A software package for Mori dream spaces

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Abstract

Mori dream spaces form a large example class of algebraic varieties, comprising the well-known toric varieties. We provide a first software package for the explicit treatment of Mori dream spaces and demonstrate its use by presenting basic sample computations. The software package is accompanied by a Cox ring database which delivers defining data for Cox rings and Mori dream spaces in a suitable format. As an application of the package, we determine the common Cox ring for the symplectic resolutions of a certain quotient singularity investigated by Bellamy–Schedler and Donten-Bury–Wiśniewski.

1. Introduction

By a *Mori dream space* we mean here a normal complete variety X defined over an algebraically closed field $\mathbb{K}$ of characteristic zero having a finitely generated divisor class group $\mathrm{Cl}(X)$ and a finitely generated *Cox ring*

$$\mathcal{R}(X) = \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)),$$

where we refer to [2] for the details of the definition. Mori dream spaces have been introduced by Hu and Keel [19] as a class of varieties with an optimal behaviour with respect to the minimal model program. Well-known examples of Mori dream spaces are toric and, more generally, spherical varieties, smooth Fano varieties [8] or Calabi–Yau varieties with a polyhedral effective cone [27]. Examples of general type can be obtained by Lefschetz-type theorems [1, 21].

An important feature of Mori dream spaces is that they allow an explicit encoding in terms of algebraic and combinatorial data [2, 6, 15] and their theory has close relations to toric geometry. This turns Mori dream spaces into natural candidates for extending the 'testing ground of algebraic geometry' given by the toric varieties. For working explicitly with toric varieties, there are meanwhile several software packages available, such as [7, 9, 14, 22, 23, 28]. Our intention is to provide with `MDSpackage` [17] such a tool also for the larger class of Mori dream spaces.

Let us give a more concrete impression. Every Mori dream space X is encoded by its Cox ring $\mathcal{R}(X)$ plus data located in the divisor class group $\mathrm{Cl}(X)$ which fix the isomorphy type among all varieties sharing $\mathcal{R}(X)$ as Cox ring. For instance, we can define a three-dimensional projective Mori dream space X by prescribing its divisor class group as $K := \mathbb{Z}^2$, its K-graded Cox ring as

$$R := \mathbb{K}[T_1, \ldots, T_6]/\langle T_1 T_2 + T_3 T_4 + T_5^2 + T_6^2 \rangle,$$
$$Q := \begin{bmatrix} -2 & 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$
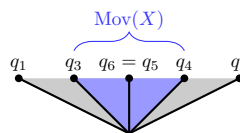
where Q has the generator degrees $\deg(T_i) \in K$ as its columns, and an ample class $w \in K$ taken from the relative interior of the above blue cone, that is, the prospective moving cone of X. The K-grading of R defined by Q gives rise to an action of the 2-torus $H = (\mathbb{K}^*)^2$ on

$$\overline{X} = V(T_1 T_2 + T_3 T_4 + T_5^2 + T_6^2) \subseteq \mathbb{K}^6$$

and the Mori dream space X is the quotient by H of the set $\widehat{X} \subseteq \overline{X}$ of semistable points associated to the weight w. As mentioned, the divisor class group and Cox ring of X are given by $\mathrm{Cl}(X) = K$ and $\mathcal{R}(X) = R$. The task of `MDSpackage` is then to extract further geometric invariants and properties from these defining data. For example, it determines the Picard group and the singularities of X as

$$\mathrm{Pic}(X) = 6\mathbb{Z} \oplus 3\mathbb{Z} \subseteq \mathbb{Z}^2 = \mathrm{Cl}(X),$$
$$\mathrm{Sing}(X) = \{\{1,5,6\}, \{1,2,5,6\}, \{1,2,6\}, \{2,3\}, \{1,4\}, \{1,2,5\}\},$$

where the output on the singularities provides information on their Cox coordinates; for example, $\{1,5,6\}$ says that there is a singular point $x \in X$ stemming from a point $z \in \widehat{X} \subseteq \mathbb{K}^6$ having precisely z_1, z_5, z_6 as non-zero coordinates.

`MDSpackage` aims to be an easy-to-use computing environment for up to medium-size computations. In the subsequent section, we present sample computations and thereby explain the syntax. The major computational ingredients come from commutative algebra and polyhedral combinatorics. The basic features of `MDSpackage` are:

- basics on finitely generated abelian groups and algebras graded by them;
- computing Picard group, local class groups, cones of effective, movable or semiample divisor classes, Mori chamber decomposition, pullback of $\mathbb{Q}$-Cartier divisors;
- computing the canonical toric ambient variety, induced orbit stratification, irrelevant ideal;
- testing (quasi-)smoothness, ($\mathbb{Q}$-)factoriality, completeness, (quasi-)projectivity;
- computing the singular locus, global resolution of singularities (approved for varieties with torus action of complexity one, experimental in the general case);
- for complete intersection Cox rings: computing intersection numbers, graph of exceptional divisors, anticanonical divisor class, Gorenstein index, testing ($\mathbb{Q}$-)Gorenstein and Fano properties;
- for varieties with a torus action of complexity one: tests for being (ε-log) terminal, almost homogeneous, computing roots of the automorphism group.

Detailed background on the algorithms and a complete manual for `MDSpackage` can be found in [**25**]. Moreover, a comprehensive online manual is available at [**17**]. Our package is implemented in the computer algebra system `Maple` [**26**] and makes use of the `convex` package by Franz [**13**].

`MDSpackage` has been essentially used in the classification of Fano varieties. In [**4**] the $\mathbb{Q}$-factorial terminal Fano threefolds of Picard number one with an effective action of a two-dimensional torus are classified; among other computations performed with `MDSpackage`, this involves more than 10^6 terminality tests for possible candidates. Other applications are the classification results on $\mathbb{K}^*$-surfaces of high Gorenstein index given in [**25**].

In order to open a broad computer-supported access to Mori dream spaces, we are building up a database of Cox rings [**16**], the entries of which can be exported in a suitable data format for `MDSpackage`. Our motivation for creating such a combined toolkit is the fruitful linking of the theory of weighted complete intersections with the *graded ring database*; see [**10**, **20**].

In Section 3 we apply `MDSpackage` to continue work of Bellamy–Schedler [**5**] and Donten-Bury–Wiśniewski [**12**] on the 81 symplectic resolutions of a certain quotient singularity: in Theorem 3.2 we determine the common Cox ring of these resolutions.

2. Working with MDSpackage

According to [6, 15], a Mori dream space X is encoded by a *bunched ring*. This basically is an integral algebra $R = \bigoplus_K R_w$ graded by a finitely generated abelian group K such that the K-homogeneous elements admit unique factorization together with a collection Φ, called 'bunch', of convex polyhedral cones in $K \otimes \mathbb{Q}$. Concretely, the K-graded algebra R is given by generators and relations and a degree map assigning to each generator its K-degree:

$$R = \mathbb{K}[T_1, \ldots, T_r]/\langle g_1, \ldots, g_s \rangle, \quad Q \colon \mathbb{Z}^r \to K, \quad e_i \mapsto \deg(T_i).$$

The bunch of cones Φ fixes the isomorphy type of X among all varieties having R as Cox ring. In case of a projective Mori dream space X, one can simply define Φ by fixing an ample class $w \in K$ from inside the moving cone.

We now demonstrate the practical work with MDSpackage [17] by means of three example computations; more can be found in [25].

EXAMPLE 2.1. The first step in defining a Mori dream space X with MDSpackage is to enter the abelian groups $E := \mathbb{Z}^r$, K and the degree map $Q \colon E \to K$:

```
> E := createAG(8);
```

$$E := AG(8, [])$$

```
> K := createAG(3, [2]);
```

$$K := AG(3, [2])$$

```
> A := cols2matrix([[1,0,1,1], [1,1,1,0], [0,1,1,1], [0,-1,1,0],
  [-1,-1,1,1], [-1,0,1,0], [2,1,1,1], [-2,-1,1,0]]);
```

$$A := \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 & 2 & -2 \\ 0 & 1 & 1 & -1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

```
> Q := createAGH(E, K, A);
```

$$Q := AGH([8, []], [3, [2]]).$$

Here we took $E = \mathbb{Z}^8$ and $K = \mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z}$ and the 4×8 matrix A fixes a map $\mathbb{Z}^8 \to \mathbb{Z}^4$ inducing the degree map $Q \colon E \to K$. The next step is to define the K-graded ring R. We have to specify variables, relations and the grading:

```
> TT := vars(8);
```

$$TT := [T[1], T[2], T[3], T[4], T[5], T[6], T[7], T[8]]$$

```
> RL := [T[1]*T[6] + T[2]*T[5] + T[3]*T[4] + T[7]*T[8]];
```

$$RL := [T[1]T[6] + T[2]T[5] + T[3]T[4] + T[7]T[8]]$$

```
> R := createGR(RL, TT, [Q], 'nocheck');
```

$$R := GR(8, 1, [3, [2]]).$$

The last output line indicates that R is given by eight generators and one relation and its grading group is isomorphic to $\mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z}$; the option `'nocheck'` speeds up the computation by omitting plausibility checks. To define X, it remains to fix a prospective ample class $w \in K \otimes \mathbb{Q}$:

```
> w := [0,0,2];
```

$$w := [0, 0, 2]$$

```
> X := createMDS(R, w);
```

$$X := MDS(8, 1, 4, [3, [2]]).$$

The last output line is similar to that for R, the only new thing is the third entry, saying that the resulting Mori dream space X is of dimension four. We are now ready for computations with X. First, we determine the Picard group of X as a subgroup of $\mathrm{Cl}(X) = K$ and the factor group $\mathrm{Cl}(X)/\mathrm{Pic}(X)$:

```
> Pic := MDSpic(X);
```

$$\mathrm{Pic} := AG(3, [])$$

```
> AGfactgrp(K, Pic);
```

$$AG(0, [2, 12, 12, 24]).$$

The first output tells us that $\mathrm{Pic}(X)$ is of rank three and torsion free, that is, we have $\mathrm{Pic}(X) \cong \mathbb{Z}^3$. The second output says that

$$\mathrm{Cl}(X)/\mathrm{Pic}(X) \;\cong\; \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/24\mathbb{Z}.$$

The cones of semiample, movable and effective divisor classes in the rational divisor class group $K \otimes_{\mathbb{Z}} \mathbb{Q}$ are computed as follows:

```
> MDSsample(X);
```

$$CONE(3, 3, 0, 8, 8)$$

```
> MDSmov(X);
```

$$CONE(3, 3, 0, 4, 4)$$

```
> MDSeff(X);
```

$$CONE(3, 3, 0, 4, 4).$$

These cones are stored in `convex` format. In particular, for the first one, the output tells us that we have a three-dimensional cone in 3-space having zero-dimensional lineality part and eight rays and eight facets; further information on the cones can be extracted via suitable `convex` commands. We compute the Mori chamber decomposition of the effective cone:

```
> F := MDSchambers(X);
```

$$F := FAN(3, 0, [0, 0, 37]).$$

Again the fan is stored in `convex` format. We refer to [**19**] for theoretical background of the Mori chamber decomposition and to [**24**] for the algorithmic aspects. We can also ask `MDSpackage` for a `povray` visualization:

```
> render(F);
```
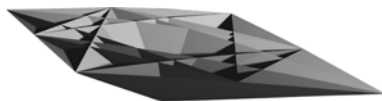

For complete intersection Cox rings like the current example, we may also check if X is a Fano variety and compute its Gorenstein index:

```
> MDSisfano(X);
```

$$true$$

```
> MDSgorensteinind(X);
```

$$4$$

EXAMPLE 2.2. We consider the Gorenstein del Pezzo surface X having Picard number one, singularity type E_6A_2 and finite automorphism group. The Cox ring is given by

$$\mathcal{R}(X) = \mathbb{K}[T_1, \dots, T_4]/\langle -T_1T_4^2 + T_2^3 + T_2T_3T_4 + T_3^3 \rangle, \quad Q = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \overline{1} & \overline{2} & \overline{0} & \overline{1} \end{bmatrix},$$

where the divisor class group $\mathrm{Cl}(X)$ is isomorphic to $K = \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$; see [**18**, Theorem 4.1]. First, we have to enter X:

```
> TT := vars(4);
```

$$TT := [T[1], T[2], T[3], T[4]]$$

```
> RL := [-T[1]*T[4]^2 + T[2]^3 + T[2]*T[3]*T[4] + T[3]^3];
```

$$RL := [-T[1]T[4]^2 + T[2]^3 + T[2]T[3]T[4] + T[3]^3]$$

```
> B := cols2matrix([[1,1], [1,2], [1,0], [1,1]]);
```

$$B := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

```
> Q := createAGH(createAG(4), createAG(1, [3]), B);
```

$$Q := AGH([4, []], [1, [3]])$$

```
> R := createGR(RL, TT, [Q]);
```

$$R := GR(4, 1, [1, [3]])$$

```
> X := createMDS(R, [1]);
```

$$X := MDS(4, 1, 2, [1, [3]]).$$

We take a look at the singularities of X and compute a resolution. Let us check smoothness properties and determine the singular locus:

> `MDSissmooth(X);`

$$false$$

> `MDSisquasismooth(X);`

$$false$$

> `L := MDSsing(X);`

$$L := [[[-2T[1]T[4] + T[2]T[3], T[2]T[4] + 3T[3]^2, 3T[2]^2 + T[3]T[4], -T[4]^2,$$
$$-T[1]T[4]^2 + T[2]^3 + T[2]T[3]T[4] + T[3]^3], \ [T[1], T[2], T[3], T[4]]], \ [\{4\}, \{1\}]].$$

The first entry of `L` is a list of generators for the ideal of the singular locus of the total coordinate space $\overline{X}$ and a list of used variables. As mentioned, the second entry tells us that the singularities are the points $[0, 0, 0, 1]$ and $[1, 0, 0, 0]$, given in Cox coordinates. The minimal resolution $X_2 \to X$ is computed by

> `X2 := MDSresolvesing(X, 'verify', 'minimal');`
  Verification successful.

$$X2 := MDS(12, 1, 2, [9, []]).$$

Since the verification was successful, this means that X_2 is indeed a smooth Mori dream space. In accordance with [**11**, p. 40, type $E_6 A_2$], we obtain for the Cox ring of X_2 a single defining equation and the following degree matrix:

> `R2 := MDSdata(X2)[1];`

$$R2 := GR(12, 1, [9, []])$$

> `GRdata(R2)[1];`

$$[-T[1]T[4]^2T[5] + T[2]^3T[12]T[7]^2T[8] + T[2]T[3]T[11]T[4]T[5]T[6]T[7]T[8]T[10]$$
$$+ T[3]^3T[9]T[11]^2T[10]]$$

> `GRdata(R2)[3];`

$$AGH([12, []], [9, []]), \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

We compute the graph of exceptional curves and a list of the self-intersection numbers of the invariant divisors $V(X; T_i)$; looking at the subgraph of (-2)-curves, we then see the E_6- and the A_2-singularities:

```
> MDSintersgraph(X2, 'latex');
```
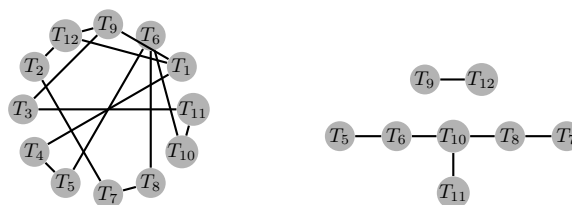


```
> MDSintersno(X2);
```

$$[-1, -1, -1, -1, -2, -2, -2, -2, -2, -2, -2, -2].$$

The plot of the intersection graph is provided by `MDSpackage` either as `Maple` objects or in LATEXformat.

EXAMPLE 2.3. We consider a variety with torus action of complexity one. The Cox ring R of any such variety Y can be simply encoded by a pair (A, P) of matrices; see for example [**2**] for details. Here is how to enter it in `MDSpackage`:

```
> P := cols2matrix([ [-2,-2,-1,-1], [1,0,0,0], [1,0,1,0], [0,1,0,1],
  [0,1,0,0] ])
```

$$P := \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

```
> A := [[1,0],[0,1],[-1,-1]];
```

$$A := [[1, 0], [0, 1], [-1, -1]]$$

```
> R := createGR(P, A);
```

$$R := GR(5, 1, [1, []])$$

```
> GRdata(R);
```

$$[[T[4]T[5] + T[2]T[3] + T[1]^2], [T[1], T[2], T[3], T[4], T[5]],$$
$$[AGH([5, []], [1, []]), [1 \quad 1 \quad 1 \quad 1 \quad 1]], \dots].$$

So far we have defined the prospective Cox ring R. To specify a variety Y with Cox ring R, we have to fix a bunch Φ. However, the unit component $\text{Aut}(Y)^0$ of the (linear algebraic) automorphism group depends only on the Cox ring and we can compute its roots as follows:

```
> MDSautroots(R);
```

$$\{[1, -1], [1, 1], [-1, -1], [-1, 1], [0, -1], [0, 1], [1, 0], [-1, 0]\}.$$

We have obtained the root system B_2 of the corresponding orthogonal group, as expected for the smooth quadric Y in $\mathbb{P}_5$.
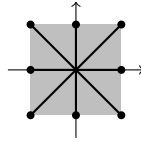
REMARK 2.4. As a support of `MDSpackage`, we build up the *Cox ring database* [**16**]. This database stores known Cox rings of varieties. Among others, it provides at the moment the Cox rings of all non-toric:
- smooth rational surfaces up to Picard number five;
- smooth rational surfaces of Picard number six admitting only trivial $\mathbb{K}^*$-actions;
- Gorenstein del Pezzo surfaces of Picard number one;
- cubic surfaces with at most rational double points;
- Gorenstein del Pezzo $\mathbb{K}^*$-surfaces;
- smooth Fano threefolds of Picard number at most two;
- terminal Fano threefolds of Picard number one with a two-torus action.

We use the representation in terms of bunched rings explained in Section 2: for a given Mori dream space X, the database stores the tuple $(r, \{g_1, \ldots, g_s\}, K, Q, \Phi)$, where the divisor class group and the Cox ring of X are given by

$$\mathrm{Cl}(X) = K, \quad \mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_r]/\langle g_1, \ldots, g_s \rangle,$$

the entry Q is the degree matrix and Φ is the bunch of cones fixing X; in case of a projective X we just replace Φ with an ample class $w \in K$. Here are some basic features of the database:
- if available, further information is stored, such as basic geometric properties (dimension, Picard number, type of singularities, etc.), reference to author and original literature;
- the web interface [**16**] allows searching the Cox ring database according to keywords, authors and geometric properties;
- the stored Cox rings and varieties can be directly exported in the necessary syntax for `MDSpackage` or LATEX.

For instance, the 40 Fano threefolds of [**4**] are found by entering `Fano AND terminal AND Picard number one AND dimension three AND Q-factorial` in the search field `description`. Searching the `id` field for 96, 97, 6 and 98 yields the Cox rings discussed in the introduction and the previous three examples.

3. *Application: resolutions of a quotient singularity*

We use `MDSpackage` to study a symplectic quotient singularity discussed by Bellamy and Schedler [**5**] and by Donten-Bury and Wiśniewski [**12**]. More precisely, we consider the subgroup $G \subseteq \mathrm{Gl}_4(\mathbb{C})$ generated by the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & \mathrm{I} & 0 & 0 \\ -\mathrm{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{I} \\ 0 & 0 & \mathrm{I} & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & \mathrm{I} \\ 0 & 0 & -\mathrm{I} & 0 \\ 0 & \mathrm{I} & 0 & 0 \\ -\mathrm{I} & 0 & 0 & 0 \end{bmatrix}.$$

The aim is to study resolutions of singularities of the quotient space $\mathbb{C}^4/G$. In Theorem 3.2, we will determine for one of them the Cox ring and show that all of its flops are as well resolutions of singularities. This continues [5, 12] and also retrieves the 81 symplectic resolutions of $\mathbb{C}^4/G$ discussed there.

Observe that the commutator subgroup $[G, G] \subseteq G$ is generated by the negative unit matrix $-E_4$ and hence is of order two. Moreover, the abelianization $G' := G/[G, G]$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$ and generated by the classes of the last four matrices. A first step is to determine the Cox ring of the quotient space $\mathbb{C}^4/G$.

PROPOSITION 3.1. *The quotient* $\mathbb{C}^4/[G, G] \to \mathbb{C}^4/G$ *by the induced action of* G' *is the characteristic space over* $\mathbb{C}^4/G$. *The Cox ring* $\mathcal{R}(\mathbb{C}^4/G) \cong \mathbb{C}[S_1, \ldots, S_4]^{[G,G]}$ *is isomorphic to* $\mathbb{C}[T_1, \ldots, T_{10}]/I$, *where the ideal* I *is generated by*

$$T_5T_6 + T_8T_9 - T_7T_{10}, \quad T_4T_6 + T_2T_7 + T_1T_8, \quad T_3T_6 - T_2T_9 - T_1T_{10},$$
$$T_2T_6 + T_4T_7 + T_3T_9, \quad T_1T_6 - T_4T_8 - T_3T_{10}, \quad T_5^2 - T_7^2 + T_9^2 + T_2^2,$$
$$T_4T_5 + T_1T_9 + T_2T_{10}, \quad T_3T_5 - T_1T_7 - T_2T_8, \quad T_2T_5 + T_3T_8 + T_4T_{10},$$
$$T_1T_5 - T_3T_7 - T_4T_9, \quad T_4^2 - T_6^2 - T_9^2 + T_{10}^2, \quad T_3T_4 + T_7T_9 - T_8T_{10},$$
$$T_2T_4 + T_6T_7 - T_5T_{10}, \quad T_1T_4 - T_6T_8 + T_5T_9, \quad T_3^2 + T_6^2 - T_7^2 + T_8^2,$$
$$T_2T_3 + T_5T_8 - T_6T_9, \quad T_1T_3 - T_5T_7 + T_6T_{10}, \quad T_2^2 - T_6^2 - T_8^2 + T_{10}^2,$$
$$T_1T_2 + T_7T_8 - T_9T_{10}, \quad T_1^2 + T_6^2 - T_7^2 + T_9^2$$

and the grading of $\mathbb{C}[T_1, \ldots, T_{10}]/I$ *by* $\mathrm{Cl}(\mathbb{C}^4/G) \cong \mathbb{X}(G') \cong (\mathbb{Z}/2\mathbb{Z})^4$ *is given by assigning to the class of* T_i *the* i*th column of the following matrix over* $\mathbb{Z}/2\mathbb{Z}$:

$$Q := \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

The variables T_i *define pairwise non-associated* $\mathbb{X}(G')$-*prime elements in* $\mathcal{R}(\mathbb{C}^4/G)$. *The above presentation of the Cox ring gives rise to a commutative diagram*

$$
\begin{array}{ccc}
\mathbb{C}^4/[G,G] & \longrightarrow & \mathbb{C}^{10} \\
{\scriptstyle /G'}\downarrow & & \downarrow{\scriptstyle /G'} \\
\mathbb{C}^4/G & \longrightarrow & Z
\end{array}
$$

where the horizontal arrows are closed embeddings. The ambient affine toric variety $Z = \mathbb{C}^{10}/G'$ *of* $\mathbb{C}^4/G$ *is of dimension ten and arises from the cone* $\sigma \subseteq \mathbb{Q}^{10}$ *generated by the vectors*

$$v_1 := e_1, \quad v_2 := e_2, \quad v_3 := e_3, \quad v_4 := e_4, \quad v_5 := e_5,$$
$$v_6 := e_1 + e_2 + e_5 + 2e_6, \quad v_7 := e_7, \quad v_8 := e_4 + e_5 + e_7 + 2e_8,$$
$$v_9 := e_1 + e_3 + e_4 + e_5 + e_7 + 2e_8, \quad v_{10} := e_1 + e_2 + e_7 + 2e_{10}.$$

There are precisely five toric orbits of dimension six in Z *that contain singularities of* $\mathbb{C}^4/G$. *Any further singularities of* $\mathbb{C}^4/G$ *are contained in the closures of these orbits. The faces of* σ *corresponding to these five toric orbits are:*

$$\mathrm{cone}(v_1, v_2, v_5, v_6), \quad \mathrm{cone}(v_4, v_5, v_7, v_8), \quad \mathrm{cone}(v_1, v_3, v_8, v_9),$$
$$\mathrm{cone}(v_4, v_6, v_9, v_{10}), \quad \mathrm{cone}(v_2, v_3, v_7, v_{10}).$$

Proof. The first sentence is a direct consequence of a more general statement due to Arzhantsev and Gaĭfullin [**3**]. We verify the claimed presentation of the Cox ring

$$\mathcal{R}(\mathbb{C}^4/G) \cong \mathbb{C}[S_1, \ldots, S_4]^{[G,G]}.$$

The algebra of $[G, G]$-invariant polynomials is generated by the quadratic monomials of $\mathbb{C}[S_1, \ldots, S_4]$. Moreover, G' acts on the ten-dimensional $\mathbb{C}$-vector subspace generated by these monomials and the following G'-homogeneous polynomials form a basis of this vector subspace:

$$S_1S_2 + S_3S_4, \quad S_1S_2 - S_3S_4, \quad S_1S_3 + S_2S_4,$$
$$S_1S_3 - S_2S_4, \quad S_1S_4 + S_2S_3, \quad S_1S_4 - S_2S_3,$$
$$\tfrac{1}{2}(S_1^2 + S_2^2 + S_3^2 + S_4^2), \quad \tfrac{1}{2}(-S_1^2 - S_2^2 + S_3^2 + S_4^2),$$
$$\tfrac{1}{2}(-S_1^2 + S_2^2 - S_3^2 + S_4^2), \quad \tfrac{1}{2}(S_1^2 - S_2^2 - S_3^2 + S_4^2).$$

According to the order of listing, the weights of these polynomials occur as columns of the matrix Q of the assertion. Thus, we have found ten $\mathbb{X}(G')$-homogeneous prime generators for $\mathbb{C}[S_1, \ldots, S_4]^{[G,G]}$.

The desired presentation of the Cox ring is now obtained by sending T_i to the ith generator and the ideal I is computed as the ideal of algebraic relations between the above generators. The remaining claims are direct applications of [**2**, Sections 3.2.5 and 3.3.1]. □

The second step is to compute the Cox ring $\mathcal{R}(X)$ for a certain resolution $X \to \mathbb{C}^4/G$ of singularities. This is done using the technique of toric ambient modifications introduced in [**15**]: we just perform an evident partial toric resolution of the ambient affine toric variety Z and show that the proper transform X of $\mathbb{C}^4/G$ is as wanted. The 81 symplectic resolutions then correspond to the full-dimensional chambers of the Mori chamber decomposition of the cone of movable divisor classes of X.

THEOREM 3.2. *Consider the polynomial ring* $\mathbb{C}[T_1, \ldots, T_{15}]$ *with the* $\mathbb{Z}^5$*-grading assigning to the variables the columns of the following matrix:*

$$\begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & -1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & -1 & 1 & 1 & -1 & -1
\end{bmatrix}$$

and the ideal $I \subseteq \mathbb{C}[T_1, \ldots, T_{15}]$ *generated by the following* $\mathbb{Z}^5$*-homogeneous polynomials:*

$$T_3T_6 - T_2T_9 - T_1T_{10}, \quad T_3T_5 - T_1T_7 - T_2T_8, \quad T_3T_4 + T_7T_9 - T_8T_{10},$$
$$T_2T_4 + T_6T_7 - T_5T_{10}, \quad T_1T_4 - T_6T_8 + T_5T_9,$$
$$T_1T_8T_{13} + T_4T_6T_{14} + T_2T_7T_{15}, \quad T_5T_8T_{12} - T_6T_9T_{14} + T_2T_3T_{15},$$
$$T_5T_7T_{12} - T_1T_3T_{13} - T_6T_{10}T_{14}, \quad T_5^2T_{12} - T_1^2T_{13} - T_6^2T_{14} + T_2^2T_{15},$$
$$T_4T_5T_{12} + T_1T_9T_{13} + T_2T_{10}T_{15}, \quad T_6^2T_{11} - T_4^2T_{12} + T_9^2T_{13} - T_{10}^2T_{15},$$
$$T_5T_6T_{11} + T_8T_9T_{13} - T_7T_{10}T_{15}, \quad T_2T_6T_{11} + T_4T_7T_{12} + T_3T_9T_{13},$$
$$T_1T_6T_{11} - T_4T_8T_{12} - T_3T_{10}T_{15}, \quad T_5^2T_{11} + T_8^2T_{13} + T_4^2T_{14} - T_7^2T_{15},$$
$$T_2T_5T_{11} + T_3T_8T_{13} + T_4T_{10}T_{14}, \quad T_1T_5T_{11} - T_4T_9T_{14} - T_3T_7T_{15},$$
$$T_2^2T_{11} - T_7^2T_{12} + T_3^2T_{13} + T_{10}^2T_{14}, \quad T_1T_2T_{11} + T_7T_8T_{12} - T_9T_{10}T_{14},$$
$$T_1^2T_{11} - T_8^2T_{12} + T_9^2T_{14} - T_3^2T_{15}.$$

Then the ideal I is homogeneous with respect to the $\mathbb{Z}^5$-grading and, thus, the factor ring $R := \mathbb{C}[T_1, \ldots, T_{15}]/I$ inherits a $\mathbb{Z}^5$-grading. Moreover, the following statements hold.

(i) *The ring R is factorial, the variables $T_1, \ldots, T_{15}$ define pairwise non-associated prime elements in R and any fourteen of the degrees of the fifteen variables generate $\mathbb{Z}^5$ as a group.*

(ii) *The moving cone $\mathrm{Mov}(R)$ of the $\mathbb{Z}^5$-graded ring R is of dimension five and contains precisely 81 five-dimensional geometric invariant theory (GIT) chambers. Each of these chambers defines a smooth variety X_i having divisor class group $\mathrm{Cl}(X_i) \cong \mathbb{Z}^5$ and Cox ring $\mathcal{R}(X_i) = R$.*

(iii) *The subring $R_0 \subseteq R$ of elements of degree zero is isomorphic to the coordinate ring of $\mathbb{C}^4/G$ and, thus, we have canonical morphisms $X_i \to \mathbb{C}^4/G$, each of which is a symplectic resolution of singularities.*

Proof. We start with $X_0 := \mathbb{C}^4/G$ and the presentation of its Cox ring $R_0 := \mathcal{R}(X_0)$ given in Proposition 3.1. We enter the relations of R_0 given there as a list `RL0` in `MDSpackage` and as well the grading matrix, which we now call `Q0mat`. Using these data, we then define the degree map $Q_0 \colon \mathbb{Z}^{10} \to (\mathbb{Z}/2\mathbb{Z})^4$, the Cox ring R_0 and X_0:

```
> Q0 := createAGH(createAG(10),createAG(0,[2,2,2,2]),Q0mat);
```

$$Q0 := AGH([10, []], [0, [2, 2, 2, 2]])$$

```
> R0 := createGR(RL0, vars(10), [Q0], 'Singular');
```

$$R0 := GR(10, 20, [0, [2, 2, 2, 2]])$$

```
> X0 := createMDS(R0,[0]);
```

$$X0 := MDS(10, 20, 4, [0, [2, 2, 2, 2]]).$$

We resolve the singularities of X_0 via a suitable toric modification of the ambient affine toric variety Z. By Proposition 3.1, there are five singular toric orbits, the closures of which form the singular locus of Z. The first evident step towards any toric resolution is to resolve the corresponding cones. This is achieved in each of the five cases by a single barycentric subdivision, that is, we insert five new rays $\mathbb{Q}_{\geqslant 0} \cdot v_1, \ldots, \mathbb{Q}_{\geqslant 0} \cdot v_5$ in total, where the list of v_i is

```
> VL := [[1,1,0,0,1,1,0,0,0,0], [0,0,0,1,1,0,1,1,0,0],
  [1,0,1,1,1,0,1,1,1,0], [1,1,1,1,1,1,1,0,1,1], [0,1,1,0,0,0,1,0,0,1]];
```

$$VL := [[1, 1, 0, 0, 1, 1, 0, 0, 0, 0], [0, 0, 0, 1, 1, 0, 1, 1, 0, 0], [1, 0, 1, 1, 1, 0, 1, 1, 1, 0],$$
$$[1, 1, 1, 1, 1, 1, 1, 0, 1, 1], [0, 1, 1, 0, 0, 0, 1, 0, 0, 1]].$$

The Cox ring R of the proper transform X of X_0 under such a partial ambient resolution can be computed via the command `MDSmodify`, which implements [18, Algorithm 3.6]:

```
> X := MDSmodify(X0, VL, 'Singular', 'verify');
```

Verification successful: all variables are prime.

$$X := MDS(15, 20, 4, [5, []])$$

```
> R := MDSdata(X)[1];
```

$$R := GR(15, 20, [5, []]).$$

The algorithm thus returns the $\mathbb{Z}^5$-graded ring R as the Cox ring $R := R_2$ of the assertion and, additionally, verifies the statements of the first assertion. For the second assertion, we compute the GIT fan using MDSpackage's implementation of [**25**, Algorithm 8]:

```
> F := GRgitfan(R, 'fan');
```

$$F := FAN(5, 0, [0, 0, 0, 0, 207])$$

```
> Mov := MDSmov(X);
```

$$\text{Mov} := CONE(5, 5, 0, 5, 5).$$

The command U := select(c -> contains(Mov,c), maximal(F)) retrieves the 81 full-dimensional chambers inside the moving cone, as claimed. We test smoothness of the varieties X_i associated to the chambers U[i] using an MDSpackage function based on the criterion [**2**, Corollary 3.3.1.12]; compare [**12**] for a related approach:

```
> w1 := relint(U[1])
```

$$w1 := [6, 1, 3, 3, 3]$$

```
> X1 := createMDS(R, w1);
```

$$X1 := MDS(15, 20, 4, [5, []])$$

```
> MDSissmooth(X1, 'magma');
```

$$true$$

All test results are positive. The computation is sped up a bit by calling magma under the surface; it takes some hours on a personal computer. Note that X is among the X_i. The third assertion is clear by the way we obtained the common Cox ring R of the X_i. □

REMARK 3.3. Having proved Theorem 3.2, a discussion of the geometry of the resolutions is possible via the methods of MDSpackage.

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