

## CENTRALLY SYMMETRIC CONFIGURATIONS OF INTEGER MATRICES

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**Abstract.** The concept of centrally symmetric configurations of integer matrices is introduced. We study the problem when the toric ring of a centrally symmetric configuration is normal and when it is Gorenstein. In addition, Gröbner bases of toric ideals of centrally symmetric configurations are discussed. Special attention is given to centrally symmetric configurations of unimodular matrices and to those of incidence matrices of finite graphs.

A *configuration* of  $\mathbb{R}^d$  is a matrix  $A \in \mathbb{Z}^{d \times n}$ , where  $n = 1, 2, \dots$ , for which there exists a hyperplane  $\mathcal{H} \subset \mathbb{R}^d$  not passing the origin of  $\mathbb{R}^d$  such that each column vector of  $A$  lies on  $\mathcal{H}$ . Let  $K$  be a field, and let  $K[\mathbf{t}, \mathbf{t}^{-1}] = K[t_1, t_1^{-1}, \dots, t_d, t_d^{-1}]$  be the Laurent polynomial ring in  $d$  variables over  $K$ . Each column vector  $\mathbf{a} = [a_1, \dots, a_d]^\top \in \mathbb{Z}^d (= \mathbb{Z}^{d \times 1})$ , where  $[a_1, \dots, a_d]^\top$  is the transpose of  $[a_1, \dots, a_d]$ , yields the Laurent monomial  $\mathbf{t}^{\mathbf{a}} = t_1^{a_1} \cdots t_d^{a_d}$ . Let  $A \in \mathbb{Z}^{d \times n}$  be a configuration of  $\mathbb{R}^d$ , with  $\mathbf{a}_1, \dots, \mathbf{a}_n$  its column vectors. The *toric ring* of  $A$  is the subalgebra  $K[A]$  of  $K[\mathbf{t}, \mathbf{t}^{-1}]$  which is generated by the Laurent monomials  $\mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_n}$ . Let  $K[\mathbf{x}] = K[x_1, \dots, x_n]$  be the polynomial ring in  $n$  variables over  $K$ , and define the surjective ring homomorphism  $\pi : K[\mathbf{x}] \rightarrow K[A]$  by setting  $\pi(x_i) = \mathbf{t}^{\mathbf{a}_i}$  for  $i = 1, \dots, n$ . We say that the kernel  $I_A \subset K[\mathbf{x}]$  of  $\pi$  is the *toric ideal* of  $A$ . Finally, we write  $\text{Conv}(A)$  for the convex hull of  $\mathbf{a}_1, \dots, \mathbf{a}_n$  in  $\mathbb{R}^d$ . Thus,  $\text{Conv}(A) \subset \mathbb{R}^d$  is an integral convex polytope, that is, a convex polytope all of whose vertices have integer coordinates. The theory of toric rings and toric ideals has application to, for example,

- triangulations of  $\text{Conv}(A)$ ,
- the Conti–Traverso algorithm for integer programming, and
- the Markov chain Monte Carlo method for contingency tables.

(For details, see [12].)

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Given a matrix  $A \in \mathbb{Z}^{d \times n}$ , which is not necessarily a configuration of  $\mathbb{R}^d$ , we introduce the *centrally symmetric configuration*  $A^\pm \in \mathbb{Z}^{(d+1) \times (2n+1)}$  of  $\mathbb{R}^{d+1}$  as follows:

$$A^\pm = \left[ \begin{array}{c|cc} 0 & & \\ \vdots & A & -A \\ \hline 0 & & \\ \hline 1 & 1 \cdots 1 & 1 \cdots 1 \end{array} \right].$$

In this article, we establish fundamental results on centrally symmetric configurations. In particular, we study centrally symmetric configurations of unimodular matrices and those of incidence matrices of finite graphs. First, in Section 1, by means of the notion of Hermite normal form of an integer matrix, we pay attention to the fact that the index  $[\mathbb{Z}^d : \mathbb{Z}A]$ , where  $\mathbb{Z}A$  is the abelian subgroup of  $\mathbb{Z}^d$  generated by the column vectors of an integer matrix  $A \in \mathbb{Z}^{d \times n}$  of rank  $d$ , is equal to the greatest common divisor of maximal minors of  $A$ . Moreover, if  $A \in \mathbb{Z}^{d \times n}$  is of rank  $d$ , then there exists a nonsingular matrix  $B$  such that  $A' = B^{-1}A$  is an integer matrix satisfying  $\mathbb{Z}A' = \mathbb{Z}^d$ . Second, in Section 2, we study the centrally symmetric configuration of a unimodular matrix. Two fundamental theorems will be given. Theorem 2.7 says that if  $A \in \mathbb{Z}^{d \times n}$  is a unimodular matrix, then there exists a reverse lexicographic order  $<$  on  $K[\mathbf{x}]$  such that  $\text{in}_<(I_{A^\pm})$  is a radical ideal. Thus, in particular,  $K[A^\pm]$  is normal. Moreover, Theorem 2.15 guarantees that the toric ring  $K[A^\pm]$  of the centrally symmetric configuration  $A^\pm$  of a unimodular matrix is Gorenstein.

On the other hand, we devote Sections 3 and 4 to the study on toric rings and toric ideals of centrally symmetric configurations of incidence matrices of finite graphs. Let  $G$  be a finite connected graph on the vertex set  $[d] = \{1, \dots, d\}$ , and suppose that  $G$  possesses no loop and no multiple edge. Let  $E(G) = \{e_1, \dots, e_n\}$  denote the edge set of  $G$ . Let  $\mathbf{e}_1, \dots, \mathbf{e}_d$  stand for the canonical unit coordinate vector of  $\mathbb{R}^d$ . If  $e = \{i, j\}$  is an edge of  $G$  with  $i < j$ , then the column vectors  $\rho(e) \in \mathbb{R}^d$  and  $\mu(e) \in \mathbb{R}^d$  are defined by  $\rho(e) = \mathbf{e}_i + \mathbf{e}_j$  and  $\mu(e) = \mathbf{e}_i - \mathbf{e}_j$ , respectively. Let  $A_G \in \mathbb{Z}^{d \times n}$  denote the matrix with column vectors  $\rho(e_1), \dots, \rho(e_n)$ , and let  $A_G^\pm \in \mathbb{Z}^{d \times n}$  denote the matrix with column vectors  $\mu(e_1), \dots, \mu(e_n)$ . Theorem 3.3 gives a combinatorial characterization on  $G$  for which  $K[A_G^\pm]$  is normal, as well as for which  $I_{A_G^\pm}$  has a square-free initial ideal. Theorem 4.4 supplies quadratic Gröbner bases of the toric ideals of the centrally symmetric configuration of a bipartite graph any of whose cycles of length greater than or equal to 6 has a

chord. Finally, we conclude this article with several examples of centrally symmetric configurations of the incidence matrices of nonbipartite graphs.

**§1. The index  $[\mathbb{Z}^d : \mathbb{Z}A]$**

Let  $A = [\mathbf{a}_1 \cdots \mathbf{a}_n] \in \mathbb{Z}^{d \times n}$  be a matrix of rank  $d$ , and let  $\mathbb{Z}A$  denote the abelian subgroup of  $\mathbb{Z}^d$  generated by the column vectors of  $A$ ; that is,

$$\mathbb{Z}A := \left\{ \sum_{i=1}^n z_i \mathbf{a}_i \mid z_i \in \mathbb{Z}, i = 1, 2, \dots, n \right\}.$$

Let  $[B \mid O]$  be the *Hermite normal form* [11, p. 45] of  $A$  which is obtained by a series of elementary unimodular column operations:

- (i) exchanging two columns,
- (ii) multiplying a column by  $-1$ , and
- (iii) adding an integral multiple of one column to another column.

Here  $B$  is a nonsingular, lower-triangular, nonnegative integer matrix, in which each row has a unique maximum entry, which is located on the main diagonal of  $B$ . As stated in [11, Proof of Corollary 4.1b], we have  $\mathbb{Z}A = \mathbb{Z}B$ . Moreover, since the g.c.d. of maximal minors is invariant under elementary unimodular column operations, the greatest common divisor of maximal minors of  $A$  is equal to  $|B|$ . Since  $[\mathbb{Z}^d : \mathbb{Z}B] = |B|$ , we have the following propositions.

**PROPOSITION 1.1.** *Let  $A \in \mathbb{Z}^{d \times n}$  be a matrix of rank  $d$ . Then the index  $[\mathbb{Z}^d : \mathbb{Z}A]$  equals the greatest common divisor of maximal minors of  $A$ .*

**PROPOSITION 1.2.** *Let  $A \in \mathbb{Z}^{d \times n}$  be a matrix of rank  $d$ . Then there exists a nonsingular matrix  $B$  such that  $A' = B^{-1}A$  is an integer matrix satisfying  $\mathbb{Z}A' = \mathbb{Z}^d$ .*

Since the centrally symmetric configuration  $A^\pm$  of a matrix  $A \in \mathbb{Z}^{d \times n}$  is brought into

$$\left[ \begin{array}{c|cc} 0 & & \\ \vdots & A & O \\ 0 & & \\ \hline 1 & 0 \cdots 0 & 0 \cdots 0 \end{array} \right]$$

by a series of elementary unimodular column operations, we have the following.

PROPOSITION 1.3. *Let  $A \in \mathbb{Z}^{d \times n}$  be a matrix of rank  $d$ . Then the index  $[\mathbb{Z}^{d+1} : \mathbb{Z}A^\pm]$  equals the index  $[\mathbb{Z}^d : \mathbb{Z}A]$ . In particular,  $\mathbb{Z}A^\pm = \mathbb{Z}^{d+1}$  if and only if  $\mathbb{Z}A = \mathbb{Z}^d$ .*

An integer matrix  $A \in \mathbb{Z}^{d \times n}$  of rank  $d$  is called *unimodular* if all nonzero maximal minors of  $A$  have the same absolute value. Let  $\delta(A)$  denote the absolute value of a nonzero maximal minor of a unimodular matrix  $A$ . For unimodular matrices, Propositions 1.1 and 1.2 are as follows.

COROLLARY 1.4. *Let  $A \in \mathbb{Z}^{d \times n}$  be a unimodular matrix of rank  $d$ . Then the index  $[\mathbb{Z}^d : \mathbb{Z}A]$  equals  $\delta(A)$ . In particular,  $\mathbb{Z}A = \mathbb{Z}^d$  if and only if  $\delta(A) = 1$ .*

COROLLARY 1.5. *Let  $A \in \mathbb{Z}^{d \times n}$  be a unimodular matrix of rank  $d$ . Then there exists a nonsingular matrix  $B$  such that  $A' = B^{-1}A$  is a unimodular matrix of  $\delta(A') = 1$ .*

**§2. Centrally symmetric configurations of unimodular matrices**

Two fundamental results (Theorems 2.7 and 2.15) on centrally symmetric configurations of unimodular matrices will be established.

First, note that the centrally symmetric configuration  $A^\pm$  of a matrix  $A$  is not unimodular even if  $A$  is unimodular.

PROPOSITION 2.1. *Let  $A \in \mathbb{Z}^{d \times n}$  be a matrix of rank  $d$ . Then  $A^\pm \in \mathbb{Z}^{(d+1) \times (2n+1)}$  is not unimodular.*

*Proof.* Let  $A = [\mathbf{a}_1 \cdots \mathbf{a}_n] \in \mathbb{Z}^{d \times n}$ . Since  $\text{rank}(A) = d$ , there exists a nonsingular submatrix  $A' = [\mathbf{a}_{i_1} \cdots \mathbf{a}_{i_d}]$  of  $A$ . Then we have

$$\begin{vmatrix} \mathbf{0} & \mathbf{a}_{i_1} & \cdots & \mathbf{a}_{i_d} \\ 1 & 1 & \cdots & 1 \end{vmatrix} = (-1)^d |A'|,$$

$$\begin{vmatrix} -\mathbf{a}_{i_1} & \mathbf{a}_{i_1} & \cdots & \mathbf{a}_{i_d} \\ 1 & 1 & \cdots & 1 \end{vmatrix} = (-1)^{d+1} |A'|.$$

Since both of them are nonzero maximal minors of  $A^\pm$ ,  $A^\pm$  is not unimodular. □

EXAMPLE 2.2. The centrally symmetric configuration

$$A^\pm = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

of the (totally) unimodular matrix  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{Z}^{2 \times 2}$  is not unimodular.

The next lemma follows easily.

LEMMA 2.3. *Let  $A \in \mathbb{Z}^{d \times n}$  be arbitrary. Then the dimension of  $\text{Conv}(A^\pm) \subset \mathbb{R}^{d+1}$  is  $\text{rank}(A)$ , and  $[0, \dots, 0, 1]^\top \in \mathbb{R}^{d+1}$  belongs to the relative interior of  $\text{Conv}(A^\pm)$ .*

Let, in general,  $A \in \mathbb{Z}^{d \times n}$  be a configuration of  $\mathbb{R}^d$ , and let  $B \in \mathbb{Z}^{d \times m}$  with  $m \leq n$  be a submatrix (or subconfiguration) of  $A$ . We say that  $K[B]$  is a *combinatorial pure subring* (see [7]) of  $K[A]$  if there exists a face  $F$  of  $\text{Conv}(A)$  such that  $B = F \cap A$ .

LEMMA 2.4. *Suppose that  $A \in \mathbb{Z}^{d \times n}$  is a configuration. Then  $K[A]$  is a combinatorial pure subring of  $K[A^\pm]$ .*

Recall that  $K[A]$  is *normal* if and only if  $\mathbb{Z}_{\geq 0}A = \mathbb{Z}A \cap \mathbb{Q}_{\geq 0}A$  (see [12, Proposition 13.5]). Here  $\mathbb{Z}_{\geq 0}$  (resp.,  $\mathbb{Q}_{\geq 0}$ ) is the set of nonnegative integers (resp., nonnegative rational numbers). It is known (see [7, Proposition 1.2]) that if  $K[B]$  is a combinatorial pure subring of  $K[A]$ , and if  $K[A]$  is normal, then  $K[B]$  is normal.

COROLLARY 2.5. *Suppose that  $A \in \mathbb{Z}^{d \times n}$  is a configuration and that  $K[A^\pm]$  is normal. Then  $K[A]$  is normal.*

Example 2.6 stated below shows that the converse of Corollary 2.5 is false.

EXAMPLE 2.6. The toric ring  $K[A]$  of the configuration

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \in \mathbb{Z}^{2 \times 3}$$

is normal. However, the toric ring  $K[A^\pm]$  of the centrally symmetric configuration

$$A^\pm = \begin{bmatrix} 0 & 2 & 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 & -2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

of  $A$  is nonnormal. In fact,  $[1, -1, 1]^\top \in \mathbb{Z}A^\pm \cap \mathbb{Q}_{\geq 0}A^\pm$  does not belong to  $\mathbb{Z}_{\geq 0}A^\pm$ .

The first fundamental result on centrally symmetric configurations of unimodular matrices is as follows.

**THEOREM 2.7.** *Let  $A \in \mathbb{Z}^{d \times n}$  be a unimodular matrix. Then there exists a reverse lexicographic order  $<$  on  $K[\mathbf{x}]$  such that the initial ideal  $\text{in}_<(I_{A^\pm})$  of  $I_{A^\pm}$  with respect to  $<$  is a radical ideal.*

*Proof.* Let  $<$  be a reverse lexicographic order on  $K[\mathbf{x}]$  such that the smallest variable corresponds to the column vector  $[0, \dots, 0, 1]^\top$  of  $A^\pm$ . Let  $\Delta$  be a pulling triangulation [12, p. 67] of  $\text{Conv}(A^\pm)$  arising from  $<$ . By [12, Proposition 8.6], the vector  $[0, \dots, 0, 1]^\top$  is a vertex of an arbitrary facet (maximal simplex)  $\sigma$  of  $\Delta$ .

Let  $\mathbb{Z}\sigma$  be the abelian subgroup of  $\mathbb{Z}A^\pm$  spanned by the vertices of  $\sigma$ . Since  $\sigma$  is a facet of  $\Delta$ , the index  $[\mathbb{Z}A^\pm : \mathbb{Z}\sigma]$  is finite. The index  $[\mathbb{Z}A^\pm : \mathbb{Z}\sigma]$  is called the *normalized volume* of  $\sigma$ . We say that  $\Delta$  is *unimodular* if the normalized volume of each facet  $\sigma$  of  $\Delta$  is equal to 1. Recall that  $\Delta$  is unimodular if and only if  $\text{in}_<(I_{A^\pm})$  is a radical ideal (see [12, Corollary 8.9]).

Now, our work is to show that  $\Delta$  is a unimodular triangulation. Let  $\mathbf{a}_1, \dots, \mathbf{a}_n$  be the column vectors of  $A$ , and let

$$\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}, \begin{bmatrix} \varepsilon_1 \mathbf{a}_{i_1} \\ 1 \end{bmatrix}, \dots, \begin{bmatrix} \varepsilon_d \mathbf{a}_{i_d} \\ 1 \end{bmatrix} \quad (\varepsilon_i \in \{1, -1\})$$

be the vertices of  $\sigma$ . One has

$$\begin{aligned} \begin{vmatrix} \mathbf{0} & \varepsilon_1 \mathbf{a}_{i_1} & \cdots & \varepsilon_d \mathbf{a}_{i_d} \\ 1 & 1 & \cdots & 1 \end{vmatrix} &= (-1)^d |\varepsilon_1 \mathbf{a}_{i_1} \ \cdots \ \varepsilon_d \mathbf{a}_{i_d}| \\ &= (-1)^d \varepsilon_1 \cdots \varepsilon_d |\mathbf{a}_{i_1} \ \cdots \ \mathbf{a}_{i_d}|. \end{aligned}$$

Since  $\sigma$  is a simplex, the above determinant cannot be zero. Hence, its absolute value is equal to  $\delta(A)$ . Thanks to Proposition 1.3 and Corollary 1.4, we have

$$[\mathbb{Z}^{d+1} : \mathbb{Z}\sigma] = \delta(A) = [\mathbb{Z}^d : \mathbb{Z}A] = [\mathbb{Z}^{d+1} : \mathbb{Z}A^\pm].$$

Consequently, one has  $\mathbb{Z}A^\pm = \mathbb{Z}\sigma$ . In other words, the normalized volume of  $\sigma$  is equal to 1. Hence,  $\Delta$  is a unimodular triangulation of  $\text{Conv}(A^\pm)$ , as required. □

By virtue of [12, Proposition 13.15], we have the following corollary.

**COROLLARY 2.8.** *If  $A \in \mathbb{Z}^{d \times n}$  is unimodular, then the toric ring  $K[A^\pm]$  of  $A^\pm$  is normal.*

Note that, even if  $A \in \mathbb{Z}^{d \times n}$  satisfies that

- every facet of  $\text{Conv}(A)$  has a unimodular triangulation, and
- $[0, \dots, 0, 1]^\top \in \mathbb{R}^d$  is a unique integer point belonging to the relative interior of  $\text{Conv}(A)$ ,

$K[A]$  is not necessarily normal.

EXAMPLE 2.9. Let  $A \in \mathbb{Z}^{4 \times 5}$  be the configuration

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

and let  $\mathcal{P} = \text{Conv}(A)$ . Then  $\mathcal{P}$  is a tetrahedron with  $\mathcal{P} \cap \mathbb{Z}^4 = A$ , and  $[0, 0, 0, 1]^\top$  is a unique integer point belonging to the relative interior of  $\mathcal{P}$ . Each facet  $\mathcal{F}$  of  $\mathcal{P}$  has a trivial unimodular triangulation since  $\mathcal{F}$  is a 2-simplex. However,  $K[A]$  is not normal since  $[1, 1, 1, 2]^\top \in \mathbb{Z}A \cap \mathbb{Q}_{\geq 0}A$  does not belong to  $\mathbb{Z}_{\geq 0}A$ . Hence, in particular,  $\mathcal{P}$  has no unimodular triangulations.

We now turn to the problem of finding Gorenstein toric rings which arise from centrally symmetric configurations of integer matrices.

Let  $\mathcal{P} \subset \mathbb{R}^d$  be an arbitrary convex polytope of dimension  $d$  such that the origin of  $\mathbb{R}^d$  belongs to the interior of  $\mathcal{P}$ . Then the *dual polytope* of  $\mathcal{P} \subset \mathbb{R}^d$  is defined to be the convex polytope  $\mathcal{P}^* \subset \mathbb{R}^d$  which consists of those  $x \in \mathbb{R}^d$  with  $\langle x, y \rangle \leq 1$  for all  $y \in \mathcal{P}$ . Here  $\langle x, y \rangle$  is the usual inner product of  $\mathbb{R}^d$ . One has  $(\mathcal{P}^*)^* = \mathcal{P}$ .

An integral convex polytope  $\mathcal{P} \subset \mathbb{R}^d$  of dimension  $d$  is called a *Fano polytope* if the origin of  $\mathbb{R}^d$  is a unique integer point belonging to the interior of  $\mathcal{P}$ . We say that a Fano polytope  $\mathcal{P} \subset \mathbb{R}^d$  is *Gorenstein* if the vertices of the dual polytope  $\mathcal{P}^*$  of  $\mathcal{P}$  have integer coordinates.

REMARK 2.10. If  $\mathcal{P} \subset \mathbb{R}^d$  is an arbitrary convex polytope of dimension  $d$  such that the origin of  $\mathbb{R}^d$  belongs to the interior of  $\mathcal{P}$ , and if the dual polytope  $\mathcal{P}^*$  is integral, then the origin of  $\mathbb{R}^d$  is a unique integer point belonging to the interior of  $\mathcal{P}$ .

Let  $\mathcal{P} \subset \mathbb{R}^N$  be an integral convex polytope of dimension  $d$ , and let  $\mathcal{A} \subset \mathbb{R}^N$  be the affine subspace spanned by  $\mathcal{P}$ . One has an invertible affine transformation  $\psi : \mathcal{A} \rightarrow \mathbb{R}^d$  with  $\psi(\mathcal{A} \cap \mathbb{Z}^N) = \mathbb{Z}^d$ . It follows that  $\psi(\mathcal{P}) \subset \mathbb{R}^d$  is an integral convex polytope of dimension  $d$ . We say that  $\psi(\mathcal{P})$  is a *standard form* of  $\mathcal{P}$ . In general, by abuse of terminology, we say that an integral

convex polytope is a Gorenstein Fano polytope if one of its standard forms is a Gorenstein Fano polytope.

LEMMA 2.11. *Suppose that  $A \in \mathbb{Z}^{d \times n}$  is a unimodular matrix of rank  $d$  with  $\delta(A) = 1$ . Then the integral convex polytope  $\text{Conv}(A^\pm) \subset \mathbb{R}^{d+1}$  of dimension  $d$  is a Gorenstein Fano polytope.*

*Proof.* Let  $\mathcal{A}$  be the affine subspace of  $\mathbb{R}^{d+1}$  spanned by  $\text{Conv}(A^\pm)$ . In other words,  $\mathcal{A}$  is the hyperplane of  $\mathbb{R}^{d+1}$  defined by the equation  $z_{d+1} = 1$ . Let  $\psi : \mathcal{A} \rightarrow \mathbb{R}^d$  be the invertible affine transformation defined by

$$\psi(z_1, \dots, z_d, z_{d+1}) = (z_1, \dots, z_d).$$

It follows from Lemma 2.3 that the standard form  $\psi(\text{Conv}(A^\pm)) \subset \mathbb{R}^d$  contains the origin of  $\mathbb{R}^d$  in its interior. Let  $\mathcal{F}$  be a facet of  $\psi(\text{Conv}(A^\pm))$ , and let  $\sigma \subset \mathcal{F}$  be an arbitrary  $(d - 1)$ -simplex. Now, by virtue of the proof of Theorem 2.7, the vertices of  $\sigma$  are a  $\mathbb{Z}$ -basis of  $\mathbb{Z}^d$ . In particular, the equation of the facet  $\mathcal{F}$  is of the form  $a_1z_1 + \dots + a_dz_d = 1$  with each  $a_i \in \mathbb{Z}$ . In other words, the dual polytope of  $\psi(\text{Conv}(A^\pm))$  is integral. Hence,  $\psi(\text{Conv}(A^\pm))$  is a Gorenstein Fano polytope. □

EXAMPLE 2.12. Let  $A \in \mathbb{Z}^{2 \times 2}$  be the unimodular matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

with  $\delta(A) = 2$ . Then the convex hull of  $A^\pm$  is a Gorenstein Fano polytope.

EXAMPLE 2.13. Let  $A \in \mathbb{Z}^{3 \times 3}$  and  $B \in \mathbb{Z}^{3 \times 3}$  be the unimodular matrices

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

with  $\delta(A) = 2$  and  $\delta(B) = 1$ . One has

$$I_{A^\pm} = I_{B^\pm} = \langle x_1^2 - x_2x_5, x_1^2 - x_3x_6, x_1^2 - x_4x_7 \rangle.$$

Lemma 2.11 says that  $\text{Conv}(B^\pm)$  is a Gorenstein Fano polytope. However,  $\text{Conv}(A^\pm)$  cannot be a Gorenstein Fano polytope.

Let  $\mathcal{P} \subset \mathbb{R}^d$  be an integral convex polytope. For  $N = 1, 2, \dots$ , we define

$$N\mathcal{P} = \{N\alpha \in \mathbb{R}^d \mid \alpha \in \mathcal{P}\}.$$



LEMMA 2.14. *Let  $A \in \mathbb{Z}^{d \times n}$  be a configuration with  $\mathbb{Z}A = \mathbb{Z}^d$ , and suppose that  $K[A]$  is normal. Then, for each  $\alpha \in N \text{Conv}(A) \cap \mathbb{Z}^d$ , there exist  $\beta_1, \dots, \beta_N \in \text{Conv}(A) \cap \mathbb{Z}^d$  such that  $\alpha = \beta_1 + \dots + \beta_N$ .*

*Proof.* Let  $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$ , and suppose that  $\alpha$  belongs to  $N \text{Conv}(A) \cap \mathbb{Z}^d$ . Since  $\alpha$  belongs to  $N \text{Conv}(A)$ , one has  $\alpha = \sum_{i=1}^n r_i \mathbf{a}_i$ , where  $0 \leq r_i \in \mathbb{Q}$  and where  $\sum_{i=1}^n r_i = N$ . In particular,  $\alpha$  belongs to  $\mathbb{Z}^d \cap \mathbb{Q}_{\geq 0}A$ . Since  $\mathbb{Z}A = \mathbb{Z}^d$  and since  $K[A]$  is normal, one has  $\mathbb{Z}^d \cap \mathbb{Q}_{\geq 0}A = \mathbb{Z}A \cap \mathbb{Q}_{\geq 0}A = \mathbb{Z}_{\geq 0}A$ . Thus,  $\alpha = \sum_{i=1}^n r_i \mathbf{a}_i = \sum_{i=1}^n z_i \mathbf{a}_i$  with each  $z_i \in \mathbb{Z}_{\geq 0}$ . Since  $A$  is a configuration, it follows easily that  $\sum_{i=1}^n z_i = N$ , as required.  $\square$

We now come to the second fundamental result on centrally symmetric configurations of unimodular matrices.

THEOREM 2.15. *Suppose that  $A \in \mathbb{Z}^{d \times n}$  is a unimodular matrix of rank  $d$ . Then the toric ring  $K[A^\pm]$  of the centrally symmetric configuration  $A^\pm$  is Gorenstein.*

*Proof.* By virtue of Corollary 1.5, we may assume that  $\delta(A) = 1$ . Lemma 2.11 says that the integral convex  $\text{Conv}(A^\pm) \subset \mathbb{R}^{d+1}$  is a Gorenstein Fano polytope. Hence, [3, Corollary 1.2] guarantees that the Ehrhart ring [5, p. 97] of  $\text{Conv}(A^\pm)$  is Gorenstein. Since  $\delta(A) = 1$ , one has  $\mathbb{Z}A^\pm = \mathbb{Z}^{d+1}$ . Moreover, the toric ring  $K[A^\pm]$  is normal. Thus, by using Lemma 2.14, it follows that  $K[A^\pm]$  coincides with the Ehrhart ring of  $\text{Conv}(A^\pm)$ . Hence,  $K[A^\pm]$  is Gorenstein, as desired.  $\square$

EXAMPLE 2.16. Let  $A \in \mathbb{Z}^{2 \times 4}$  be the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix},$$

which is not unimodular. The toric ring  $K[A^\pm]$  of

$$A^\pm = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

is normal and Gorenstein.

### §3. Centrally symmetric configurations of finite graphs

Let  $G$  be a finite connected graph on the vertex set  $[d] = \{1, \dots, d\}$ , and suppose that  $G$  possesses no loop and no multiple edge. Let  $E(G) =$

$\{e_1, \dots, e_n\}$  denote the edge set of  $G$ . Let  $\mathbf{e}_1, \dots, \mathbf{e}_d$  stand for the canonical unit coordinate vector of  $\mathbb{R}^d$ . If  $e = \{i, j\}$  is an edge of  $G$  with  $i < j$ , then the column vectors  $\rho(e) \in \mathbb{R}^d$  and  $\mu(e) \in \mathbb{R}^d$  are defined by  $\rho(e) = \mathbf{e}_i + \mathbf{e}_j$  and  $\mu(e) = \mathbf{e}_i - \mathbf{e}_j$ , respectively. Let  $A_G \in \mathbb{Z}^{d \times n}$  denote the matrix with column vectors  $\rho(e_1), \dots, \rho(e_n)$ , and let  $A_{\vec{G}} \in \mathbb{Z}^{d \times n}$  denote the matrix with column vectors  $\mu(e_1), \dots, \mu(e_n)$ . Thus,  $A_G$  is a configuration of  $\mathbb{R}^d$ . However,  $A_{\vec{G}}$  is not necessarily a configuration of  $\mathbb{R}^d$ .

The  $(0, 1)$ -polytope  $\text{Conv}(A_G) \subset \mathbb{R}^d$  is called the *edge polytope* of  $G$  (see [8]). We say that  $\text{Conv}(A_G^\pm) \subset \mathbb{R}^{d+1}$  is the *symmetric edge polytope* of  $G$ . The symmetric edge polytope  $\text{Conv}(A_{\vec{G}}^\pm) \subset \mathbb{R}^{d+1}$  is of dimension  $d - 1$ . The dimension of  $\text{Conv}(A_G)$  is given in [8, Proposition 1.3]. Note that  $\dim(\text{Conv}(A_G^\pm)) = 1 + \dim(\text{Conv}(A_G))$ . If  $G$  is nonbipartite graph, then the dimension of  $\text{Conv}(A_G^\pm) \subset \mathbb{R}^{d+1}$  is  $d$ . If  $G$  is bipartite, then the dimension of  $\text{Conv}(A_G^\pm) \subset \mathbb{R}^{d+1}$  is  $d - 1$ .

Recall that an integer matrix is *totally unimodular* [11, p. 266] if every square submatrix has determinant 0, 1, or  $-1$ . In particular, every entry of a totally unimodular matrix belongs to  $\{0, 1, -1\}$ . It is known that  $A_{\vec{G}}$  is totally unimodular. In addition,  $A_G$  is totally unimodular if and only if  $G$  is a bipartite graph.

The following is based on Theorem 2.7 and Corollary 2.8.

**COROLLARY 3.1.** *Let  $G$  be a finite connected graph. Then there exists a reverse lexicographic order  $<$  on  $K[\mathbf{x}]$  such that  $\text{in}_{<}(I_{A_G^\pm})$  is a radical ideal. Thus, in particular,  $K[A_G^\pm]$  is normal.*

On the other hand, the following is known (see [8, Corollary 2.3]).

**PROPOSITION 3.2.** *Let  $G$  be a finite connected graph. Then  $K[A_G]$  is normal if and only if, for each two odd cycles  $C_1$  and  $C_2$  of  $G$  having no common vertex, there exists an edge of  $G$  which joins a vertex of  $C_1$  and a vertex of  $C_2$ .*

We now discuss when the toric ring  $K[A_G^\pm]$  is normal.

**THEOREM 3.3.** *Let  $G$  be a finite connected graph. Then the following conditions are equivalent:*

- (i)  $K[A_G^\pm]$  is normal,
- (ii)  $I_{A_G^\pm}$  has a square-free initial ideal,

- (iii)  $A_G$  is a unimodular matrix (by deleting a redundant row if  $G$  is bipartite),
- (iv) any two odd cycles of  $G$  possess a common vertex.

*Proof.* First, (ii)  $\implies$  (i) is known (see [12, Proposition 13.15]). Second, (iii)  $\iff$  (iv) is discussed in, for example, [4]. Third, (iii)  $\implies$  (ii) follows from Theorem 2.7.

Now, in order to show that (i)  $\implies$  (iv), suppose that  $K[A_G^\pm]$  is normal and that there exist two odd cycles  $C_1 = (i_1, \dots, i_r)$  and  $C_2 = (j_1, \dots, j_s)$  of  $G$  having no common vertex. Thanks to Corollary 2.5 together with Proposition 3.2, there exists an edge  $e$  of  $G$  which joins a vertex of  $C_1$  and a vertex of  $C_2$ . Let, say,  $e = \{i_r, j_s\}$ . Then

$$\begin{aligned} \alpha &= \sum_{k=1}^{\frac{r-1}{2}} (\mathbf{e}_{i_{2k-1}} + \mathbf{e}_{i_{2k}} + \mathbf{e}_{d+1}) + \sum_{\ell=1}^{\frac{s-1}{2}} (-\mathbf{e}_{j_{2\ell}} - \mathbf{e}_{j_{2\ell+1}} + \mathbf{e}_{d+1}) \\ &\quad + (-\mathbf{e}_{j_s} - \mathbf{e}_{j_1} + \mathbf{e}_{d+1}) + (\mathbf{e}_{i_r} + \mathbf{e}_{j_s} + \mathbf{e}_{d+1}) - \mathbf{e}_{d+1} \\ &= \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \dots + \mathbf{e}_{i_r} - \mathbf{e}_{j_1} - \mathbf{e}_{j_2} - \dots - \mathbf{e}_{j_s} + \frac{r+s}{2} \mathbf{e}_{d+1} \end{aligned}$$

belongs to  $\mathbb{Z}A_G^\pm$ . Let  $i_{r+1} = i_1$ , and let  $j_{s+1} = j_1$ . It then follows that

$$\alpha = \sum_{k=1}^r \frac{1}{2} (\mathbf{e}_{i_k} + \mathbf{e}_{i_{k+1}} + \mathbf{e}_{d+1}) + \sum_{\ell=1}^s \frac{1}{2} (-\mathbf{e}_{j_\ell} - \mathbf{e}_{j_{\ell+1}} + \mathbf{e}_{d+1}) \in \mathbb{Q}_{\geq 0} A_G^\pm.$$

Since  $K[A_G^\pm]$  is normal,  $\alpha$  belongs to  $\mathbb{Z}_{\geq 0} A_G^\pm$ . Thus,

$$\alpha = z_0 \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} + \sum_{i=1}^n z_i \begin{bmatrix} \rho(e_i) \\ 1 \end{bmatrix} + \sum_{j=1}^n z'_j \begin{bmatrix} -\rho(e_j) \\ 1 \end{bmatrix},$$

where  $0 \leq z_i, z'_j \in \mathbb{Z}$ . Since both  $r$  and  $s$  are odd and since  $\{i_1, \dots, i_r\} \cap \{j_1, \dots, j_s\} = \emptyset$ , it follows that  $(r+1)/2 \leq \sum_{i=1}^n z_i$  and that  $(s+1)/2 \leq \sum_{j=1}^n z'_j$ . Hence,

$$\frac{r+1}{2} + \frac{s+1}{2} \leq z_0 + \sum_{i=1}^n z_i + \sum_{j=1}^n z'_j = \frac{r+s}{2},$$

a contradiction. □

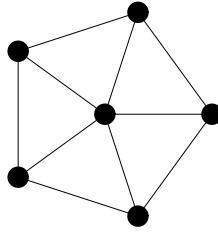


Figure 1: The wheel graph  $W_6$ .

It follows from the theory of totally unimodular matrices that the toric ring  $K[A_G^\pm]$  of the centrally symmetric configuration  $A_G^\pm$  of a finite connected graph  $G$  is Gorenstein. Moreover, if  $G$  is bipartite, then  $K[A_G^\pm]$  is Gorenstein.

**THEOREM 3.4.** *Let  $G$  be a finite connected graph, and suppose that any two odd cycles of  $G$  possesses a common vertex. Then the toric ring  $K[A_G^\pm]$  of the centrally symmetric configuration  $A_G^\pm$  is Gorenstein.*

*Proof.* By virtue of the equivalence of Theorem 3.3(iii) and (iv), it follows that the matrix  $A_G$  is unimodular. Hence, Theorem 2.15 guarantees that  $K[A_G^\pm]$  is Gorenstein, as desired.  $\square$

**EXAMPLE 3.5.** Let  $W_d$  be the wheel graph on  $[d]$ . For example,  $W_6$  is as shown in Figure 1.

Thanks to Theorems 3.3 and 3.4,  $K[A_{W_d}^\pm]$  is normal and Gorenstein. One can compute the Hilbert series

$$H(K[A_{W_d}^\pm], \lambda) = \sum_{j=0}^{\infty} \dim_K(K[A_{W_d}^\pm])_j \lambda^j$$

of  $K[A_{W_d}^\pm]$  by using the software CoCoA (see [2]). See Table 1.

**EXAMPLE 3.6.** Let  $G$  be the graph on the vertex set  $\{1, \dots, 6\}$  with the edge set

$$E(G) = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{4, 6\}\}.$$

By virtue of Theorem 3.3,  $K[A_G^\pm]$  is not normal. On the other hand, by CoCoA, one can check that  $K[A_G^\pm]$  is Cohen–Macaulay and not Gorenstein.

Table 1: The Hilbert series of the toric ring of the wheel graph  $W_d$ .

$d$	$(1 - \lambda)^{d+1} H(K[A_{W_d}^\pm], \lambda)$
4	$1 + 8\lambda + 14\lambda^2 + 8\lambda^3 + \lambda^4$
5	$1 + 11\lambda + 32\lambda^2 + 32\lambda^3 + 11\lambda^4 + \lambda^5$
6	$1 + 14\lambda + 65\lambda^2 + 104\lambda^3 + 65\lambda^4 + 14\lambda^5 + \lambda^6$
7	$1 + 17\lambda + 105\lambda^2 + 249\lambda^3 + 249\lambda^4 + 105\lambda^5 + 17\lambda^6 + \lambda^7$

CONJECTURE 3.7. *Let  $G$  be a finite connected graph. Then,  $K[A_G^\pm]$  is Gorenstein if and only if any two odd cycles of  $G$  possess a common vertex.*

QUESTION 3.8. *Let  $G$  be a finite connected graph such that  $A_G$  is not unimodular. What is the necessary and sufficient condition for  $K[A_G^\pm]$  to be Cohen–Macaulay?*

**§4. Centrally symmetric configurations of bipartite graphs**

In this section, we study toric ideals of centrally symmetric configurations arising from bipartite graphs.

PROPOSITION 4.1. *Let  $G$  be a bipartite graph. Then, we have  $I_{A_G^\pm} = I_{A_G^\mp}$ .*

*Proof.* If  $G$  is bipartite, the rows of  $A_G$  equal the rows of  $A_G^\rightarrow$  up to  $-1$  multiples. Hence, the rows of  $A_G^\pm$  equal the rows of  $A_G^\mp$  up to  $-1$  multiples. Thus, we have  $\text{Ker } A_G^\pm = \text{Ker } A_G^\mp$ . By [12, Corollary 4.3], it follows that  $I_{A_G^\pm} = I_{A_G^\mp}$ . □

The following is known from [9].

PROPOSITION 4.2. *Let  $G$  be a connected bipartite graph. Then the following conditions are equivalent:*

- (i) *every cycle of length  $\geq 6$  in  $G$  has a chord,*
- (ii)  *$I_{A_G}$  has a quadratic Gröbner basis,*
- (iii)  *$K[A_G]$  is Koszul, and*
- (iv)  *$I_{A_G}$  is generated by quadratic binomials.*

By Lemma 2.4, since  $A_G$  is a configuration,  $K[A_G]$  is a combinatorial pure subring of  $K[A_G^\pm]$ . It then follows from [6, Theorem 1.3] and [7, Proposition 1.1] that, if  $\mathcal{G}$  is the reduced Gröbner basis (resp., a set of binomial

generators) of  $I_{A_G^\pm}$ , and if  $I_{A_G}$  is an ideal of  $K[\mathbf{x}]$ , then  $\mathcal{G} \cap K[\mathbf{x}]$  is the reduced Gröbner basis (resp., a set of binomial generators) of  $I_{A_G}$ .

**COROLLARY 4.3.** *Let  $G$  be a connected bipartite graph. If  $G$  has a cycle of length greater than or equal to 6 having no chord, then  $I_{A_G^\pm}$  ( $= I_{A_G^\pm}$ ) is not generated by quadratic binomials.*

Let  $G$  be a connected bipartite graph on the vertex set  $\{1, \dots, p\} \cup \{1', \dots, q'\}$ . Suppose that every cycle of  $G$  of length greater than or equal to 6 has a chord. Then, by the same argument as in [9, Theorem], we may assume that  $G$  satisfies the condition

$$(*) \quad \{i, \ell'\}, \{j, k'\}, \{j, \ell'\} \in E(G) \implies \{i, k'\} \in E(G)$$

for each  $1 \leq i < j \leq p, 1 \leq k < \ell \leq q$ . Let  $\mathcal{R} = K[s_1^{\pm 1}, \dots, s_p^{\pm 1}, t_1^{\pm 1}, \dots, t_q^{\pm 1}, u]$ , and let  $K[A_G^\pm] = K[\{u\} \cup \{s_i t_j^{-1} u \mid \{i, j'\} \in E(G)\} \cup \{s_i^{-1} t_j u \mid \{i, j'\} \in E(G)\}] \subset \mathcal{R}$ . Let  $\mathcal{S} = K[\{x_{ij}\}_{\{i, j'\} \in E(G)} \cup \{y_{ij}\}_{\{i, j'\} \in E(G)} \cup \{z\}]$  be a polynomial ring over  $K$ . Then the toric ideal  $I_{A_G^\pm}$  ( $= I_{A_G^\pm}$ ) is the kernel of surjective homomorphism  $\varphi : \mathcal{S} \rightarrow K[A_G^\pm]$  defined by  $\varphi(x_{ij}) = s_i t_j^{-1} u, \varphi(y_{ij}) = s_i^{-1} t_j u$ , and  $\varphi(z) = u$ . Let  $<$  denote the reverse lexicographic order on  $\mathcal{S}$  induced by the ordering

$$z < y_{11} < x_{11} < y_{12} < x_{12} < \dots < y_{1q} < x_{1q} < y_{21} < x_{21} < \dots < y_{pq} < x_{pq}.$$

**THEOREM 4.4.** *Let  $G$  be a connected bipartite graph. Suppose that every cycle in  $G$  of length greater than or equal to 6 has a chord. Let  $\mathcal{G}$  be the set consists of the following binomials:*

$$\begin{aligned} & x_{ik}y_{ik} - z^2 \quad \{i, k'\} \in E(G), \\ & x_{il}x_{jk} - x_{ik}x_{j\ell} \quad \{i, k'\}, \{i, \ell'\}, \{j, k'\}, \{j, \ell'\} \in E(G), \quad i < j, k < \ell, \\ & x_{il}y_{j\ell} - x_{ik}y_{jk} \quad \{i, k'\}, \{i, \ell'\}, \{j, k'\}, \{j, \ell'\} \in E(G), \quad i < j, k < \ell, \\ & y_{j\ell}x_{jk} - x_{ik}y_{i\ell} \quad \{i, k'\}, \{i, \ell'\}, \{j, k'\}, \{j, \ell'\} \in E(G), \quad i < j, k < \ell, \\ & x_{j\ell}y_{jk} - y_{ik}x_{i\ell} \quad \{i, k'\}, \{i, \ell'\}, \{j, k'\}, \{j, \ell'\} \in E(G), \quad i < j, k < \ell, \\ & y_{i\ell}x_{j\ell} - y_{ik}x_{jk} \quad \{i, k'\}, \{i, \ell'\}, \{j, k'\}, \{j, \ell'\} \in E(G), \quad i < j, k < \ell, \\ & y_{i\ell}y_{jk} - y_{ik}y_{j\ell} \quad \{i, k'\}, \{i, \ell'\}, \{j, k'\}, \{j, \ell'\} \in E(G), \quad i < j, k < \ell, \end{aligned}$$

where the initial monomial of each binomial is the first monomial and square-free. Then  $\mathcal{G}$  is the reduced Gröbner basis of  $I_{A_G^\pm}$  with respect to  $<$ .

*Proof.* It is easy to see that  $\mathcal{G} \subset I_{A_{\frac{\pm}{G}}}$  and that the initial monomial of each binomial in  $\mathcal{G}$  is the first monomial and square-free.

Suppose that  $\mathcal{G}$  is not the reduced Gröbner basis of  $I_{A_{\frac{\pm}{G}}}$  with respect to  $<$ . Then there exists an irreducible binomial  $f = m_1 - m_2 \in I_{A_{\frac{\pm}{G}}}$  such that, for  $i = 1, 2$ ,  $m_i$  is not divisible by the initial monomial of any binomial in  $\mathcal{G}$ .

Suppose that the biggest variable appearing in  $m_1 m_2$  is  $x_{j\ell}$ . We may assume that  $m_1$  is divided by  $x_{j\ell}$  and that  $m_2$  is not divided by  $x_{j\ell}$ . Since  $m_1$  is not divided by  $x_{j\ell} y_{j\ell}$ ,  $m_1$  is not divided by  $y_{j\ell}$ . Let  $\varphi(m_1) = s_1^{\alpha_1} \dots s_p^{\alpha_p} t_1^{\beta_1} \dots t_q^{\beta_q} u^\gamma$ .

*Case 1:*  $m_1$  is divided by  $y_{jk}$  for some  $k < \ell$ .

Suppose that  $\{i, \ell'\} \in E(G)$  for some  $i < j$ . Thanks to condition  $(*)$ , we have  $\{i, k'\} \in E(G)$ . Hence,  $m_1$  is divided by the initial monomial of the binomial  $x_{j\ell} y_{jk} - y_{ik} x_{i\ell}$ , and this is a contradiction. Thus,  $\{i, \ell'\} \notin E(G)$  for all  $i < j$ . Then we have  $\beta_\ell < 0$ . Since  $\varphi(m_1) = \varphi(m_2)$ ,  $m_2$  is divided by  $x_{\lambda\ell}$  for some  $\lambda < j$ . Then,  $\{\lambda, \ell'\} \in E(G)$ , and this is a contradiction.

*Case 2:*  $m_1$  is divided by  $y_{i\ell}$  for some  $i < j$ .

Similar to Case 1, suppose that  $\{j, k'\} \in E(G)$  for some  $k < \ell$ . Thanks to condition  $(*)$ , we have  $\{i, k'\} \in E(G)$ . Hence,  $m_1$  is divided by the initial monomial of the binomial  $y_{i\ell} x_{j\ell} - y_{ik} x_{jk}$ , and this is a contradiction. Thus,  $\{j, k'\} \notin E(G)$  for all  $k < \ell$ . Then we have  $\alpha_j > 0$ . Since  $\varphi(m_1) = \varphi(m_2)$ ,  $m_2$  is divided by  $x_{j\mu}$  for some  $\mu < \ell$ . Then,  $\{j, \mu'\} \in E(G)$ , and this is a contradiction.

*Case 3:*  $m_1$  is not divided by  $y_{jk}$  for all  $k < \ell$  and not divided by  $y_{i\ell}$  for all  $i < j$ .

It then follows that  $\alpha_j > 0$  and  $\beta_\ell < 0$ . Since  $\varphi(m_1) = \varphi(m_2)$ ,  $m_2$  is divided by  $x_{i\ell}$  for some  $i < j$  and by  $x_{jk}$  for some  $k < \ell$ . Thanks to condition  $(*)$ , we have  $\{i, k'\} \in E(G)$ . Hence,  $m_2$  is divided by the initial monomial of the binomial  $x_{i\ell} x_{jk} - x_{ik} x_{j\ell}$ , and this is a contradiction.

On the other hand, if the biggest variable appearing in  $m_1 m_2$  is  $y_{j\ell}$ , then, by the similar argument (changing the role of  $x$  and  $y$ ) as above, a contradiction arises. □

**COROLLARY 4.5.** *Let  $G$  be a connected bipartite graph, and let  $A = A_{\frac{\pm}{G}}$ . Then the following conditions are equivalent:*

- (i) every cycle of length greater than or equal to 6 in  $G$  has a chord,
- (ii)  $I_A$  has a quadratic Gröbner basis,

- (iii)  $K[A]$  is Koszul,
- (iv)  $I_A$  is generated by quadratic binomials.

**§5. Examples of nonbipartite graphs**

In the previous section, we showed that, if  $G$  is a bipartite graph, then the following conditions are equivalent:

- (i)  $I_{A_G}$  is generated by quadratic binomials,
- (ii)  $I_{A_G^\pm}$  is generated by quadratic binomials,
- (iii)  $I_{A_G^\pm}$  is generated by quadratic binomials.

In this section, we study toric ideals of centrally symmetric configurations arising from nonbipartite graphs. Since  $K[A_G]$  is a combinatorial pure subring of  $K[A_G^\pm]$ , (ii)  $\implies$  (i) holds for a nonbipartite graph  $G$ . However, if  $G$  is not bipartite, then none of the other directions hold. (Computation below is done by CoCoA.)

EXAMPLE 5.1. Let  $G$  be a graph on the vertex set  $\{1, \dots, 6\}$  together with the edge set  $E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{1, 6\}, \{1, 3\}, \{1, 5\}, \{2, 6\}\}$ . Then,  $I_{A_G}$  has a Gröbner basis consists of three quadratic binomials. However, neither  $I_{A_G^\pm}$  nor  $I_{A_G^\pm}$  is generated by quadratic binomials. Thus, neither (i)  $\implies$  (ii) nor (i)  $\implies$  (iii) holds.

EXAMPLE 5.2. Let  $G$  be a complete 3-partite graph on the vertex set  $\{1, 2\} \cup \{3, 4\} \cup \{5, 6\}$ . Then,  $I_{A_G^\pm}$  is generated by quadratic binomials. However,  $I_{A_G^\pm}$  is not generated by quadratic binomials. Thus, (ii)  $\implies$  (iii) does not hold.

EXAMPLE 5.3. Let  $G$  be a graph on the vertex set  $\{1, \dots, 5\}$  together with the edge set  $E(G) = \{\{1, 5\}, \{3, 5\}, \{1, 3\}, \{2, 5\}, \{4, 5\}, \{2, 4\}\}$ . Then,  $I_{A_G^\pm}$  is generated by quadratic binomials. However,  $I_{A_G} = \langle x_1x_2x_6 - x_3x_4x_5 \rangle$ , and hence  $I_{A_G^\pm}$  is not generated by quadratic binomials. Thus, neither (iii)  $\implies$  (i) nor (iii)  $\implies$  (ii) holds.

REMARK 5.4. A combinatorial criterion for the toric ideal  $I_{A_G}$  of a finite connected graph  $G$  to be generated by quadratic binomials is given in [10, Theorem 1.2].

PROPOSITION 5.5. *Let  $G$  be a nonbipartite graph. Suppose that there exists a vertex  $v$  of  $G$  such that any odd cycle of  $G$  contains  $v$ . Then there exists a bipartite graph  $G'$  such that  $I_{A_G} = I_{A_{G'}}$  and  $I_{A_G^\pm} = I_{A_{G'}^\pm}$ .*



*Proof.* Let  $V(G) = \{1, 2, \dots, d\}$ . Suppose that any odd cycle of  $G$  contains the vertex  $d$ . Then the induced subgraph  $G''$  of  $G$  on the vertex set  $\{1, 2, \dots, d - 1\}$  is a bipartite graph. (Note that  $G''$  is not necessarily connected.) Let  $\{1, 2, \dots, d - 1\} = V_1 \cup V_2$  denote a partition of the vertex set of  $G''$ . Let  $G'$  be a bipartite graph on the vertex set  $\{1, 2, \dots, d, d + 1\}$  together with the edge set

$$E(G'') \cup \{\{i, d\} \in E(G) \mid i \in V_1\} \cup \{\{i, d + 1\} \mid i \in V_2, \{i, d\} \in E(G)\}.$$

It then follows that  $A_{G'}$  is brought into  $A_G$  by adding the  $(d + 1)$ th row to the  $d$ th row and deleting the  $(d + 1)$ th row, which is redundant. □

**EXAMPLE 5.6.** Let  $G$  be the nonbipartite graph in Example 5.3, and let  $G'$  be a cycle of length 6. Then  $G$  and  $G'$  satisfy the condition in Proposition 5.5. The corresponding matrices are

$$A_G = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad A_{G'} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

**EXAMPLE 5.7.** The wheel graph  $W_d$  on  $[d]$  satisfies the condition in Proposition 5.5 if and only if  $d$  is odd. On the other hand, it is known (see [10, Example 2.1]) that  $I_{A_{W_6}}$  is generated by quadratic binomials and has no quadratic Gröbner basis. Thanks to Proposition 4.2, there exists no bipartite graph  $G'$  such that  $I_{A_{W_6}} = I_{A_{G'}}$ .

**REMARK 5.8.** If  $G$  is the complete graph on the vertex set  $[d]$ , then  $A_G^\pm$  coincides with the configuration  $M'_{A_d}$  studied in [1].

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## REFERENCES

- [1] F. Ardila, M. Beck, S. Hoşten, J. Pfeifle, and K. Seashore, *Root polytopes and growth series of root lattices*, SIAM J. Discrete Math. **25** (2011), 360–378. MR 2801233. DOI 10.1137/090749293.
- [2] CoCoATeam, CoCoA: A System for Doing Computations in Commutative Algebra, <http://cocoa.dima.unige.it> (accessed 29 December 2014).
- [3] E. De Negri and T. Hibi, *Gorenstein algebras of Veronese type*, J. Algebra **193** (1997), 629–639. MR 1458806. DOI 10.1006/jabr.1997.6990.
- [4] J. W. Grossman, D. M. Kulkarni, and I. E. Schochetman, *On the minors of an incidence matrix and its Smith normal form*, Linear Algebra Appl. **218** (1995), 213–224. MR 1324059. DOI 10.1016/0024-3795(93)00173-W.
- [5] T. Hibi, *Algebraic Combinatorics on Convex Polytopes*, Carlsaw Publications, Glebe, Australia, 1992. MR 3183743.
- [6] H. Ohsugi, *A geometric definition of combinatorial pure subrings and Gröbner bases of toric ideals of positive roots*, Comment. Math. Univ. St. Pauli **56** (2007), 27–44. MR 2356748.
- [7] H. Ohsugi, J. Herzog, and T. Hibi, *Combinatorial pure subrings*, Osaka J. Math. **37** (2000), 745–757. MR 1789447.
- [8] H. Ohsugi and T. Hibi, *Normal polytopes arising from finite graphs*, J. Algebra **207** (1998), 409–426. MR 1644250. DOI 10.1006/jabr.1998.7476.
- [9] ———, *Koszul bipartite graphs*, Adv. in Appl. Math. **22** (1999), 25–28. MR 1657721. DOI 10.1006/aama.1998.0615.
- [10] ———, *Toric ideals generated by quadratic binomials*, J. Algebra **218** (1999), 509–527. MR 1705794. DOI 10.1006/jabr.1999.7918.
- [11] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, Chichester, 1986. MR 0874114.
- [12] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, Univ. Lecture Ser. **8**, Amer. Math. Soc., Providence, 1996. MR 1363949.

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