Appendix E
Differential Forms on Infinite-Dimensional Manifolds

E.1 Introduction

In this appendix we give a short introduction to differential forms on infinite-dimensional manifolds. For more information on differential forms on infinite-dimensional manifolds and their application, we refer the interested reader to Beggs (1987), as well as Glöckner and Neeb (forthcoming). The main difference between the finite-dimensional (or Banach) and our setting is that it is, in general, impossible to interpret differential forms as (smooth) sections into certain bundles of linear forms. The reason for this is again that the topology on spaces of linear forms breaks down beyond the Banach setting (see Proposition A.19). Even worse, the many equivalent ways to define differential forms in finite dimensions become inequivalent in the infinite-dimensional setting (see Kriegl and Michor, 1997, Section 33 for a thorough discussion of this phenomenon). Most notably, there is no useful way to describe differential forms as a sum of differential forms coming from a local coordinate system.

We begin with the definition of a differential form. This definition is geared towards avoiding any reference to topologies on spaces of linear mappings. This again is a continuity problem and in the inequivalent convenient setting of global analysis, differential forms can be described as sections in suitable bundles; see Kriegl and Michor (1997, 33.22 Remark). Furthermore, we need to avoid arguments involving the existence of (smooth) bump functions (which in general do not exist; see Appendix A.4).

E.1 Definition Let M be a manifold and E be a locally convex space and $p \in \mathbb{N}_0$. An *E-valued p-form* ω on M is a function ω which associates to each $x \in M$ a p-linear alternating map $\omega_x \colon (T_x M)^p \to E$ such that for each chart (U, φ) of M, the map

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$$\omega_\varphi \colon V_\varphi \times F_\varphi^p \to E, \quad \omega_\varphi(x,v_1,\ldots,v_p) := \omega_{\varphi^{-1}(x)}(T_x\varphi^{-1}(v_1),\ldots,T_x\varphi^{-1}(v_p)) \tag{E.1}$$

is smooth. We write $\Omega^p(M,E)$ for the *space of smooth E-valued p-forms* on M. Note that $\Omega^0(M,E) = C^\infty(M,E)$.

E.2 Example For $p = 0$, we have already seen that smooth functions are differential forms. If $f \in C^\infty(M,E)$, then the derivative $df \colon TM \to E$, $v \mapsto \mathrm{pr}_2 \circ Tf(v)$ is a smooth E-valued 1-form.

Constructing the derivative of a smooth function can be generalised to a differential on the space of p-forms, the so-called *exterior differential*

$$\mathrm{d} \colon \Omega^p(M,E) \to \Omega^{p+1}(M,E)$$

which we discuss now. On an infinite-dimensional manifold there is no generalisation of local coordinates in a vector basis. Hence the finite-dimensional approach defining the exterior differential in a local coordinate frame is not available. Recall some standard notation useful in the present context: In (D.1) we defined a derivative $X.f$ of a smooth function f in the direction of X. If $\omega \in \Omega^p(M,E)$ and $U \Subset M$, we define for vector fields $X_1,\ldots X_p \in \mathcal{V}(U)$ a smooth map

$$\omega(X_1,\ldots,X_p) \colon U \to E, \quad m \mapsto \omega_m(X_1(m),\ldots X_p(m)).$$

Finally, we write $\omega(X_0,\ldots,\hat{X}_i,\ldots,X_p)$ to indicate that the ith component is to be omitted from the formula.

E.3 Proposition *For $\omega \in \Omega^p(M,E)$ there exists a smooth $p+1$-form $\mathrm{d}\omega \in \Omega^{p+1}(M,E)$ which for any $U \Subset M$ and vector fields $X_1,\ldots,X_p \in \mathcal{V}(U)$, satisfies*

$$\begin{aligned}\mathrm{d}\omega(X_0,X_1,\ldots,X_p)(m) &= \sum_{i=0}^{p}(-1)^i(X_i.\omega(X_0,\ldots,\hat{X}_i,\ldots,X_p))(m)\\ &+\sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots\hat{X}_j,\ldots,X_p)(m).\end{aligned} \tag{E.2}$$

Proof Consider $m \in M$ and $v_1,\ldots,v_p \in T_mM$. To define $(\mathrm{d}\omega)_m(v_1,\ldots,v_p)$ we pick an open neighbourhood U of m together with vector fields $X_i \in \mathcal{V}(U)$ such that $X_i(m) = v_i, i = 0,1,2,\ldots,p$. Note that such vector fields always exist as we can take the constant vector fields in a chart neighbourhood (in particular, the definition does not require us to globalise these fields, which would require bump functions which may not exist). Then

$$(\mathrm{d}\omega)_m(v_0,v_1,\ldots,v_p) := \mathrm{d}\omega(X_0,X_1,\ldots,X_p)(m), \tag{E.3}$$

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where the right-hand side has been defined via (E.2) for our choice of vector fields.

Step 1: $(\mathrm{d}\omega)_m(v_1,\dots,v_p)$ does not depend on the choice of vector fields in (E.3). We have to show that the expression (E.2) becomes 0 if $X_k(m) = 0$ for at least one k. Assuming that X_k vanishes in m, we may without loss of generality assume that we are working in local coordinates. We will suppress the chart identification in the formulae and also identify each vector field X_i with its principal part on some $U \Subset F$ (where F is a locally convex space). Exploit now that ω is alternating and linear to see that the contributions in (E.2) which do not directly vanish are

$$\sum_{i\neq k}^{p}(-1)^i(X_i.\omega(X_0,\dots,\hat{X}_i,\dots,X_p))(m) \tag{E.4}$$

$$+\sum_{i<k}(-1)^{i+k}\omega([X_i,X_k],X_0,\dots,\hat{X}_i,\dots,\hat{X}_k,\dots,X_p)(m) \tag{E.5}$$

$$+\sum_{k<i}(-1)^{i+k}\omega([X_k,X_i],X_0,\dots,\hat{X}_k,\dots,\hat{X}_i,\dots,X_p)(m). \tag{E.6}$$

Apply the definition of the differential form ω on the open subset of a locally convex space (E.1). In this presentation ω is a function of $p+1$-variables and p-linear in the last p-variables. Hence we can compute the derivative for a summand in (E.4) explicitly as

$$\begin{aligned}
&X_i.\omega(X_0,\dots,\hat{X}_i,\dots,X_p)(m)\\
&\quad= d_1\omega(m,X_1(m),\dots\hat{X}_i(m),\dots,X_p(m);X_i(m))\\
&\qquad+\sum_{j<i}\omega_m(X_0(m),\dots,dX_j(m;X_i(m)),\dots,\hat{X}_i(m),\dots,X_p(m))\\
&\qquad+\sum_{i<j}\omega_m(X_0(m),\dots,\hat{X}_i(m),dX_j(m;X_i(m)),\dots,X_p(m)).
\end{aligned}$$

As X_k vanishes, we see that for every $i > k$ only

$$\omega_m(X_0(m),\dots,dX_k(m;X_i(m)),\dots,\hat{X}_i(m),\dots,X_p(m))$$

survives and as $X_k(m) = 0$, we have

$$dX_k(m,X_i(m)) = dX_k(m;X_i(m)) - dX_i(m;X_k(m)) = [X_i,X_k](m).$$

Using the fact that ω is alternating, we have

$$\begin{aligned}
&(-1)^k\omega([X_k,X_i],X_0,\dots,\hat{X}_k,\dots,\hat{X}_i,\dots,X_p)(m)\\
&\quad= -\omega_m(X_0(m),\dots,dX_k(m;X_i(m)),\dots,\hat{X}_i(m),\dots,X_p(m)),
\end{aligned}$$

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hence the corresponding terms in (E.4) and (E.5) cancel. Similar arguments show that this also happens for the parts in (E.4) and (E.6) if $i < k$. We conclude that (E.2) vanishes at a point if one of the vector fields vanishes at the point. This shows, in particular, that (E.3) is independent of the choices of vector fields.

Step 2: $\mathrm{d}\omega$ is a smooth $p+1$-form. For smoothness we work again locally in a chart as in Step 1 and pick all vector fields X_i to be constant. As the Lie bracket of constant vector fields vanishes, (E.3) reduces to

$$(\mathrm{d}\omega)_m(v_0,\ldots,v_p) = \sum_{i=0}^{p}(-1)^i d_1\omega(m,v_0,\ldots,\hat{v}_i,\ldots,v_p;v_i). \tag{E.7}$$

Now by definition ω induces a smooth function in all charts, whence we see that $\mathrm{d}\omega$ is also smooth in $(m,v_0,v_1,\ldots,v_o)$. To see that $(\mathrm{d}\omega)_m$ is alternating, observe that the summands in (E.7) are alternating. Assume now that $v_i = v_j$ for some $i < j$. It is easy to see that (E.7) vanishes (we leave this as Exercise E.1.1). □

We thus obtain for every $p \geq 0$ an exterior differential on the space of p-forms. The usual proof (see Kriegl and Michor, 1997, Theorem 33.18, or Lang, 1999, V. Proposition 3.3) then shows that $\mathrm{d}^2 = \mathrm{d}\circ\mathrm{d} = 0$ in every degree. Hence as in the finite-dimensional (or the Banach) setting, the exterior differential gives rise to a cochain complex of differential forms

$$C^\infty(M,E) = \Omega^0(M,E) \xrightarrow{\mathrm{d}} \Omega^1(M,E) \xrightarrow{\mathrm{d}} \Omega^3(M,E) \xrightarrow{\mathrm{d}} \cdots .$$

Starting from this complex, one can define and study de Rham cohomology on the (infinite-dimensional) manifold M. We will not pursue this route here and refer instead to Beggs (1987) or Kriegl and Michor (1997, Chapter 34) for more information.

Differential forms of higher order are typically constructed using the wedge product. The definition is as in the finite-dimensional setting (note, however, that there are several conventions as to the coefficients; we chose to follow Lang, 1999).

E.4 Definition Let E_i, $i = 1,2,3$ be locally convex spaces and $\beta\colon E_1 \times E_2 \to E_3$ be a continuous bilinear map. Fix $p,q \in \mathbb{N}_0$ and denote by S_{p+q} the symmetric group of all permutations of $\{1,2,\ldots p+q\}$. For $\omega \in \Omega^p(M,E_1)$ and $\eta \in \Omega^q(M,E_2)$, define the *wedge product* $\omega \wedge \eta \in \Omega^{p+q}(M,E_3)$ via $(\omega \wedge \eta)_x := \omega_x \wedge \eta_x$ for $x \in M$, where

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$$(\omega_x \wedge \eta_x)(v_1, \ldots, v_{p+q})$$
$$:= \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \operatorname{sgn}(\sigma)\beta(\omega_x(v_{\sigma(1)}, \ldots, v_{\sigma(p)}), \eta_x(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)})).$$

Then

$$\wedge\colon \Omega^p(M, E_1) \times \Omega^q(M, E_2) \to \Omega^{p+q}(M, E_3), \quad (\omega, \eta) \mapsto \omega \wedge \eta$$

is a bilinear map.

E.5 Example If $s\colon \mathbb{R} \times E \to E$ is the scalar multiplication and $f \in C^\infty(M, \mathbb{R})$, then the wedge product of f and $\omega \in \Omega^p(M, E)$ is given by $(f \wedge \omega)_x = f(x)\omega_x$. This is usually abbreviated by $f\omega := f \wedge \omega$ and it is easy to see that $\Omega^p(M, E)$ becomes a $C^\infty(M, \mathbb{R})$-module.

E.6 Example Let $(E, [\cdot, \cdot])$ be a locally convex Lie algebra. Then $[\cdot, \cdot]\colon E \times E \to E$ is bilinear and we can construct the wedge product $\wedge$ with respect to the Lie bracket. For this special situation we define for $\omega \in \Omega^p(M, E)$ and $\eta \in \Omega^q(M, E)$ the bracket

$$[\omega, \eta]_\wedge := \omega \wedge \eta.$$

We will now define several standard operations on differential forms such as the pullback of p-forms by smooth mappings.

E.7 Definition Let $\varphi\colon M \to N$ be a smooth map between manifolds. Then we define for $\omega \in \Omega^p(N, E)$ a p-form $\varphi^\omega \in \Omega^p(M, E)$, the *pullback of ω by φ* via

$$(\varphi^*\omega)_x(v_1, \ldots v_p) := \omega_{\varphi(x)}(T_x\varphi(v_1), \ldots T_x\varphi(v_p)).$$

Due to the chain rule we immediately have the following rules for the computation of pullbacks.

E.8 The following rules hold for smooth maps and p-forms

$$\mathrm{id}_M^* \omega = \omega, \quad \varphi_1^*(\varphi_2^*\omega) = (\varphi_2 \circ \varphi_1)\omega^*, \quad \varphi^*(\omega \wedge \eta) = \varphi^*\omega \wedge \varphi^*\eta.$$

If $p = 0$, that is, $\omega = f \in C^\infty(M, E)$, then $\varphi^* f = f \circ \varphi$ and we recover the pullback discussed in the context of manifolds of mappings.

Finally, we define the Lie derivative of a differential form by a vector field. Before we begin, note that the definition of the Lie derivative has to diverge from the usual definition on finite-dimensional or Banach manifolds. This is due to the fact that the common description of the Lie derivative (see e.g. Lang, 1999, V. §2) uses the differential of a flow of a vector field. However, as flows of vector fields are the solutions to certain ordinary differential equations, it is unclear whether the flow of a vector field would exist on the more general

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manifolds we consider (see Appendix A.6 for a discussion of this problem). Nevertheless, Lang (1999, V.5 Proposition 5.1) shows that for Banach manifolds the following definition coincides with the classical one involving flows.

E.9 Definition Let M be a manifold and E a locally convex space. For $X \in \mathcal{V}(M)$ and $\omega \in \Omega^p(M,E), p \in \mathbb{N}_0$ we define the *Lie derivative* $\mathcal{L}_Y\omega \in \Omega^p(M,E)$ as follows:

$$
\begin{aligned}
&(\mathcal{L}_Y\omega)_m(v_1,\ldots,v_p)\\
&\quad = Y.\omega(X_1,\ldots,X_p)(m) - \sum_{j=1}^{p} \omega(X_1,\ldots,[Y,X_j],\ldots X_p)(m)\\
&\quad = Y.\omega(X_1,\ldots,X_p)(m) + \sum_{j=1}^{p} (-1)^j \omega([Y,X_j],X_1,\ldots,\hat{X}_j,\ldots X_p)(m),
\end{aligned}
$$

where the X_i are smooth vector fields defined in a neighbourhood of m such that $X_i(m) = v_i$. That the Lie derivative is well defined will be checked in Exercise E.1.4.

Note that for $\omega \in \Omega^0(M,E) = C^\infty(M,E)$ the formula of the Lie derivative reduces to $\mathcal{L}_Y\omega = d\omega \circ Y$. This was precisely the formula for the Lie derivative described in Definition D.10 for functions.

Exercises

E.1.1 Check that the exterior differential $\mathrm{d}\omega$ of a p-form is an alternating $p+1$-form.

E.1.2 Check the details in Definition E.4. Show that

- (a) the wedge product of a p-form and a q-form is indeed a $p+q$-form;
- (b) the wedge product defines a bilinear map between spaces of differential forms;
- (c) $\Omega^p(M,E)$ is a $C^\infty(M,\mathbb{R})$-module (see Example E.5);
- (d) for $p = q = 1$ we have $\omega_x \wedge \eta_x(v_1,v_2) = \beta(\omega_x(v_1),\eta_x(v_2)) - \beta(\omega_x(v_2),\eta_x(v_2))$.

E.1.3 Prove that for $\omega \in \Omega^p(M,E)$, $\eta \in \Omega^q(M,F)$ and any wedge product, the following formula holds:

$$\mathrm{d}(\omega \wedge \eta) = (\mathrm{d}\omega) \wedge \eta + (-1)^p \omega \wedge (\mathrm{d}\eta).$$

Furthermore, show that for $f\colon N \to M$ smooth, we have

$$f^*\mathrm{d}\omega = \mathrm{d}f^*\omega.$$

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Hint: The assertions are local, whence they can be solved using the local formula for the exterior differential.

E.1.4 In this exercise we let $X \in \mathcal{V}(M)$ and $\omega \in \Omega^p(M,E), p \in \mathbb{N}_0$ for M a manifold and E a locally convex space. Show that the definition of the Lie derivative $\mathcal{L}_Y\omega$ does not depend on the choice of vector fields X_i in Definition E.9. Conclude that $\mathcal{L}_Y\omega$ is a smooth p-form.
Hint: It suffices to show that $\mathcal{L}_Y\omega$ vanishes if $X_i(m) = 0$, and this can be checked locally.

E.2 The Maurer–Cartan Form on a Lie Group

For Lie groups there are two important differential forms induced by the Lie group structure.

E.10 Example Let G be a Lie group with Lie algebra $\mathbf{L}(G)$. Then we define the *right Maurer–Cartan form* $\kappa^r \in \Omega^1(G,\mathbf{L}(G))$ via

$$(\kappa^r)_g : T_g G \to \mathbf{L}(G), \quad v \mapsto T_g \rho_{g^{-1}}(v),$$

where $\rho_{g^{-1}}(h) = hg^{-1}$.

Now if $f \in C^\infty(M,G)$, we can define its right logarithmic derivative via

$$\delta^r f : M \to \mathbf{L}(G), \quad \delta^r f := f^*\kappa^r.$$

Similarly, one can define the *left Maurer–Cartan form* $\kappa^\ell \in \Omega^1(G,\mathbf{L}(G))$ and a left logarithmic derivative by replacing right multiplication with left multiplication in the definition of κ^r. This generalises the construction of the logarithmic derivatives for curves from 3.31. One can show (see Exercise E.2.1) that the left logarithmic derivative of any function satisfies the right Maurer–Cartan equation

$$\mathrm{d}\delta^\ell f + \frac{1}{2}[\delta^\ell f,\delta^\ell f]_\wedge = 0, \tag{E.8}$$

where $[\delta^\ell f,\delta^\ell f]_\wedge = \delta^\ell f \wedge \delta^\ell f$ for the wedge product induced by the Lie bracket.

E.11 Definition Let G be a Lie group with Lie algebra $\mathbf{L}(G)$ and $\omega \in \Omega^1(M,\mathbf{L}(G))$ for some smooth manifold M. Then ω is called

(a) *integrable* if there exists $f \in C^\infty(M,G)$ with $\omega = \delta^\ell f$;
(b) *locally integrable* if for every $m \in M$ there is $m \in U \Subset M$ such that $\omega|_U$ is integrable.

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Equivalently we could have defined (local) integrability using the right logarithmic derivative. With this definition it is possible to formulate the fundamental theorem for Lie group-valued functions with values in regular Lie groups; see §3.3.

E.12 Proposition (Fundamental theorem for Lie group-valued functions) *Let G be a regular Lie group with Lie algebra $\mathbf{L}(G)$ and $\omega \in \Omega^1(M, \mathbf{L}(G))$. If ω satisfies the right Maurer–Cartan equation* (E.8)*, then ω is locally integrable. If, in addition, M is simply connected, then ω is integrable.*

The proof of Proposition E.12 needs concepts (e.g. connections on principal bundles) which we will not introduce here. Instead we refer the interested reader either to the classical proofs for the finite-dimensional setting, for example, Sharpe (1997, 3.§6–7), or to the infinite-dimensional sources Neeb (2006, Theorem III.2.1) as well as Kriegl and Michor (1997, Theorem 40.2).

E.13 Lemma *Let M be a connected manifold and $\varphi, \psi \in C^\infty(M, G)$, where G is a Lie group. Then $\delta^\ell \varphi = \delta^\ell \psi$ is equivalent to the existence of $g \in G$ with $\varphi = g \cdot \psi$.*

Proof If there exists $g \in G$ with $\varphi = g \cdot \psi$, then a straightforward calculation shows that $\delta^\ell \varphi = \delta^\ell \psi$ holds. Assume conversely that $\delta^\ell \varphi = \delta^\ell \psi$. Then define the map $\Psi := \varphi \cdot \psi^{-1}$ (where the product and inverse are taken pointwise in G). Exercise E.2.3 yields

$$\delta(\gamma) = T_e \lambda_\psi T_e \rho_{\psi^{-1}} (\delta^\ell \varphi - \delta^\ell \psi) = 0,$$

or in other words, the map γ is locally constant by Corollary 1.19. As M is connected, we conclude that $g := \gamma(m) \in G$ (for any $m \in M$) satisfies $g \circ \psi = \varphi$. □

E.14 Proposition (Lie II for regular Lie groups) *Let G, H be Lie groups with Lie algebras $\mathbf{L}(G)$ and $\mathbf{L}(H)$, respectively. Let $f : \mathbf{L}(G) \to \mathbf{L}(H)$ be a morphism of locally convex Lie algebras. If H is a regular Lie group and G is connected and simply connected, then there exists a unique morphism of Lie groups $\varphi : G \to H$ with $\mathbf{L}(\varphi) = f$.*

Proof Since f is continuous we can consider the smooth 1-form $\alpha := f \circ \kappa^\ell \in \Omega^1(G, \mathbf{L}(H))$, where κ^ℓ is the left Maurer–Cartan form on G. Moreover, since the Maurer–Cartan form is left invariant, so is α. We consider the Maurer–Cartan equation on H (to mark this we label the bracket operation $[\cdot,\cdot]^H_\wedge$ by H). As the wedge is induced by the Lie bracket on $\mathbf{L}(H)$, we can exploit that f is a Lie algebra morphism and compute with Exercise E.1.3 as follows:

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$$\mathrm{d}\alpha+\frac{1}{2}[\alpha,\alpha]^H_\wedge=\mathrm{d}(f\circ\kappa^\ell)+\frac{1}{2}[f\circ\kappa^\ell,f\circ\kappa^\ell]^H_\wedge=f\circ\left(\mathrm{d}\kappa^\ell+\frac{1}{2}[\kappa^\ell,\kappa^\ell]_\wedge\right)=0,$$

where the unlabelled bracket is the one induced by the Lie bracket of $\mathbf{L}(G)$ and we exploited that the Maurer–Cartan form satisfies the Maurer–Cartan equation on G by Exercise E.2.1. Now the fundamental theorem, Proposition E.12, implies that there is a mapping $\varphi\colon G\to H$ with $\delta^\ell\varphi=\alpha$. Fixing $\varphi(e_G)=e_H$, this mapping is unique by Lemma E.13. Consider now $g\in G$ and $\varphi\circ\lambda_g$. Then we pick $v\in T_kG$ and evaluate the differential form

$$\delta^\ell(\varphi\circ\lambda_g)(v)=T\lambda_{\varphi(gk)^{-1}}T_{gk}\varphi T\lambda_g(v)=\delta^\ell\varphi(T\lambda_g(v))=\lambda_g^*\alpha=\alpha.$$

Now applying Lemma E.13 again, the maps $\varphi\circ\lambda_g$ and φ differ only by left translation with an element which we compute as $\varphi\circ\lambda(e_G)=\varphi(g)$. In other words, $\varphi(gk)=\varphi(g)\varphi(k)$ for all $k\in G$. Since $g\in G$ was arbitrary, we see that φ is indeed a morphism of Lie groups. □

Exercises

E.2.1 Let G be a Lie group with Lie algebra $\mathbf{L}(G)$ and left Maurer–Cartan form κ^ℓ. Show that κ^ℓ:

(a) is an $\mathbf{L}(G)$-valued differential form on G which is left invariant in the sense that $\lambda_g^*\kappa^\ell=\kappa^\ell$ for each $g\in G$ (where $\lambda_g(h):=gh$;

(b) satisfies the right Maurer–Cartan equation

$$\mathrm{d}\kappa^\ell+\frac{1}{2}[\kappa^\ell,\kappa^\ell]_\wedge=0.$$

Hint: Compute the exterior derivative locally using (E.3). It suffices to prove the formula using left invariant vector fields (why?).

Remark: The ‘right Maurer–Cartan equation’ is related to the right principal action of G on itself by multiplication. There is also a corresponding left Maurer–Cartan equation for the right Maurer–Cartan form, where the bracket in the equation gets a negative sign.

(c) Deduce that for a smooth function f, $\delta^\ell f$ also satisfies the right Maurer–Cartan equation.

E.2.2 Let $\varphi\colon G\to H$ be a morphism of Lie groups and κ^ℓ the (left) Maurer–Cartan form on G. Show that:

(a) $\delta^\ell\varphi=\mathbf{L}(\varphi)\circ\kappa^\ell$;

(b) if $\psi\colon G\to H$ is another Lie group morphism with $\delta^\ell\varphi=\delta^\ell\psi$, then $\varphi=\psi$.

E.2.3 Let $f,g \in C^\infty(M,G)$ be smooth maps to a Lie group. Establish the following quotient rule for the left logarithmic derivative:

$$\begin{aligned}\delta^\ell(f \cdot g^{-1})(m) &= T_{\mathbf{1}_G}\lambda_{g(m)} T_{\mathbf{1}_G}\rho_{g^{-1}(m)}(\delta^\ell f(m) - \delta^\ell g(m)) \\ &= \mathrm{Ad}_{g(m)}(\delta^\ell f(m) - \delta^\ell g(m)),\end{aligned}$$

where $\mathbf{1}_G \in G$ is the identity element, products and inverses are taken pointwise and λ (resp. ρ) denotes left (resp. right) multiplication in the Lie group.
Hint: Apply Lemma 3.12.

E.3 Supplement: Volume Form and Classical Differential Operators

In this short supplement we will record some well-known facts on differential forms on finite-dimensional (compact) manifolds. Many of these notions are needed in Chapter 7 and we recall them for the reader's convenience. Thus detailed proofs will, in general, be omitted in this section. However, all of these results are readily available in the standard finite-dimensional literature (which we will reference).

Conventions We fix (M,g) a compact (thus finite-dimensional) and connected manifold with Riemannian metric g. Furthermore, we denote by $d = \dim M$ the dimension of the model space of M.

Let us first recall that there is another canonical way to define differential forms.

E.15 (Differential forms as sections; see Klingenberg, 1995, 1.4; Abraham et al., 1988, Section 6) Starting with the tangent bundle TM we can construct the bundle of alternating k-forms $\mathcal{A}^k(M) \to M$ for $k \geq 0$. The fibre of $\mathcal{A}^k(M)$ over $x \in M$ is given by the space of alternating k-linear mappings $(T_xM)^k \to \mathbb{R}$ which we denote by $L^k_a(T_xM,\mathbb{R})$ and topologise as a subspace of the k-linear mappings (which carry the usual norm topology induced by the operator norm for k-linear maps; see Lang, 1999, I. §2). Further, every chart (U,φ) of M induces a vector bundle trivialisation of $\mathcal{A}^k(M)$ over U via

$$\kappa_\varphi(x,\omega) := (\varphi(x), \omega \circ (T_x\varphi^{-1} \times T_x\varphi^{-1} \times \cdots \times T_x\varphi^{-1})).$$

Comparing the construction with (E.1), it becomes clear that for the finite-dimensional manifold M (indeed for any Banach manifold) differential k-forms are just smooth sections of $\mathcal{A}^k(M)$. In other words, we obtain

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$$\Omega^k(M) = \Gamma(\mathcal{A}^k(M)), \quad \text{for all } k \in \mathbb{N}_0.$$

Moreover, this allows us to topologise the space of differential k-forms as a locally convex vector space via C.7. Namely if we pick a family of open sets $(U_i)_{i\in I}$ which covers M such that on each U_i there is a bundle trivialisation $(\kappa_i)^{-1}\colon \mathcal{A}^k(M)|_{U_i} \to U_i \times \mathbb{R}^d$, $x \mapsto ((\kappa_i^1)^{-1}(x), (\kappa_i^2)^{-1}(x))$, then the mapping

$$\Omega^k(M) \to \prod_{i\in I} C^\infty(U_i, L_a^k(\mathbb{R}^d,\mathbb{R})), \quad \omega \mapsto \omega \circ (\kappa_i^1, \kappa_i^2, \ldots, \kappa_i^2)$$

is an embedding of $\Omega^k(M)$ as a closed locally convex subspace of the product on the right-hand side.

Recall that the dimension of the spaces $L_a^k(T_xM,\mathbb{R})$ depends on the dimension $d = \dim T_xM$. In particular, $L_a^d(T_xM,\mathbb{R})$ is one-dimensional and a differential form $\mu \in \Omega^d(M)$ which vanishes nowhere is called a *volume form*. On Riemannian manifolds there is a convenient way to construct a volume form associated to the Riemannian metric. Recall that a manifold M is orientable, if it admits an atlas $(U_i, \varphi_i)_{i\in I}$ such that the Jacobians of all change of charts are positive. One can prove (Gallot et al., 2004, Theorem 1.127) that a Riemannian manifold is orientable if and only if it admits a volume form μ (induced by the Riemannian metric). Volume forms are the tool of choice to define integration on manifolds; see Lang (1999, Part III) or Abraham et al. (1988, Section 7). In particular, we can define the L^2-metric on $\mathcal{V}(M)$ in the presence of a volume form symbolically without defining the integral as follows:

$$g_{L^2}(X,Y) = \int_M g(X,Y)\mathrm{d}\mu, \qquad X,Y \in \mathcal{V}(M). \tag{E.9}$$

In Chapter 5 we often considered only integration on $\mathbb{S}^1$ since a global parametrisation allowed us to hide the dependence on a volume form and (E.9) reduces for $M = \mathbb{S}^1$ to (5.1).

Classical Differential Operators on a Riemannian Manifold

We will now assume that there is a volume form μ associated to the Riemannian metric on M. Let us then recall the following classical differential operators on M.

E.16 Definition (Abraham et al., 1988, Sections 6.5 and 7.5) For a compact Riemannian manifold (orientable in case we need a volume form μ), we will consider the following differential operators.

- For a vector field X, there is a unique smooth function $\operatorname{div} X \colon M \to \mathbb{R}$, the *divergence* of X such that

$$\mathcal{L}_X \mu = (\operatorname{div} X)\mu.$$

- If $f \in C^\infty(M,\mathbb{R})$ we exploit that the Riemannian metric induces an isomorphism $TM \cong T^*M$ (see Proposition 4.5). Thus the following formula uniquely determines the *gradient of f with respect to g*:

$$g_m(\operatorname{grad} f(m), v_m) = df(v_m) \quad \text{for all } m \in M, v_m \in T_m M.$$

- The *(Hodge)Laplacian* $\Delta = \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$ is associated to the metric (see Lang, 1999, p. 423). Here d is the exterior differential from Proposition E.3 and d^* is the codifferential defined via the Hodge star (this is a finite-dimensional construction which depends on the Riemannian metric g; we refer to Lee, 2013, p. 464 for more information).

Having defined the necessary differential operators, we recall two decompositions which are, for example, relevant in geometric hydrodynamics.

E.17 Proposition (Helmholtz decomposition, see Modin (2019), Lemma 1.2) *Let (M,g) be a compact oriented Riemannian manifold with volume form μ and $X \in \mathcal{V}(M)$. Then there exist $V \in \mathcal{V}(M)$ and $f \in C^\infty(M,\mathbb{R})$ such that*

$$X = V + \operatorname{grad} f \qquad \textit{and} \ \operatorname{div} V = 0.$$

Moreover, V and $\operatorname{grad} f$ *are orthogonal with respect to the L^2-metric* (E.9)*, that is,*

$$g_{L^2}(V, \operatorname{grad} f) = \int_M g(V, \operatorname{grad} f)\mathrm{d}\mu = 0.$$

Note that since the differential d and the codifferential d^* make sense for arbitrary k-forms, we can also extend the Hodge Laplacian to k-forms. This induces the Hodge decomposition of k-forms; see, for example, Taylor (2011, Proposition 8.2). We will not recall it here, but would like to mention that it is an important ingredient to establish the Lie group structure of the groups of volume-preserving diffeomorphisms and the symplectomorphism group; Example 3.10. As we now have the necessary notation in place, let us very briefly sketch the idea of the proof.

E.18 (Submanifold structure of volume-preserving diffeomorphisms (sketch)) Define the map $\Psi_\mu \colon \operatorname{Diff}(M) \to \Omega^d(M)$, $\phi \mapsto \phi^*\mu$. Since M is compact, we endow $\Omega^d(M)$ via E.15 with a locally convex vector space structure. With some work one can show that Ψ_μ is a smooth map with derivative $T_\phi \Psi_\mu(V_\phi) = \phi^*(\mathcal{L}_{V_\phi}\omega)$ (this follows somewhat similarly to the proof that the pullback with smooth functions is smooth). Then one needs to prove that Ψ_μ is a submersion onto the cohomology class $[\mu] = \mu + \mathrm{d}\Omega^{d-1}(M) \subseteq \Omega^d(M)$ of μ. The proof uses the Hodge decomposition of d-forms to construct a splitting of the kernel of Ψ_μ. Further, one needs to work in a Sobolev completion of $\mathrm{Diff}(M)$, whence this is beyond the techniques we are developing in this book. We refer the interested reader to Smolentsev (2007), along with Ebin and Marsden (1970). Then the volume-preserving diffeomorphism group is simply the preimage $\Psi_\mu^{-1}(\mu)$ of the singleton μ. In particular, $\mathrm{Diff}_\mu(M)$ is a submanifold and thus a Lie subgroup of $\mathrm{Diff}(M)$.

Note that it is apparent from the derivative of Ψ_μ and Exercise 1.7.4 that the Lie algebra of $\mathrm{Diff}_\mu(K)$ is $\mathcal{V}_\mu(M) = \{X \in \mathcal{V}(M) \mid \operatorname{div} X = 0\}$, the Lie algebra of divergence-free vector fields.

Exercises

E.3.1 Show that the structure described in E.15 yields a vector bundle $\mathcal{A}^k(M) \to M$.

E.3.2 Prove that the characterisation of differential forms via the bundle in E.15 coincides with the one from Definition E.1.

E.3.3 Consider $\mathbb{R}^d$ as a Riemannian manifold with the standard Euclidean metric. Convince yourself that div, grad and Δ are 'the usual' differential operators from vector calculus in this case.

E.3.4 Work out the details for E.18 (note that this requires the Hodge decomposition theorem, Abraham et al., 1988, Theorem 7.5.3). Show that:

(a) Ψ_μ is smooth with surjective derivative;

(b) the kernel of $T_\eta \Psi_\mu$ is a split subspace of $T_\eta \mathrm{Diff}(M)$.

Remark: If $\mathrm{Diff}(M)$ were a Banach manifold the above would imply that Ψ_μ is a submersion. This is one reason why manifolds of finitely often differentiable mappings enter the picture here: The same statements as in the C^∞-case can be proven and these manifolds turn out to be Banach manifolds.

E.3.5 Let (M, g) be a Riemannian manifold with metric derivative ∇. Show that for vector fields $X, Y, Z \in \mathcal{V}(M)$, the following formula holds:

$$g(Z, \operatorname{grad} g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

Then deduce that this implies $g(\nabla_X Y, Y) = \frac{1}{2} g(X, \operatorname{grad} g(Y, Y))$.

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