Canad. Math. Bull. Vol. **58** (1), 2015 pp. 144–149
http://dx.doi.org/10.4153/CMB-2014-043-x

Localization and Completeness in $L_2(\mathbb{R})$

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Abstract. We give a necessary and sufficient condition for a sequence to be a localization set for a determining average sampler.

1 Introduction

This note is devoted to the problem of uniqueness of reconstruction of a signal, *i.e.*, a square integrable function on the real line, from its values, which are measured in average.

The problem goes back to the classical results on determination of functions with bounded spectrum by a discrete sampling.

It is well known that if the spectrum of f lies on a segment $[-\sigma, \sigma]$ (that is, the Fourier transform $\widehat{f}$ vanishes outside $[-\sigma, \sigma]$), then, by the Whittaker–Kotelnikov–Shannon theorem (see *e.g.*, [H]), f can be uniquely determined by its values at the equidistant net $\frac{\pi}{\sigma}\mathbb{Z}$.

For non-equidistant sampling, the problem was solved in a seminal paper by Beurling and Malliavin [BM] on distributions of zeros of entire functions of exponential type.

The condition of boundedness of the spectrum, which implies analyticity of f, is essential: an arbitrary function, even smooth and rapidly decreasing, cannot be identified by its values at a discrete set.

An alternative approach to the reconstruction problem is to replace point sampling by average sampling, *i.e.*, taking values of inner products of f and a countable set of test-functions $\{\psi_n\}$ localized around given points.

Of course, "localization" does not mean that every ψ_n is supported only in a small neighbourhood of the corresponding point: reconstruction would be impossible in this case.

Let $g(t)$ be an even positive bounded function with finite integral, decreasing on $[0, +\infty)$. We call such an object *a hat-function*. The following definitions are given according to [NST].

A set of functions $\Psi = \{\psi_n\}$, $n \in \mathbb{Z}$, is said to be *localized* at points $\{\lambda_n\}$ with respect to a hat-function $g(t)$, or (g, Λ)-*localized*, if

$$|\psi_n(t)| \le c\, g(t - \lambda_n), \quad t \in \mathbb{R},$$

where c is a constant not depending on n.

Received by the editors June 2, 2014.
Published electronically September 20, 2014.
AMS subject classification: **42C30**, 94A20.
Keywords: localization, completeness, average sampling.

The sequence $\Lambda = \{\lambda_n\}$ is assumed to be increasing and uniformly discrete (that is, $\inf(\lambda_{n+1} - \lambda_n) > 0$). As usual, the inner product and the norm in $L_2(\mathbb{R})$ are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$.

A system Ψ is called *an average sampler* if the operator $f \mapsto \{\langle f, \psi_n \rangle\}$ acts from $L_2(\mathbb{R})$ into l_2, *i.e.*, there is a constant C such that for any $f \in L_2(\mathbb{R})$

$$\sum |\langle f, \psi_n \rangle|^2 \leq C\|f\|^2.$$

In this case Ψ is also known as a Bessel system.

A sampler Ψ is called *determining* if the above operator is one-to-one, *i.e.*, f can be uniquely determined by the sequence $\{\langle f, \psi_n \rangle\}$, which means that Ψ is a complete set of functions in $L_2(\mathbb{R})$.

So the problem can be stated as follows: given a function g and a set Λ, does a normalized (g, Λ)-localized determining average sampler (complete Bessel system) exist?

Based on the deep result from [BM], the authors in [NST] gave a positive answer to this question for a certain hat-function $g(t)$, and mentioned as an open problem whether one can achieve localization with an exponentially decreasing hat-function.

2 Statement of the Results

Our first goal is to solve this problem positively. Moreover, we suggest quite an elementary approach that allows us to construct a localized determining sampler for a hat-function arbitrarily rapidly decreasing at infinity.

Theorem 1 *For any hat-function $g(t)$ there exists a normalized $(g, \mathbb{Z})$-localized complete Bessel system in $L_2(\mathbb{R})$.*

Actually, the result holds under much weaker requirement on the localization set Λ. It can be arbitrarily sparse provided that some of its points have a mate at a bounded distance.

Theorem 2 *Let a sequence Λ contain infinitely many pairs (λ, λ') with bounded $|\lambda - \lambda'|$. Then for any hat-function $g(t)$ there exists a normalized (g, Λ)-localized complete Bessel system Ψ in $L_2(\mathbb{R})$.*

It turns out that the above condition is also necessary: the twin structure of a localization set cannot be avoided.

Theorem 3 *Let $\Lambda = \{\lambda_n\}$ be an increasing sequence with $\lambda_{n+1} - \lambda_n \to \infty$ for $n \to \pm\infty$. Suppose $\Psi = \{\psi_n(t)\}$ is a normalized in $L_2(\mathbb{R})$ system of functions, (g, Λ)-localized with respect to some hat-function $g(t)$. Then Ψ is not complete.*

Therefore, Theorems 2 and 3 together give a complete characterization of localization sets for determining average samplers. It might be interesting to observe that the characterization does not depend on a hat-function.

3 Proof of Theorem 1

As a basis for our construction we take the complete orthonormal system of functions in $L_2(\mathbb{R})$

$$\phi_{k,m}(t) = e^{2\pi ikt}\mathbf{1}_{[m,m+1]}(t), \quad k,m \in \mathbb{Z},\ t \in \mathbb{R},$$

obtained by integer shifts of a standard set of exponents on $[0,1]$.

Decompose this basis into two systems: the former one will consist of the functions

$$\phi_n(t) = \mathbf{1}_{[2n,2n+1]}(t), \quad n \in \mathbb{Z},$$

and the latter one will contain all the rest, enumerated in arbitrary order. We denote them by $\phi'_n(t)$, $n \in \mathbb{Z}$ (the prime is not a derivative herein).

For every n there is a corresponding number $m = m(n)$ such that the segment $[m, m+1]$ is the support of ϕ'_n. Let

$$\alpha_n = \min\{g(m(n) - 2n), g(m(n) - 2n - 1)\}.$$

Define now a set of functions $\Psi = \{\psi_n\}$, $n \in \mathbb{Z}$, as follows:

$$\psi_{2n} = \phi_n; \quad \psi_{2n+1} = \frac{\phi_n + \alpha_n\phi'_n}{\sqrt{1+\alpha_n^2}}.$$

Clearly, Ψ is normalized. Observe that the pair of functions ψ_{2n} and ψ_{2n+1} generate the same subspace in $L_2(\mathbb{R})$ as does the pair ϕ_n and ϕ'_n. Therefore, the system Ψ is complete.

The above subspaces, let us call them S_n, are pairwise orthogonal, which gives us the Bessel property of Ψ as well. Indeed, denote a projection of a function $f \in L_2(\mathbb{R})$ on S_n by f_n. Then

$$\sum |\langle f, \psi_n\rangle|^2 = \sum \big(|\langle f_n, \psi_{2n}\rangle|^2 + |\langle f_n, \psi_{2n+1}\rangle|^2\big) \le 2\sum \|f_n\|^2 = 2\|f\|^2.$$

Hence, it remains to check localization. It follows from the definition of a hat-function that

$$\mathbf{1}_{[0,1]}(t) \le \frac{g(t)}{g(1)}, \quad \mathbf{1}_{[-1,0]}(t) \le \frac{g(t)}{g(1)}$$

for all $t \in \mathbb{R}$. Consequently,

$$\mathbf{1}_{[k,k+1]}(t) = \mathbf{1}_{[0,1]}(t-k) \le \frac{g(t-k)}{g(1)}, \quad \mathbf{1}_{[k,k+1]}(t) \le \frac{g(t-k-1)}{g(1)}.$$

For similar reasons, for $m > k$ and for $m < k$ respectively, we have

$$\mathbf{1}_{[m,m+1]}(t) \le \frac{g(t-k)}{g(m-k+1)} \quad \text{and} \quad \mathbf{1}_{[m,m+1]}(t) \le \frac{g(t-k)}{g(m-k)}.$$

Therefore,

$$|\psi_{2n}(t)| = \mathbf{1}_{[2n,2n+1]}(t) \le \frac{g(t-2n)}{g(1)},$$

$$|\psi_{2n+1}(t)| < |\phi_n(t) + \alpha_n\phi'_n(t)| \le |\phi_n(t)| + \alpha_n|\phi'_n(t)| =$$

$$= \mathbf{1}_{[2n,2n+1]}(t) + \alpha_n\mathbf{1}_{[m(n),m(n)+1]} \le \Big(\frac{1}{g(1)} + 1\Big)\, g(t-(2n+1)).$$

So the constructed system Ψ is $(g,\mathbb{Z})$-localized, with $c = g^{-1}(1) + 1$. ∎

https://doi.org/10.4153/CMB-2014-043-x Published online by Cambridge University Press

4 Proof of Theorem 2

This proof is a modification of the previous one, so we present it briefly, omitting some technical details. Let $\Lambda_0 = \{(\lambda_{n_k}, \lambda'_{n_k})\}$ be a subsequence of Λ such that for some $d > 0$

$$|\lambda_{n_k} - \lambda'_{n_k}| < d, \quad k \in \mathbb{Z}.$$

Take a complete orthonormal system Φ in $L_2(\mathbb{R})$ such that each function $\phi \in \Phi$ satisfies

$$|\phi(t)| = \mathbf{1}_{I(\phi)},$$

where $I(\phi) = [m, m+1]$ for some $m \in \mathbb{Z}$. Such a set can be obtained, *e.g.*, by the appropriate translations of a standard Fourier basis on $[0, 1]$.

Denote by $J(\lambda)$ an interval of the form $[m, m+1]$ containing λ. Without loss of generality, we may assume that at most one point of $\{\lambda_{n_k}\}$ lies on any such interval.

For every k fix a function $\phi_k \in \Phi$ such that

$$I(\phi_k) = J(\lambda_{n_k}).$$

There are infinitely many such functions, so we take any of them and refer to the chosen countable set $\{\phi_k\}$ as main elements of Φ.

Now decompose the system Φ into pairs (ϕ_k, ϕ'_k), where ϕ_k is a main function, and ϕ'_k is not. In each two-dimensional subspace generated by such a pair we form a new basis, normalized but no longer orthogonal, as follows:

$$\psi_k = \phi_k, \quad \psi'_k = \frac{\phi_k + \alpha_k \phi'_k}{\sqrt{1 + \alpha_k^2}}.$$

Arguing as in the preceding proof, we deduce that for sufficiently small α_k and sufficiently large c (depending on d)

$$|\psi_k(t)| \leq c\, g(t - \lambda_{n_k}), \quad |\psi'_k(t)| \leq c\, g(t - \lambda'_{n_k}).$$

Therefore, joining all the functions ψ_k and ψ'_k together, we obtain a complete normalized Bessel system in $L_2(\mathbb{R})$, localized with respect to the hat-function $g(t)$ and the set Λ_0, which is even slightly more than desired. ■

Remark Using the Riesz interpolation theorem, it is not difficult to show that the localization by itself implies the Bessel property.

5 Proof of Theorem 3

By the definition, $g(t)$ is decreasing on $[0, +\infty)$ and $\int g(t)dt < \infty$. Take $a > 1$ such that

$$c^2 \int_a^\infty g^2(t)dt < \frac{1}{32},$$

and fix $A > a$ such that

$$\sum_{n=1}^{\infty} g(nA - a) < \frac{1}{12ac}.$$

Due to the condition on Λ, we can choose a number N such that

$$|\lambda_k| > A \quad \text{and} \quad \lambda_{k+1} - \lambda_k > A, \quad \text{for all } |k| > N.$$

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Denote by S_N the closed linear span of the vectors ψ_k, $|k| > N$, in the space $L_2(\mathbb{R})$. These functions are "far away" from zero and from each other: the distances between their localization points λ_k are greater than A.

Every remaining ψ_k, with $|k| \le N$, can be decomposed as

$$\psi_k = \psi_k' + \psi_k'',$$

where $\psi_k' \in S_N$ and $\psi_k'' \perp S_N$.

Suppose by contradiction that the set Ψ is complete. Then $L_2(\mathbb{R})$ is a direct sum of S_N and a finite-dimensional subspace generated by the vectors ψ_k'', $|k| \le N$.

Let $f(t)$, $\|f\| = 1$, be a function supported on $[0,1]$ and orthogonal to all ψ_k''. We are going to prove that f cannot be approximated in S_N, which gives us the desired contradiction.

For any finite linear combination

$$h(t) = \sum_{|k|>N} c_k \psi_k(t)$$

we have

$$|\langle f, h\rangle| = \Big|\sum_{|k|>N} c_k \langle f, \psi_k\rangle\Big| \le \max_k |c_k| \cdot \sum_{|k|>N} |\langle f, \psi_k\rangle|.$$

To estimate these two factors, first let us show that

$$\max_k |c_k| < 2\|h\|.$$

Indeed, assume that $\max|c_k| = M$ is attained at certain k_0, and let J denote $[\lambda_{k_0} - a, \lambda_{k_0} + a]$. It follows from the definition of localization and from our choice of a that

$$\|\psi_{k_0} \cdot \mathbf{1}_{\mathbb{R}\setminus J}\| \le \Big(2c^2 \int_a^\infty g^2(t)dt\Big)^{\frac{1}{2}} < \frac{1}{4},$$

which implies

$$\|\psi_{k_0} \cdot \mathbf{1}_J\| > \frac{3}{4}.$$

For $k \ne k_0$ (still $|k| > N$), we have

$$\sup_{t\in J} |\psi_k(t)| \le c\, g(|\lambda_k - \lambda_{k_0}| - a) \le c\, g(|k - k_0|A - a),$$

hence

$$\Big\|\sum_{k\ne k_0} c_k\, \psi_k \cdot \mathbf{1}_J\Big\| \le M \cdot \sqrt{2a} \cdot 2c \sum_{n=1}^{\infty} g(nA - a) < \frac{M}{4}.$$

Therefore,

$$\|h\| \ge \|h \cdot \mathbf{1}_J\| \ge \|M\psi_{k_0} \cdot \mathbf{1}_J\| - \Big\|\sum_{k\ne k_0} c_k\, \psi_k \cdot \mathbf{1}_J\Big\| > \frac{M}{2},$$

as desired.

By the same arguments, for $|k| > N$ we have

$$|\langle f, \psi_k\rangle| \le \|f\| \cdot \|\psi_k \cdot \mathbf{1}_{[0,1]}\| \le c\, g(|\lambda_k| - 1),$$

thus

$$\sum_{|k|>N} |\langle f, \psi_k\rangle| \le 2c \sum_{n=1}^{\infty} g(nA - 1) < \frac{1}{4},$$

which finally implies

$$|\langle f, h\rangle| < 2\|h\| \cdot \frac{1}{4} = \frac{\|h\|}{2}.$$

Consequently,

$$\|f-h\|^2 = \|f\|^2 + \|h\|^2 - 2\mathrm{Re}\langle f, h\rangle \geq 1 + \|h\|^2 - \|h\| \geq \frac{3}{4}. \qquad \blacksquare$$

Acknowledgment The results were partially presented at the MGUPI conferences [O]. The author is also grateful to the organizers of the University of Alberta seminar for the kind invitation to give a talk on the subject.

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