \rho\, dx - \rho u\, dt.$$

Along a trajectory $x = x(t)$ of a gas particle in the (x,t)-plane we have $dx/dt = u$ and consequently ψ is constant along such a trajectory. The curves $\psi(x, t)$ is constant in the (x, t)-plane are accordingly the trajectories of gas particles and the function ψ may therefore be termed the *trajectory function.*

One readily sees that the above system can be given the form

$$d\bar{\xi} = -\, p\, dt + u d\psi, \qquad d\psi = \rho\, dx - \rho u\, dt,$$

which, in turn can be replaced by

$$d\xi = u d\psi + t dp, \qquad d\psi = \rho\, dx - \rho u\, dt, \tag{2}$$

provided we set $\xi = \bar{\xi} + pt$.

The form of this system suggests that ψ, p be taken for independent variables. This amounts to introducing the trajectories and isobars in the (x, t)-plane as a system of curvilinear coordinates in this plane. Obviously this cannot be done if the trajectories and isobars coincide and we accordingly exclude from consideration those flows in which each gas particle retains a constant pressure

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during its motion.[1] The introduction of ψ, p as independent variables into the system (2) yields the relations

$$u = \xi_\psi, \quad t = \xi_p, \quad x_\psi = \xi_\psi \, \xi_{\psi p} + \tau, \quad x_p = \xi_\psi \, \xi_{pp},$$

where $\tau = \rho^{-1} = \tau(\psi, p)$, is the specific volume. When x is eliminated by partial differentiation from the last two equations above we find that $\xi = \xi(\psi, p)$ is a solution of the Monge-Ampère partial differential equation.[2]

$$\xi_{\psi\psi} \, \xi_{pp} - \xi_{\psi p}^{\;2} = \tau_p. \tag{3}$$

Here $\tau = \tau(\psi, p)$ is an arbitrary function whose appearance reflects the underdeterminedness of the system (1). Once $\tau(\psi, p)$ has been specified and a solution $\xi = \xi(\psi, p)$ of this equation has been found an unsteady flow is presented by

$$x = x(\psi,p) = \int \{(\xi_\psi \, \xi_{\psi p} + \tau)d\psi + \xi_\psi \, \xi_{pp} \, dp\}, \quad t = \xi_p(\psi,p), \; u = \xi_\psi(\psi,p). \tag{4}$$

For a given ψ the first two equations present the trajectory of a gas particle in the (x, t)-plane parametrically in terms of the pressure p; the remaining equation provides the velocity u at each point of the trajectory, and the density may be obtained from $\tau = \tau(\psi, p)$.

The arbitrariness in the function $\tau(\psi, p)$ entering into (3) may be limited by further hypotheses of a physical nature.

To begin with, let us suppose that the gas is polytropic with the caloric equation of state [**1**, p. 10]

$$p = e^{(S-S_0)/c_v} \cdot \tau^{-\gamma},$$

which we write in the form

$$\tau = e^{(S-S_0)/c_p} \cdot p^{-n}, \qquad n = 1/\gamma.$$

Secondly, from the hypothesis of conservation of energy it follows that the specific entropy S is constant along a trajectory [**1**, pp. 15-16], i.e., $S = S(\psi)$, the function $S(\psi)$ being termed the *entropy distribution function*. Thus the arbitrary function $\tau(\psi, p)$ must take the special form

$$\tau = \delta(\psi)p^{-n}, \qquad n = 1/\gamma, \tag{5}$$

where $\delta(\psi)$ is an arbitary function, determined by the choice of the entropy distribution function.

2. Progressive condensation shock. For a condensation shock, carrying in back of it the values u, τ, p of the velocity, specific volume, and pressure, traveling into a quiet atmosphere in which these quantities have the constant values u_0, τ_0, p_0, the shock conditions take the form [**3**, pp. 512-513]

[1]Although I have not investigated the point, I suspect that flows with this property form a very restricted class.

[2]This equation was obtained earlier by the author as a minor formal modification of a method developed for the treatment of steady plane flows. See [**2**, pp. 149–150].

$$(6)\qquad \begin{aligned} &\text{(i)} \quad u = u_0 + \sqrt{(p - p_0)(\tau_0 - \tau)}, \\ &\text{(ii)} \quad \dot{x} = u_0 + \tau_0\sqrt{(p - p_0)/(\tau_0 - \tau)}, \\ &\text{(iii)} \quad \frac{p}{p_0} = \frac{(\gamma + 1)\tau_0 - (\gamma - 1)\tau}{(\gamma + 1)\tau - (\gamma - 1)\tau_0}, \end{aligned}$$

where $\dot{x}$ denotes the velocity of propagation of the shock with respect to a fixed plane.

We shall prove the following theorem for a polytropic gas.

THEOREM. *Once the entropy distribution function $S(\psi)$ is specified, the determination of a condensation shock moving into a quiet atmosphere (and the flow immediately in back of it) reduces to the solution of a problem of Cauchy for the Monge-Ampère partial differential equation*

$$\xi_{\psi\psi}\,\xi_{pp} - \xi_{\psi p}^2 + n\delta(\psi)p^{-n-1} = 0.$$

If we substitute τ from (5) into (6 iii), we find

$$(7)\qquad \mu^2\tau_0\, p^{n+1} - \delta(\psi)p + p_0\,\tau_0\, p^n - \mu^2\delta(\psi)p_0 = 0, \qquad \mu^2 = (\gamma - 1)/(\gamma + 1).$$

This equation defines p as a function of ψ and the curve $p = p(\psi)$ in the (ψ, p)-plane is termed the *carrier*.

From (4) and (6 i) the values of ξ_ψ are prescribed along the carrier by

$$\xi_\psi = u_0 + \sqrt{\{\tau_0 - \delta(\psi)p^{-n}(\psi)\}\{p(\psi) - p_0\}}.$$

To obtain the values prescribed for ξ_p along the carrier, we employ (6 ii). First we observe, from (4), that

$$\dot{x} = \xi_\psi + \tau\Big/\frac{d\xi_p}{d\psi},$$

and consequently, from (6 ii),

$$\frac{d\xi_p}{d\psi} = \sqrt{\{\tau_0 - \delta(\psi)p^{-n}(\psi)\}/\{p(\psi) - p_0\}},$$

to determine ξ_p along the carrier up to an arbitrarily additive constant. Of the three shock conditions (6), the last serves to determine the carrier in the (ψ, p)-plane, and the first two provide the required Cauchy data upon the carrier.

If $\xi = \xi(\psi, p)$ is a solution of this problem of Cauchy, the shock curve in the (x, t)-plane and the flow behind it is obtained, at least locally, by mapping one side of the neighbourhood of the carrier in the (ψ, p)-plane on to the (x, t)-plane by the first two formulae in (4).

REFERENCES

1. R. Courant and K. O. Friedrichs, *Supersonic flow and shock waves* (New York, 1948).
2. M. H. Martin, *A new approach to problems in two dimensional flow*, Quart. Appl. Math., *8* (1950).
3. B. Riemann and H. Weber, *Die partiellen Differentialgleichungen der mathematischen Physik*, vol. 2, sixth ed. (Braunschweig, 1919).

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