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COMPLEX APPROXIMATION AND SIMULTANEOUS INTERPOLATION ON CLOSED SETS

P. M. GAUTHIER AND W. HENGARTNER

Let f be a complex-valued function defined on a closed subset F of the finite complex plane $\mathbf{C}$, and let $\{z_n\}$ be a sequence on F without limit points. We wish to find an analytic function g which simultaneously approximates f uniformly on F and interpolates f at the points $\{z_n\}$.

Let E denote the set of entire functions, $E|F$ the restriction of E to F, and $\bar{E}(F)$ the uniform closure of $E|F$. $A(F)$ denotes, as usual the set of functions continuous on F and holomorphic on the interior F^0. We denote by $\mathscr{M}$ the set of functions meromorphic on $\mathbf{C}$ and by $\bar{\mathscr{M}}(F)$ the uniform limits on F of functions in $\mathscr{M}$ having no poles on F. Let $\bar{\mathbf{C}}$ denote the closed plane $\mathbf{C} \cup \{\infty\}$. Arakeljan [**1**] has shown that $\bar{E}(F) = A(F)$ if and only if $\mathbf{C}\backslash F$ is connected and locally connected. The analogous problem for meromorphic approximation was solved by Nersesian [**7**] and Roth (**9**). On the other hand, by well known uniqueness theorems, further conditions must be imposed in order to simultaneously interpolate on an infinite sequence $\{z_n\}$. The condition we shall impose is that $\{z_n\}$ "avoids" the interior of F.

In this direction L. Hoischen [**6**] shows that if f is infinitely differentiable on the real line, then one can simultaneously approximate f and finitely many derivatives, as well as interpolate them at a sequence $\{x_n\}$ without finite limit points.

Our approach is somewhat different in that we shall not require differentiability of the function f which is to be approximated. Nevertheless, we require that finitely many derivatives of the approximating function g have preassigned values at given points. In this respect, our results resemble more closely those of Rubel and Venkateswaran [**10**] rather than those of Hoischen.

In seeking to specify the derivatives of the approximating function while the function to be approximated is not necessarily differentiable, we are reminded of the well known phenomenon that on a closed interval an interpolating sequence of polynomials may fail to converge, yet convergence can be achieved by specifying values for the derivatives of the polynomials at certain points.

Our principal result is the following:

THEOREM. *Let F be a closed subset of $\mathbf{C}$ such that $\bar{E}(F) = A(F)$ and $\{z_n\}$ a sequence in $F\backslash(F^0)^-$ without finite limit points. Then for each $f \in A(F)$, $\epsilon > 0$,*

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and arbitrarily given complex numbers $w_n^{(\nu)}$, $\nu = 1, 2, \ldots, \nu(n)$; $n = 1, 2, \ldots,$ *there is a* $g \in E$ *such that*

(1) $|f(z) - g(z)| < \epsilon, \quad z \in F,$

(2) $g(z_n) = f(z_n), \quad n = 1, 2, \ldots,$ *and*

(3) $g^{(\nu)}(z_n) = w_n^{(\nu)}, \quad \nu = 1, 2, \ldots, \nu(n), n = 1, 2, \ldots.$

Rubel and Venkateswaran [**10**] obtain this result with the additional assumptions that $F^0 = \emptyset$ and $\nu(n)$ is bounded.

The proof we shall give of our theorem can be modified to yield many similar theorems on simultaneous approximation and interpolation. We list some of these without proof.

1. Instead of considering entire functions one can consider meromorphic functions. That is, if $\overline{\mathcal{M}}(F) = A(F)$, similar conclusions hold where g of course is in $\mathcal{M}$.

2. Instead of working with E and $\mathcal{M}$ one can let F be a (relatively) closed subset of a plane domain G. Then one considers the possibility of simultaneous approximation and interpolation by functions holomorphic (meromorphic) on G. One obtains similar results.

3. Roth [**8**] has considered the possibility of approximation within ϵ by meromorphic functions, where ϵ is a positive continuous function. Our techniques also allow simultaneous interpolation in this context.

Before proving our theorem we state some lemmas. The well known Walsh lemma has been generalized by various authors. The following version is due to Deutsch [**3**]:

LEMMA 1. *Let* Y *be a dense linear subspace of a topological vector space* X. *Then for every* $x \in X$, *neighbourhood* U *of* x, *and* $T_1, T_2, \ldots, T_n \in X'$ *(topological dual), there is a* $y \in Y$ *such that:*

$$y \in U \quad \text{and} \quad T_j(x) = T_j(y), \quad j = 1, 2, \ldots, n.$$

Let ϵ be a positive continuous function on F. We shall say that $E|F$ is ϵ-*dense* in $A(F)$ provided that for each $f \in A(F)$, there is a $g \in E$ such that:

$$|f(z) - g(z)| < \epsilon(z), \quad z \in F.$$

Note that if $E|F$ is ϵ-dense in $A(F)$ and t is a positive constant, then $E|F$ is also $t \cdot \epsilon$-dense in $A(F)$. Let us now denote by $A_\epsilon(F)$, the set of $f \in A(F)$ such that

$$||f||_\epsilon = \sup_F |f(z)/\epsilon(z)|$$

is finite. Then $A_\epsilon(F)$ is a normed space. Moreover, we endow $A(F)$ with the compact open topology. The corresponding topological dual spaces are denoted by $A_\epsilon'(F)$ and $A'(F)$ respectively.

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LEMMA 2. *Suppose Y is an ϵ-dense subspace of $A(F)$ and $T_j \in A'(F)$, $j = 1, 2, \ldots, n$. Then for each $f \in A(F)$, there is a $g \in Y$ such that:*

$$|f(z) - g(z)| < \epsilon(z), \quad z \in F,$$

and

$$T_j(f) = T_j(g), \quad j = 1, 2, \ldots, n.$$

Proof. There is an $h \in Y$ so that: $|f(z) - h(z)| < \epsilon(z)$, $z \in F$. Hence, $f - h \in A_\epsilon(F)$. Since $A'(F) \subset A_\epsilon'(F)$, $T_j \in A_\epsilon'(F)$. Also $Y \cap A_\epsilon(F)$ is dense in $A_\epsilon(F)$. So by the Walsh Lemma 1, there is a $g_0 \in Y$ such that:

$$|g_0(z) - f(z) + h(z)| < \epsilon(z), z \in F, \quad \text{and}$$
$$T_j(g_0) = T_j(f - h), \quad j = 1, 2, \ldots, n.$$

Thus $g = g_0 + h$ has the required properties.

In the following lemma, we denote by $\bar{R}(K)$, the uniform closure on K of rational functions having no poles on K. Of course, for K compact, $\bar{R}(K)$ and $\bar{M}(K)$ coincide. We recall that a *Lyapunov domain* is a domain whose boundary in finitely many Lyapunov arcs whose interiors are disjoint. For the definition of a *Lyapunov arc* see [**11**]. For the application of Lemmas 3 and 4 that we have in mind, it is sufficient to think of a Lyapunov arc as a circular arc.

LEMMA 3. *Let K be a compact subset of the extended plane and let D be a Lyapunov domain. Then,*

$$A(K) = \bar{R}(K) \Rightarrow A(K \cup \bar{D}) = \bar{R}(K \cup \bar{D}).$$

Proof. Our proof is modeled on [**4**, p. 52]. Let $\Gamma = \Gamma^1 \cup \Gamma^2 \cup \ldots \cup \Gamma^n$, be the boundary of D, where Γ^j are Lyapunov arcs with disjoint interiors.

Let $f \in A(K \cup \bar{D})$ and let $\epsilon > 0$. According to Vitushkin [**11**, Lemma 1, p. 185], f can be approximated uniformly within ϵ/n by a continuous function f_1 which is holomorphic in $K^0 \cup D \cup \Gamma_\epsilon^1$, where Γ_ϵ^1 is a neighbourhood of Γ^1. Again by Vitushkin, f_1 may be approximated uniformly within ϵ/n by a continuous function f_2 which is holomorphic in

$$K^0 \cup D \cup \Gamma_\epsilon^1 \cup \Gamma_\epsilon^2,$$

where Γ_ϵ^2 is a neighbourhood of Γ^2. Repeating this argument finitely many times, we see that f may be approximated uniformly within ϵ by a continuous function f_n which is holomorphic on

$$K^0 \cup D \cup \Gamma_\epsilon^1 \cup \ldots \cup \Gamma_\epsilon^n.$$

Now by the Bishop localization theorem (see [**4**, p. 51]),

$$f_n|K \cup \bar{D} \in \bar{R}(K \cup \bar{D}).$$

We have used here the assumption that $A(K) = \bar{R}(K)$. Thus f may be approximated uniformly on $K \cup \bar{D}$ within ϵ by a rational function. This proves the lemma.

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LEMMA 4. *Let F be a closed subset of* $\mathbf{C}$ *and D a Lyapunov domain. Then*

$$A(F) = \bar{M}(F) \Rightarrow A(F \cup \bar{D}) = \bar{M}(F \cup \bar{D}).$$

Proof. From the necessity part of the Nersesian-Roth theorem ([**7**] and [**9**]),

$$(*)\ A(F) = \bar{M}(F) \Rightarrow A(F \cap K) = \bar{R}(F \cap K) \quad \text{for each closed disc } K.$$

Now from the sufficiency of this same theorem, all we have to show is that

$$A[(F \cup \bar{D}) \cap K] = \bar{R}[(F \cup \bar{D}) \cap K]$$

for each closed disc K. Since

$$(F \cup \bar{D}) \cap K = (F \cap K) \cup (\bar{D} \cap K),$$

this follows from (*) and Lemma 3, since $\bar{D} \cap K$ is a Lyapunov domain.

We now prove our theorem. Let p_n be a polynomial with:

(4) $\quad p_n(z_n) = f(z_n)$ and

(5) $\quad p_n^{(\nu)}(z_n) = w_n^{(\nu)}, \quad \nu = 1, 2, \ldots, \nu(n),$

for $n = 1, 2, \ldots$, and let K_k, $k = 1, 2, \ldots$ be an exhaustion of C by closed discs: $\mathbf{C} = \bigcup_1^\infty K_k$, $K_k \supset \overset{0}{K}_{k+1}$, such that K_1 and each ∂K_k meet no z_n and such that each point $a \in C \backslash (F \cup K_{k+1})$ can be connected to ∞ by an arc $\sigma(a)$ in $C \backslash (F \cup K_k)$. This is possible by Arakeljan's theorem [**1**].

The desired function g will be the limit of a sequence $g_k \in \mathscr{M}$ which will be "constructed" inductively to satisfy, for $k = 1, 2, \ldots$: g_k has no poles on K_{k-1} and

(6) $\quad |g_k(z) - g_{k-1}(z)| < \epsilon \cdot 2^{-k}, \quad z \in K_{k-2},$

(7) $\quad |g_k(z) - f(z)| < \epsilon \sum_1^k 2^{-j}, \quad z \in F,$

(8) $\quad g_k(z_n) = f(z_n)$, and

(9) $\quad g_n^{(\nu)}(z_n) = w_n^{(\nu)},$

for $\nu = 1, 2, \ldots, \nu(n)$, and z_n in K_k.

In (6) we shall set $K_{-1} = K_0 = \emptyset$ and $g_0 = g_1$, where g_1 is found as follows. By hypothesis, there is an entire function g_1, with $|g_1(z) - f(z)| < \epsilon \cdot 2^{-1}$, $z \in F$ and hence g_1 satisfies (7). The conditions (6), (8) and (9) are vacuously satisfied. Using Lemma 2, it is also easy to construct g_2.

Suppose g_j meromorphic, for $j = 1, \ldots, k-1$, have been found satisfying (6), to (9). We shall now find g_k. First we construct an auxiliary function ϕ (which depends on k): On $K_{k-1} \cup (F^0)^-$, set $\phi = g_{k-1}$. Now about each z_n in $(K_k \backslash K_{k-1})$, we construct disjoint closed discs D_n, $D_n \subset K_k \backslash (K_{k-1} \cup (F^0)^-)$. These discs can be chosen so small by (4) that: $|p_n(z) - f(z)| < \epsilon \sum_1^{k-1} 2^{-j}$, $z \in F \cap D_n$. Now set $\phi = p_n$ on such D_n. Set $\psi = \phi - f$. By Tietze's theorem

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we can extend ψ and hence ϕ so that $\phi \in A(F_k)$, and $|\phi(z) - f(z)| < \epsilon \sum_1^{k-1} 2^{-j}$, $z \in F$. where

$$F_k = F \cup K_{k-1} \cup \bigcup \{D_n : z_n \in K_k \backslash K_{k-1}\}.$$

From Lemma 4, it follows that there is a $\gamma \in \mathscr{M}$ (γ depends on k) for which:

$$|\gamma(z) - \phi(z)| < \epsilon \cdot 2^{-(k+1)}, \quad z \in F_k.$$

Moreover, by Lemma 2, where Y is the set of $f \in \mathscr{M}$ with no poles on F and $\epsilon(z) \equiv \epsilon$, we may assume that

$$\gamma(z_n) = \phi(z_n) \quad \text{and}$$
$$\gamma^{(\nu)}(z_n) = \phi^{(\nu)}(z_n), \quad \nu = 1, \ldots, \nu(n), \text{ for } z_n \in K_k.$$

From the way in which ϕ was constructed, it follows that γ satisfies

(10) $|\gamma(z) - g_{k-1}(z)| < \epsilon \cdot 2^{-(k+1)}, \quad z \in K_{k-1},$

(11) $|\gamma(z) - f(z)| < \epsilon\left[\sum_1^{k-1} 2^{-j} + 2^{-(k+1)}\right], \quad z \in F,$

(12) $\gamma(z_n) = f(z_n)$, and

(13) $\gamma^{(\nu)}(z_n) = w_n{}^{(\nu)}, \quad \nu = 1, 2, \ldots, \nu(n)$, for $z_n \in K_k$.

Now γ has only finitely many poles $a_1, \ldots, a_\mu$ in K_k and they all lie in $K_k \backslash (K_{k-1} \cup F)$. From our choice of the exhaustion $\{K_j\}_{j=1}^\infty$, we may construct a path $\sigma(a_j)$ from a_j to ∞ which misses $F \cup K_{k-2}$ and each z_n. In fact we may construct disjoint, open, simply connected neighbourhoods S_j of $\sigma(a_j)$ respectively which miss $F \cup K_{k-2}$ and whose closures miss each z_n. Let $r = r_k$ be the principal part of γ on K_k. Then there is an entire function $h = h_k$ such that

$$|r - h| < \epsilon\, 2^{-(k+1)}, \quad z \notin \bigcup_{j=1}^{\mu} S_j,$$

$$h(z_n) = r(z_n), \quad \text{and}$$

$$h^{(\nu)}(z_n) = r^{(\nu)}(z_n),$$

for $\nu = 1, \ldots, \nu(n)$, and $z_n \in K_k$. Now set $g_k = \gamma - r + h$. Then g_k satisfies (6), (7), (8), and (9).

Thus $\{g_k\}$ is inductively "constructed". From (6), and since g_k has no poles on K_{k-1}, we see that g_k converges to some $g \in E$, and from (7), (8) and (9) we see that g satisfies (1), (2), and (3). This completes the proof of the theorem.

On open Riemann surfaces, the usual conditions for uniform approximation are still necessary, but they are not sufficient [**5**]. However if $F^0 = \emptyset$, then the same conditions as in the planar case prevail and even allow tangential approximation. In the planar case this is a corollary of Arakeljan's theorem. However on Riemann surfaces, a direct proof is required (using the Bishop-Mergelian theorem [**2**] for example.) In this context ($F^0 = \emptyset$) Carleman

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approximation and simultaneous interpolation is again possible, but of course (3) must be formulated in terms of charts.

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Université de Montréal,
Montréal, Québec;
Université Laval,
Québec, Québec

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