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Invariant subsets of expanding mappings of the circle

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Abstract The continuity of Hausdorff dimension of closed invariant subsets K of a C^2-expanding mapping g of the circle is investigated If $g|K$ satisfies the specification property then the equilibrium states of Holder continuous functions are studied It is proved that if f is a piecewise monotone continuous mapping of a compact interval and ϕ a continuous function with $P(f, \phi) > \sup(\phi)$, then the pressure $P(f, \phi)$ is attained on one-dimensional 'Smale's horseshoes', and some results of Misiurewicz and Szlenk [**M–Sz**] are extended to the case of pressure

1 The main aim of this paper is to extend the results of [**U**] refering to the continuity of Hausdorff dimension and topological entropy to the case of an arbitrary C^2-expanding mapping g and wider classes of closed invariant subsets In order to do this, one needs to deal with pressure instead of entropy Approximating continuous functions by piecewise constant functions we are able to develop the methods from [**M–Sz**] to the case of pressure Consequently it permits us to find wide classes of closed invariant subsets of g, the Hausdorff dimension function restricted to which is continuous In particular, as a corollary we obtain the existence of closed invariant subsets of Hausdorff dimension t for every $0 \le t \le 1$

In § 3 we deal with the family $\{K(\varepsilon)\}_{\varepsilon \in [0\ 1]}$ defined in [**U**] We distinguish the set of parameters of Hausdorff dimension 1 for which the mappings $g|K(\varepsilon)$ satisfy the specification property which enables us to make use of the results from [**P–U–Z**] to study the equilibrium states of Holder continuous functions The author would like to thank M Misiurewicz and F Przytycki for inspiration and helpful discussions around the subject of this paper

Now we want to introduce the basic notation and definitions used The circle S^1 is always assumed to have length 1 If $x, y \in S^1$, $x \neq y$, then $(x, y) \subset S^1$ denotes the open arc anticlockwise oriented from x to y and is called *the open interval from x to y* The symbols $[x, y]$ and $[x, y)$ are understood in a similar way X is always assumed to be a closed subset of either the circle or a compact interval of the real line We will call a subset a of X *an interval* mod (X) iff a is the intersection of X and an interval A mapping f $X \to X$ has by definition *the Darboux property* on $Y \subset X$ iff $f|Y$ maps intervals mod (X) onto intervals mod (X) If X is a closed subset of a compact interval of the real line then the monotonicity of $f|Y$ is understood in the standard way If X is a closed subset of the circle then $f|Y$ is said to be monotone iff diam(Y), diam$(f(Y))<\frac{1}{2}$ and if $Y'\supset Y$, $Y''\supset f(Y)$ are intervals of the circle with diam(Y'), diam$(Y'')<\frac{1}{2}$ then $f|Y$ is monotone with respect to the orders on Y' and Y'' Observe that this definition does not depend on the choice of Y' and Y'' This follows because of our restriction of diameters, and in fact in this paper we will need monotonicity only on sets of arbitrarily small diameters

Definition 1 f $X\to X$ is called a *mapping of Misiurewicz-Szlenk* iff X can be covered by a finite number of intervals mod(X) on which f is continuous monotone and satisfies the Darboux property

If moreover ϕ $X\to\mathbb{R}$ is piecewise constant i e X can be expressed as a union of a finite number of intervals mod(X) on which ϕ is constant, then let $\mathscr{A}$ denote a partition into intervals on which f is continuous, monotone, has the Darboux property and ϕ is constant We will call it an *admissible partition* for f and ϕ

Write $\mathscr{A}=\{a_1, \quad, a_k\}$ and define the function $\imath$ $X\to\{1, \quad, k\}$ by $\imath(x)=\jmath$ if $x\in a_\jmath\in\mathscr{A}$ and s $X\to\Sigma_+=\{1, \quad, k\}^\infty$ by $s(x)=\imath(x)\imath(f(x))$

Now it is easy to see that the set $\Sigma_f=\mathrm{cl}(s(X))\subset\Sigma_+$ is σ-(the shift mapping)-invariant and $\sigma\circ s=s\circ f$ Let us observe that since ϕ is constant on each element of $\mathscr{A}$, one can define the function $\tilde{\phi}$ $\Sigma_f\to\mathbb{R}$ putting $\tilde{\phi}(\{\imath_\jmath\}_{\jmath=1}^\infty)=\phi(a_{\imath_1})$

$\tilde{\phi}$ depends only on the first coordinate, so we will sometimes simply write $\tilde{\phi}(\jmath)$ or $\phi(a_\jmath)$ $\jmath=1, \quad, k$, and is continuous

Definition 2 $P(f,\phi)\stackrel{\mathrm{def}}{=}\sup_{\mu\in M_f}(h_\mu(f)+\int\phi\,d\mu)$ where M_f is the set of all f-invariant, ergodic probability measures on X

$P(f,\phi,\mathscr{A})$ is defined as the usual pressure $P(\sigma,\tilde{\phi})$ (see for instance [**M**])

As an immediate consequence of this definition and the variational principle for pressure [**M**] we get

(1) $P(f,\phi,\mathscr{A})=\limsup_{n\to\infty}1/n\log(\sum_{a\in\mathscr{A}^n}\exp(\phi(a)+\quad+\phi(f^{n-1}(a))))$

(2) $P(f,\phi,\mathscr{A})=\sup_{\mu\in M_\sigma}(h_\mu+\int\tilde{\phi}\,d\mu)$

LEMMA 1 *Let M_f^+, M_σ^+ denote the subsets of measures with positive entropy of M_f and M_σ respectively Then the mapping $M_f^+\ni\mu\mapsto s_*(\mu)$ is a bijection between M_f^+ and M_σ^+ and s establishes a metric isomorphism between (X,f,μ) and $(\Sigma_f,\sigma,s_*(\mu))$*

Proof We will follow F Hofbauer [**H**$_1$], [**H**$_2$] Let $\mu\in M_f^+$, $x,y\in X$ and suppose that $s(x)=s(y)=\bar{x}$ i e $f^k(x)$ and $f^k(y)$ are in the same $a_\imath$ for every $k\geq0$ If z is in the interval with endpoints x and y, it follows that $f^k(z)$ is in the interval with endpoints $f^k(x)$ and $f^k(y)$ So $s(z)=\bar{x}$ This means that $s^{-1}(\bar{x})$ is a subinterval of X Let

$$H=\{\bar{x}\in\Sigma_f\ \ s^{-1}(\bar{x})\text{ is a non-one point interval}\}$$

As there can be only countably many disjoint subintervals of X with positive length, H is at most countable Since $s_*(\mu)$ is ergodic, if H were of positive measure we would find a periodic point $w\in H$, say $\sigma^k(w)=w$, such that $s_*(\mu)(\{w, \quad, \sigma^{k-1}(w)\})=1$ But then $\mu(\{s^{-1}(w), \quad, s^{-1}(\sigma^{k-1}(w))\})=1$ and for every $0\leq\jmath\leq k-1$, f^k $s^{-1}(\sigma^\jmath(w))\to s^{-1}(\sigma^\jmath(w))$ is monotone So $h_\mu(f^k)=0$

and consequently $h_\mu(f)=0$ - the contrary Hence $\mu(s^{-1}(H))=0$ and s is injective modulo a set of measure 0 Thus s $(X, f, \mu)\to(\Sigma_f, \sigma, s_*(\mu))$ is an isomorphism and $h_{s_*(\mu)}(\sigma)>0$

Now, let $\nu\in M_\sigma^+$ Hofbauer [$\mathbf{H_2}$] proved that the set $\Sigma_f\backslash s(X)$ is countable So $\nu(s(X)\backslash H)=1$ and hence $s_*^{-1}(\nu)$ is well defined Now $s_*(s_*^{-1}(\nu))=\nu$ and the lemma is proved

LEMMA 2 *If $P(f,\phi)>\sup(\phi)$ then for any admissible partition $\mathscr{A}$, $P(f,\phi,\mathscr{A})=P(f,\phi)$*

Proof Lemma 1 implies that $\sup_{\mu\in M_\sigma^+}(h_\mu(\sigma)+\int\tilde\phi\, d\mu)\le P(f,\phi)$ If $\mu\in M_\sigma\backslash M_\sigma^+$ then

$$h_\mu(\sigma)+\int\tilde\phi\, d\mu=\int\tilde\phi\, d\mu\le\sup(\tilde\phi)=\sup(\phi)<P(f,\phi)$$

Therefore (2) implies that $P(f,\phi,\mathscr{A})\le P(f,\phi)$ On the other hand the inequality $P(f,\phi)>\sup(\phi)$ shows that

$$\sup_{\mu\in M_f\backslash M_f^+}\left(h_\mu(f)+\int\phi\, d\mu\right)<P(f,\phi)$$

Hence, using lemma 1 and (2),

$$P(f,\phi)=\sup_{\mu\in M_f\backslash M_f^+}\left(h_\mu(f)+\int\phi\, d\mu\right)\le P(\sigma,\tilde\phi)=P(f,\phi,\mathscr{A})$$

By the variational principle for pressure and definition 2 we get immediately

LEMMA 3 *If f $X\to X$ is a continuous mapping of Misiurewicz-Szlenk and ϕ $X\to R$ a continuous function such that $P(f,\phi)>\sup(\phi)$ then for a piecewise constant function $\bar\phi$ sufficiently near to ϕ in the supremum metric, $P(f,\bar\phi)>\sup(\bar\phi)$*

Remark 1 Observe that because of uniform continuity, every continuous function on X admits an approximation by piecewise constant functions

Now we will give some sufficient conditions for a piecewise constant function ϕ to attain the pressure on 'Smales horseshoes' Our considerations here are a modification of considerations of Misiurewicz and Szlenk from [**M-Sz**]

Again let f $X\to X$ be a mapping of Misiurewicz-Szlenk and ϕ $X\to R$ a piecewise constant function Let $\mathscr{A}$ denote an admissible partition For a family $\mathscr{B}\subset\mathscr{A}^n$, we write $\Sigma_n(\mathscr{B})$ for the number

$$\sum_{b\in\mathscr{B}}\exp(\phi(b)+\phi\circ f(b)+\quad+\phi\circ f^{n-1}(b))$$

If $\phi\equiv0$ this is simply Card $(\mathscr{B})$

Following [**M-Sz**] we define

$$E=\left\{a\in\mathscr{A}\ \ \limsup_{n\to\infty}\frac{1}{n}\log\Sigma_n(\mathscr{A}^n|a)\right\}=P(f,\phi,\mathscr{A})$$

By the definition of pressure this family is non-empty Exactly as lemma 6 from [**M-Sz**] we obtain the following

(1) $\limsup_{n\to\infty}\dfrac{1}{n}\log\Sigma_n(E^n|a)=P(f,\phi,\mathscr{A})$ for every $a\in E$

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Now for any $a, b \in E$ we set

$$\tilde{\gamma}(a, b, n) = \Sigma_n(\{e \in E^n \mid a \quad f^n(e) \supset b\})$$

LEMMA 4 *If* $P(f, \phi, \mathscr{A}) > \log 3 + \sup(\phi)$, *then there exists an* $a_0 \in E$ *such that* $\limsup_{n\to\infty} 1/n \log \tilde{\gamma}(a_0, a_0, n) = P(f, \phi, \mathscr{A})$

Proof Let us fix a set $a \in E$ and a real number u such that $\log 3 + \sup(\phi) < u < P(f, \phi, \mathscr{A})$ In view of (1) it is easy to see (see also [**M-Sz**]) that for every number p there exists an integer $n \geq p$ such that

$$(2) \quad \frac{1}{n} \log \Sigma_n(E^n \mid a) > u \text{ and } \Sigma_{n+1}(E^{n+1} \mid a) \geq 3\, e^{\sup(\phi)} \Sigma_n(E^n \mid a)$$

Fix a set $e \in E^n \mid a$ The set $f^n(e)$ is an interval mod (X) and therefore if it has non-empty intersections with r elements of E, then it contains at least $r-2$ of them But $r = \text{Card}(E^{n+1} \mid e)$ Thus $\text{Card}(\{b \in E \quad f^n(e) \supset b\}) \geq \text{Card}(E^{n+1} \mid e) - 2$ Hence changing the order of summation we obtain

$$\begin{aligned}
\sum_{b\in E} \tilde{\gamma}(a, b, n) &= \sum_{e\in E^n \mid a} \text{Card}(\{b \in E \quad f^n(e) \supset b\})\, e^{S_n(\phi)(e)} \\
&\geq \sum_{e\in E^n \mid a} (\text{Card}(E^{n+1} \mid e) - 2)\, e^{S_n(\phi)(e)} \\
&= \sum_{e\in E^n \mid a} \text{Card}(E^{n+1} \mid a)\, e^{S_n(\phi)(e)} - 2\Sigma_n(E^n \mid a) \\
&\geq \sum_{e\in E^n \mid a} \sum_{c\in E^{n+1} \mid e} e^{-\sup(\phi)}\, e^{S_n(\phi)(e) + \phi(f^n(c))} - 2\Sigma_n(E^n \mid a) \\
&= e^{-\sup(\phi)} \Sigma_{n+1}(E^{n+1} \mid a) - 2\Sigma_n(E^n \mid a) \geq \Sigma_n(E^n \mid a)
\end{aligned}$$

By (2) we get

$$\limsup_{n\to\infty} \frac{1}{n} \log\left(\sum_{b\in E} \tilde{\gamma}(a, b, n)\right) \geq u$$

Since u is an arbitrary number less than $P(f, \phi, \mathscr{A})$ we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \log\left(\sum_{b\in E} \tilde{\gamma}(a, b, n)\right) \geq P(f, \phi, \mathscr{A})$$

Now in exactly the same way as in [**M-Sz**] we find an $a_0 \in E$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log(\tilde{\gamma}(a_0, a_0, n)) \geq P(f, \phi, \mathscr{A})$$

The converse inequality immediately follows from our definition of pressure

THEOREM 0 *Let* $f \quad X \to X$ *be a mapping of Misiurewicz–Szlenk,* $\phi \quad X \to \mathbb{R}$ *a piecewise constant function such that* $P(f, \phi) > \sup(\phi)$, $\mathscr{A}$ *an admissible partition Then there exist*

(i) *an interval* mod (X), J,

(ii) *A sequence* $\{k_n\}_{n=1}^{\infty}$ *of positive integers,*

(iii) *A sequence* $\{D_n\}_{n=1}^{\infty}$ *of partitions of* J *by intervals* mod (X) *which belong to* $\mathscr{A}^{k_n}$ *such that*

$$\lim_{n\to\infty} \frac{1}{k_n} \log \Sigma_{k_n}(D_n) = P(f, \phi)$$

and $f^{k_n}(d) \supset J$ for any $d \in D_n$

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Proof If we take $r > \log 3/(P(f, \phi) - \sup(\phi))$ then

$$P(f^r, S_r(\phi)) = rP(f, \phi) > \log 3 + r\sup(\phi) > \log 3 + \sup(S_r(\phi))$$

and because of lemma 2 we can apply lemma 4 to the mapping f^r, function $S_r(\phi)$ and partition $\mathcal{A}^r$ So we get an interval mod (X), a_0, and a sequence $\{m_n\}_{n=1}^{\infty}$ of integers such that

$$\lim_{n\to\infty} \frac{1}{m_n} \log \tilde{\gamma}(a_0, a_0, m_n) = P(f^r, S_r(\phi), \mathcal{A}^r)) = P(f^r, S_r(\phi) = rP(f, \phi)$$

where the second equality is again due to lemma 2 And now it is sufficient to set $J = a_0$, $k_n = rm_n$ and D_n the partition by those elements d of $E_{f^r}^{m_n} | a_0$ for which $f^{rm_n}(d) \supset a_0$

Remark 2 Observe that the assumption $P(f, \phi) > \sup(\phi)$ is valid if for instance $h_{\text{top}}(f) > \sup(\phi) - \inf(\phi)$

2 Now we are able to prove several facts about continuity of Hausdorff dimension considered as a function of compact invariant subsets of expanding mappings of the circle

Throughout the whole of this section $g \; S^1 \to S^1$ will be a C^2-expanding mapping of the circle i e for some positive integer n, $|(g^n)'| > 1$, or equivalently there exists a Riemannian metric, say ρ, in which already $|g'| > 1$ Throughout the whole of this paper we will work only with this metric assuming that the length of the whole circle in it is also equal to 1 Let $\mathcal{K}$ be the class of all compact invariant subsets of g equipped with the Hausdorff metric ρ_H We have the following

LEMMA 5 *If $\phi \; S^1 \to \mathbb{R}$ is continuous then the function $P(g|(\ )\phi|(\ ))$ $\mathcal{K} \to \mathbb{R}$ is upper semi-continuous*

Proof Let $\mathcal{K} \ni F_n \to F$ and let μ_n be an equilibrium state for $g|F_n$ and $\phi|F_n$ i e

$$P(g|F_n, \phi|F_n) = h_{\mu_n} + \int_{F_n} \phi \, d\mu_n$$

and μ any weak accumulation point of μ_n treated as measures on S^1, say $\mu = \lim_{k\to\infty} \mu_{n_k}$ Observe that since $F_n \to F$, $\mu(F) = 1$ and μ is an invariant measure for $g|F$ Since $g \; S^1 \to S^1$ is expansive, the function $\nu \mapsto h_\nu(g)$ is upper semi-continuous and therefore

$$\lim_{k\to\infty} P(g_n, \phi|F_n) = \lim_{k\to\infty}\left(h(\mu_{n_k}) + \int_{F_{n_k}} \phi \, d\mu_{n_k}\right) \le h_\mu(g) + \int_{S^1} \phi \, d\mu$$
$$= h_\mu(g|F) + \int_F \phi \, d\mu \le P(g|F, \phi|F)$$

This completes the proof because μ is an arbitrary accumulation point

We remark that we used compactness of the space $\mathcal{K}$

COROLLARY 1 *The functions $\mathcal{K} \ni F \mapsto h_{\text{top}}(g|F)$, HD (F) are upper semi-continuous*

Proof The upper semi-continuity of topological entropy follows immediately from lemma 5 if we set $\phi \equiv 0$ To prove it for Hausdorff dimension let $\mathcal{K} \ni F_n \to F$ and let t_n denote the unique non-negative number such that $P(g|F_n, -t\phi_n^u)=0$, where $\phi_n^u = \log(|g'||F_n)$ t_n turns out to be also HD (F_n) (see [U, th 3], cf also [**McC-M**], [$\mathbf{B}_3$]) Let $s = \limsup_{n\to\infty} t_n$ and $\varepsilon > 0$ There exists a subsequence of integers $\{n_k\}_{k=1}^{\infty}$, $\lim_{k\to\infty}(n_k)=\infty$ such that for every k, $t_{n_k} \geq s-\varepsilon$ and hence $P(g|F_{n_k}, -(s-\varepsilon)\phi_{n_k}^u) \geq 0$ So, in view of lemma 5

$$0 \leq \limsup_{n\to\infty} P(g|F_n, (-(s-\varepsilon)\phi_n^u) \leq P(g|F, -(s-\varepsilon)\phi^u|F)$$

Since $\varepsilon>0$ can be taken arbitrarily small, this implies that $P(g|F, -s\phi^u|F)\geq 0$ which means that $s \leq \mathrm{HD}(F)$

THEOREM 1 *If $F \in \mathscr{K}$ then the following conditions are equivalent*

(a) HD $(F)=0$,

(b) $h_{\mathrm{top}}(g|F)=0$,

(c) *the function* HD $\mathscr{K}\to\mathbb{R}$ *is continuous at* F,

(d) *the function* h_{top} $\mathscr{K}\to\mathbb{R}$ *is continuous at* F

Proof In view of theorem 4 from [U] there exists a measure μ such that HD $(F)=h_\mu(g|F)/\chi_\mu$ where χ_μ denotes the Lyapunov exponent of μ Therefore $h_{\mathrm{top}}(g|F)=0$ also implies that HD $(F)=0$ If HD $(F)=0$ then let m be an ergodic measure with maximal entropy for $g|F$ Hence $0=\mathrm{HD}(m)=h_m/\chi_m = h_{\mathrm{top}}(g|F)/\chi_m$ and consequently $h_{\mathrm{top}}(g|F)=0$ So (a)$\Leftrightarrow$(b)

The implications (a)$\Rightarrow$(c) and (b)$\Rightarrow$(d) are immediate consequences of corollary 1 To prove the implications (c)$\Rightarrow$(a) and (d)$\Rightarrow$(b) it is enough to show that F is the limit of a sequence of finite sets from $\mathscr{K}$ To do this fix $\varepsilon>0$ and choose points $x_1, x_2, \ldots, x_k$ from F making an $\varepsilon/2$-net in F i e for every $x\in F$ there exists $1\leq j\leq k$ such that $\rho(x, x_j)<\varepsilon/2$ Now, by a version of the well-known Anosov's closing lemma (see for example [$\mathbf{B}_1$]) we can find $\delta>0$ so that the following holds if $x\in S^1$ and $\rho(g^n(x), x)<\delta$, then there is an $x'\in S^1$ with $g^n(x')=x'$ and $\rho(g^i(x), g^i(x'))<\varepsilon/2$ for all $0\leq i\leq n$ Since S^1 is compact, for every $1\leq j\leq k$ there exist positive integers $m_j<n_j$ such that $\rho(g^{m_j}(x_j), g^{n_j}(x_j))<\delta$ So there is an $x_j'\in S^1$ with $g^{n_j-m_j}(x_j')=x_j'$ and $\rho(g^i(x_j'), g^i(g^{m_j}(x_j)))<\varepsilon/2$ for all $k\in[0, n_j-m_j]$ Now let $y_j = g_\nu^{-m_j}(x_j')$ where the branch $g_\nu^{-m_j}$ is taken so that $g_\nu^{-m_j}(g^{m_j}(x_j))=x_j$ So the set $\{g^i(y_j)\}_{1\leq j\leq k, 0\leq i\leq n_j}$ is finite g-invariant and, since g increases ρ-distances, its Hausdorff distance to F is less than ε

As a consequence of this theorem, its proof, corollary 1 and a theorem of Baire we get the following

COROLLARY 2 *The space of zero-dimensional closed g-invariant subsets of g is dense and of type G_δ in $\mathscr{K}$*

In view of theorem 1 to obtain some results about continuity of Hausdorff dimension which would involve subsets of positive dimension we have to restrict to smaller classes than $\mathscr{K}$ And indeed, it is possible to find natural wide subclasses of $\mathscr{K}$ (containing for example, for every number $0\leq t\leq 1$ closed invariant subsets of Hausdorff dimension t (see corollary 4)) the Hausdorff dimension function restricted to which is already continuous To do this, again let $F \in \mathcal{K}$ Then $S^1 \backslash F = \bigcup_{k=1}^{n(F)} (x_k, y_k)$ where $\{(x_k, y_k)\}_{k=1}^{n(F)}$ ($n(F)$ is an integer or ∞) is a family of open pairwise disjoint intervals, in fact connected components of $S^1 \backslash F$ We will call the family

$$\tau(F) = \{(x_k, y_k) \quad F \cap g((x_k, y_k)) \neq \varnothing\}$$

generating for the set F Let us observe that $S^1 \backslash F = \bigcup_{i=0}^{\infty} \bigcup_{a \in \tau(F)} g^{-i}(a)$ or in other words F is the set of those points whose forward trajectory omits all the intervals from $\tau(F)$ We define

$$\mathcal{K}_f = \{F \in \mathcal{K} \quad \tau(F) \text{ is finite}\}$$

and for $n \geq 1$

$$\mathcal{K}_n = \{F \in \mathcal{K} \quad \tau(F) \leq n\}$$

The classes $\{\mathcal{K}_n\}_{n=1}^{\infty}$ are increasing and they are just those classes which were announced above (cf th 3)

PROPOSITION 1 *If $F \in \mathcal{K}_f$ then $g|F$ is a mapping of Misiurewicz-Szlenk*

Proof It is clear that $g \quad S^1 \to S^1$ and consequently also $g|F$ has a finite partition by intervals of monotonicity So we only need to find a finite partition by intervals mod (F) with the Darboux property We claim that the partition of $S^1 \backslash \bigcup_{a \in \tau(F)} a$ by connected components restricted to F has the required property Indeed, since $\tau(F)$ is finite, this partition consists of a finite number of closed intervals mod (F), and let c be an arbitrary subinterval of a connected component of $S^1 \backslash \bigcup_{a \in \tau(F)} a$ If $y \in F \cap g(c)$ then let $x \in c$ denote a point such that $g(x) = y$ So $x \notin \bigcup_{a \in \tau(F)} a$ and $g(x) = y \notin \bigcup_{i=0}^{\infty} \bigcup_{a \in \tau(F)} g^{-i}(a)$, which means that $x \in F$ and consequently $g(F \cap c) = F \cap g(c)$ is an interval mod (F)

THEOREM 2 *Let $m \geq 1$ be an arbitrary integer, $F \in \mathcal{K}_m$, $\phi \quad S^1 \to \mathbb{R}$ a continuous function such that $P(g|F, \phi|F) > \sup(\phi)$ Then the function $P(g| \ , \phi| \ ) \ \mathcal{K}_m \to \mathbb{R}$ is continuous at F*

Proof (cf also [**M-Sz**, th 5]) In view of lemma 5 it is sufficient to prove the lower semi-continuity of this function at F Take any $0 < \varepsilon \leq (P(g|F, \phi|F) - \sup(\phi))$ and approximate ϕ by a piecewise constant function $\bar{\phi}$ such that $\|\bar{\phi} - \phi\| \leq \varepsilon/4$ and lemma 3 holds So, due to this lemma we can apply theorem 0 to the mapping $g|F$ and the function $\bar{\phi}|F$ Hence for n large enough we have

$$(1) \quad \frac{1}{k_n} \log \Sigma_{k_n}(D_n) \geq P(g|F, \bar{\phi}|F) - (\varepsilon/4) \geq P(g|F, \phi|F) - (\varepsilon/2)$$

Now, let $\tilde{D}_n \subset D_n$ be an arbitrary subpartition such that Card $(D_n \backslash \tilde{D}_n) \leq 5mk_n$ We have

$$\frac{1}{k_n} \log \Sigma_{k_n}(\tilde{D}_n) - \frac{1}{k_n} \log \Sigma_{k_n}(D_n) = \frac{1}{k_n} \log \frac{\Sigma_{k_n}(D_n) - \Sigma_{k_n}(D_n \backslash \tilde{D}_n)}{\Sigma_{k_n}(D_n)}$$

$$= \frac{1}{k_n} \log \left(1 - \frac{\Sigma_{k_n}(D_n \backslash \tilde{D}_n)}{\Sigma_{k_n}(D_n)}\right)$$

But

$$\Sigma_{k_n}(D_n \backslash \tilde{D}_n) \le 5mk_n\, e^{k_n \sup(\bar{\phi})}$$
$$\le 5mk_n \exp\left(k_n\left(\sup(\phi) + \frac{\varepsilon}{4}\right)\right)$$
$$\le 5mk_n \exp\left(k_n(P(g \mid F, \phi \mid F) - \tfrac{3}{4}\varepsilon)\right)$$

and by (1) for n large enough

$$\Sigma_{k_n}(D_n) \ge \exp\left(k_n\left(P(g \mid F, \phi \mid F) - \frac{\varepsilon}{2}\right)\right)$$

Hence

$$\frac{\Sigma_{k_n}(D_n \backslash \tilde{D}_n)}{\Sigma_{k_n}(D_n)} \le 5mk_n \exp\left(k_n\left(P(g \mid F, \phi \mid F) - \tfrac{3}{4}\varepsilon - P\left(g \mid F, \phi \mid F\right) + \frac{\varepsilon}{2}\right)\right)$$
$$= 5mk_n \exp\left(-\frac{\varepsilon}{4} k_n\right) \to 0 \qquad \text{when } n \text{ tends to } \infty$$

Thus for n large enough

$$(2) \quad \left| \frac{1}{k_n} \log \Sigma_{k_n}(\tilde{D}_n) - \frac{1}{k_n} \log \Sigma_{k_n}(D_n) \right| \le \varepsilon/4$$

Now let $\delta > 0$ be small enough so that for every $z \in S^1$ every $0 \le j \le k_n - 1$, $g^j(B(z, \delta))$ intersects at most two elements of the partition D_n, and consider a subset $G \in \mathcal{K}_m$ such that $\rho_H(F, G) < \delta$ So the partition

$$D'_n = \left\{ a \in D_n \;\; a \cap \bigcup_{(x,y) \in \tau(G)} \bigcup_{j=0}^{k_n - 1} g^j(B(x, \delta) \cup B(y, \delta)) = \varnothing \right\}$$

consists of at least $(\mathrm{Card}\,(D_n) - 4mk_n)$ elements

Since for every a, D'_n, $g^{k_n}(a) \supset J \supset \bigcup_{b \in D'_n} b$ and $g^{k_n} \mid a$ is monotone(!) we can slightly decrease the elements of D'_n (perhaps after subtracting two end elements) to be compact and for every element c of the new partition $\bar{D}_n$ obtained the formula $g^{k_n}(c) \supset \bigcup_{d \in \bar{D}_n} d$ still holds So $\bar{D}_n$ consists of at least

$$(\mathrm{Card}\,(D_n) - 4mk_n - 2) \ge \mathrm{Card}\,(D_n) - 5mk_n$$

elements and consequently satisfies formula (2)

Now the set $X = \bigcap_{i=0}^{\infty} (g^{k_n})^{-i}(\bigcup_{b \in \bar{D}_n} b)$ is compact g^{k_n}-invariant, $\bar{D}_n \mid X$ is an open partition and for any $d \in \bar{D}_n \mid X$, $g^{k_n}(d) = X$ Hence it is easy to check that

$$(3) \quad P(g^{k_n} \mid X, S_{k_n}(\phi)) \ge P(g^{k_n} \mid X, S_{k_n}(\phi), \bar{D}_n \mid X)$$
$$\ge \log_{d \in \bar{D}_n \mid X} \exp\left(S_{k_n}(\bar{\phi} - (\varepsilon/4)(d)\right) = \log \Sigma_{k_n}(\bar{D}_n) - (\varepsilon/4)k_n$$

Now we check that $X \subset G$ Obviously $X \subset F$ and let $z \in X \cap (S^1 \backslash G)$ It means that there exists an integer $k \ge 0$ such that $g^k(z) \in (x, y) \in \tau(G)$ But then $g^k(z) \in F$ and $\rho_H(G, F) < \delta$ imply that $g^k(z) \in B(x, \delta) \cup B(y, \delta)$, and if $0 \le r \le k_n - 1$ is the unique number such that $k_n \mid k + r$, then $g^{k+r}(z) \in X$ Hence $g^{k+r}(z) \in a \in \bar{D}_n$ for some $a \in \bar{D}_n$ and $g^{k+r}(z) \in g^r(B(x, \delta) \cup B(y, \delta))$, which gives $a \cap g^r(B(x, \delta) \cup B(y, \delta)) \ne \varnothing$ – a contradiction Therefore by (3)

$$P(g|G, \phi|G) = \frac{1}{k_n} P(g^{k_n}|G, S_{k_n}(\phi|G)) \geq \frac{1}{k_n} P(g^{k_n}|X, S_{k_n}(\phi|X))$$
$$\geq \frac{1}{k_n} \log(\Sigma_{k_n}(D_n)) - \frac{\varepsilon}{4}$$

and further by (2) and (1)

$$P(g|G, \phi|G) \geq P(g|F, \phi|F) - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} - \frac{\varepsilon}{2} = P(g|F, \phi|F) - \varepsilon$$

This completes the proof of the theorem

As an immediate consequence of this theorem and theorem 1 we obtain the following

COROLLARY 3 *For every* $m \geq 1$ *the function* h_{top} $\mathcal{K}_m \to \mathbb{R}$ *is continuous*

We can also prove the following

THEOREM 3 *If* g $S^1 \to S^1$ *is a* C^2*-expanding mapping then for every* $m \geq 1$ *the function* HD $\mathcal{K}_m \to \mathbb{R}$ *is continuous*

Proof Let $\mathcal{K}_m \ni F = \lim_{n\to\infty} F_n$, $F_n \in \mathcal{K}_m$, $n = 1, 2,$ By theorem 1 we can assume that $s = \text{HD}(F) > 0$ and let $0 \leq t < s \leq 1$ be an arbitrary real number So $P(g|F, -t\phi^u) \geq 0 > \sup(-t\phi^u)$ where $\phi^u(z) = \log|g(z)|$ and using theorem 2 we get

$$\lim_{n\to\infty} P(g|F_n, -t\phi^u|F_n) = P(g|F, -t\phi^u|F) > 0$$

Hence for n large enough, $P(g|F_n, -t\phi^u|F_n) > 0$, and since the function $r \mapsto P(g|F_n, -r\phi^u)$ is decreasing, $t \leq \text{HD}(F_n)$ Consequently $\liminf_{n\to\infty} \text{HD}(F_n) \geq s = \text{HD}(F)$ This and corollary 1 complete the proof of the theorem

The following simple example shows that this theorem and corollary 3 are no longer true if h_{top} and HD are treated as functions from $\mathcal{K}_f$ Indeed, since periodic points of an expanding g are dense in S^1 we can find a sequence $\{F_n\}_{n=1}^{\infty}$ of sets consisting of a finite number of periodic orbits such that $\lim_{n\to\infty} F_n = S^1$ Obviously F_n are closed, invariant, $\text{HD}(F_n) = h_{\text{top}}(g|F_n) = 0$, and since the sets $S^1 \backslash F_n$ have only finitely many connected components, they belong to $\mathcal{K}_f$

Now let us study some other properties of closed invariant subsets of g First we shall give an effective criterion for two arbitrary sets in $\mathcal{K}_f$ to be close in the sense of the Hausdorff metric ρ_H It will be expressed in terms of their generating families and the standard Riemannian metric on the circle

Let $F, G \in \mathcal{K}_f$ and $B(F, G)$ denote the set of all bijections from $\tau(F)$ to $\tau(G)$ The Λ-*distance* between F and G is defined as

$$\Lambda(F, G) = \inf\left\{ \sum_{A\in\tau(F)} \lambda(j(A) - A) \quad j \in B(F, G) \right\} \qquad \text{if } B(F, G) \neq \varnothing$$

and we put $\Lambda(F, G) = \infty$ if $B(F, G) = \varnothing$ λ denotes here Lebesgue measure (induced by standard Riemannian metric) and $-$ symmetric subtraction

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LEMMA 6 *There is a constant L which depends only on g, such that $\rho_H(F, G) \leq L\Lambda(F, G)$*

Proof Since the metric ρ and consequently the corresponding Λ_ρ-distance is Lipschitz equivalent to the standard one, it is sufficient to prove the lemma using Λ_ρ-distance instead of Λ Let $j \in B(F, G)$ be chosen so that $\Lambda_\rho(F, G) = \sum_{A \in \tau(F)} \lambda_\rho(j(A) - A)$ and let $x \in G$ We want to find a point $y \in F$ whose distance from x does not exceed $2\Lambda_\rho(F, G)$ Let $n(x) \geq 0$ be the smallest integer such that $g^{n(x)}(B(x, 2\Lambda_\rho(F, G))) \cap (\bigcup_{A \in \tau(F)} A) \neq \emptyset$, say $A_0 \cap g^{n(x)}(B(x, 2\Lambda_\rho(F, G))) \neq \emptyset$ Since $\lambda_\rho(A_0 \backslash j(A_0)) \leq \Lambda_\rho(F, G)$, $g^{n(x)}(x) \notin j(A_0)$ and

$$g^{n(x)}(B(x, 2\Lambda_\rho(F, G))) \supset B(g^{n(x)}(x), 2\Lambda_\rho(F, G)))$$

this implies that at least one endpoint of A_0, say a, belongs to $g^{n(x)}(B(x, 2\Lambda_\rho(F, G)))$ Since $a \in F$, it is sufficient to take as y a point from $B(x, 2\Lambda_\rho(F, G)) \cap g^{-n(x)}(a)$ We proceed analogously if we start with a point in F So $\rho_H(F, G) \leq 2\Lambda_\rho(F, G)$ and the lemma is proved

Now we want to investigate sets made up in a way similar to that from [U] That is, let $(a_1, b_1), \quad, (a_k, b_k)$ be a family of open disjoint intervals of the circle and

$$K((a_i, b_i)_{i=1}^k) = \bigcap_{i=1}^{k} \bigcap_{n=0}^{\infty} (S^1 \backslash g^{-n}((a_i, b_i)))$$

be the set of those points whose forward trajectory avoids the intervals $(a_1, b_1), \quad, (a_k, b_k)$ We are interested in what can happen if the endpoints of the intervals change continuously The first remark is that the corresponding invariant sets do not have to vary continuously Indeed, let us consider, as in [U] the sets $K((0, a))$ where 0 is a fixed point of g If we take a so that $a \in K((0, a))$ and for some $n \geq 1$, $g^n(a) = 0$, then it is easy to see that a is an isolated point of $K(0, a))$ and so, for any $\varepsilon > 0$, $K((0, a + \varepsilon))$ does not intersect a fixed neighbourhood of a However the following theorem is true

THEOREM 4 *If the endpoints of the intervals $(a_1, b_1), \quad, (a_k, b_k)$ change continuously then the topological entropy and Hausdorff dimension of corresponding invariant subsets also change continuously*

Proof For $i = 1, \quad, k$, let $a_i(\varepsilon)$, $b_i(\varepsilon)$, $\bar{a}_i(\varepsilon)$, $\bar{b}_i(\varepsilon)$ be defined as in figure 1

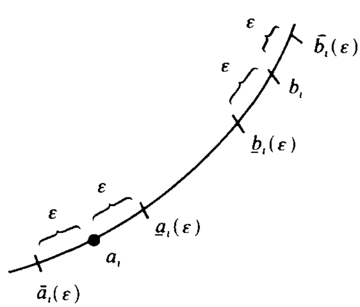

FIGURE 1

To prove our theorem it is sufficient to show that

$$\lim_{\varepsilon \searrow 0} \mathrm{HD}\,(K((a_i(\varepsilon), b_i(\varepsilon))_{i=1}^{k})) = \mathrm{HD}\,(K(a_i, b_i)_{i=1}^{k}))$$
$$= \lim_{\varepsilon \searrow 0} \mathrm{HD}\,(K((\bar{a}_i(\varepsilon), \bar{b}_i(\varepsilon))_{i=1}^{k}))$$

The limits exist because the sequences are monotone The first equality can be proved in the same way as proposition 1 from [U], so we will concentrate on the second one In order to prove it we only need to find a sequence of compact invariant subsets which do not intersect some thickness of all generating intervals of $K((a_i, b_i)_{i=1}^{k})$ and whose Hausdorff dimension tends to $\mathrm{HD}\,(K((a_i, b_i)_{i=1}^{k}))$ To do this we return to the proofs of theorems 2 and 3 Let us treat the set X constructed in the proof of theorem 2 as a function of $\varepsilon - X(\varepsilon)$ It follows from the construction that the set $\tilde{X}(\varepsilon) = \bigcup_{i=1}^{k_n - 1} g^i(X(\varepsilon))$ is compact, g-invariant and does not intersect the $\delta(\varepsilon)$-thickness of the intervals from $\tau(K((a_i, b_i)_{i=1}^{k}))$ (we have replaced the set F from the proof of theorem 2 by $K((a_i, b_i)_{i=1}^{k})$ Now proceeding as in the proof of theorem 3 we get that $\lim_{\varepsilon \searrow 0} \mathrm{HD}\,(\tilde{X}(\varepsilon)) = \mathrm{HD}\,(K((a_i, b_i)_{i=1}^{k})$ The case of topological entropy is analogous and simpler

Corollary 4 *For every $0 \le \varepsilon \le 1$ there exists a compact g-invariant subset of S^1 whose Hausdorff dimension is equal to ε*

Proof The function $\varepsilon \mapsto \mathrm{HD}\,(K((0, \varepsilon)))$, where 0 is a fixed point of g is, by theorem 4, continuous Besides, $\mathrm{HD}\,(K(0,0)) = \mathrm{HD}\,(S^1) = 1$ and $\mathrm{HD}\,(K((0,1))) = \mathrm{HD}\,(\{0\}) = 0$

We want to finish this section with the following two simple propositions

Proposition 2 *For every $F \in \mathcal{K}$ the set $\bigcup_{i=0} \bigcup_{(a,b) \in \tau(F)} g^{-i}(\{a, b\})$ is dense in F*

Proposition 3 *For every uncountable $F \in \mathcal{K}_f$ there is a unique set F_0 homeomorphic to the Cantor set such that $\tau(F_0) \le \tau(F)$ and $F \backslash F_0$ is at most countable*

To prove the last proposition use the Cantor–Bendixon theorem (see [**K**])

3 In this section we shall deal with the family $K((0, \varepsilon))$ (abbreviated to $K(\varepsilon)$), where 0 is a fixed point of an orientation preserving C^2-expanding mapping of the circle, g This family was the main object of interest in [U] Now using the methods developed in [**P–U–Z**] we want to give a more detailed description of equilibrium states of Holder continuous functions Let $C(g)$ denote the set of those points at which the function $\varepsilon \mapsto K(\varepsilon)$ is not locally constant It follows from [U] that we get the same set if we take into account the function $\varepsilon \mapsto h_{\mathrm{top}}(g \mid K(\varepsilon))$, and $C(g) = \{\varepsilon \;\; g^n(\varepsilon) \ge \varepsilon$ for every $n \ge 0\}$ Throughout this section we will only consider the points $\varepsilon \in C(g)$ (observe that for every $0 \le \varepsilon \le 1$ there is an $\varepsilon \le \varepsilon' \in C(g)$ such that $K(\varepsilon') = K(\varepsilon)$

Proposition 4 *The set of all periodic points is dense in $K(\varepsilon)$ iff $0 \notin \{g^n(\varepsilon) \;\; n \ge 0\}$ or $\varepsilon = 0$*

Proof If $g^m(\varepsilon) = 0$ for certain $m \ge 1$ and $\varepsilon \ne 0$ then ε is isolated in $K(\varepsilon)$ and is not periodic And if $0 \notin \{g^n(\varepsilon) \;\; n \ge 0\}$ then using symbolic representation $\phi \;\; \Sigma_q^+ \to$

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S^1 ($q = \deg(g)$) of g, constructed in [U], we easily see that for every sequence $\{x_n\}_{n=1}^{\infty}$ in $\phi^{-1}(K(\varepsilon))$ and every $m \geq 1$ there exists M so large that the periodic point $(x_1, \quad, x_m(q-1)^M)^{\infty}$ belongs to $\phi^{-1}(K(\varepsilon))$ The case of $\varepsilon = 0$ is obvious

Let $\mathrm{SP}(g) = \{\varepsilon \in C(g) \ \ 0 \notin \mathrm{cl}(\{g^n(\varepsilon) \ \ n \geq 0\})\}$

PROPOSITION 5 $0 \neq \varepsilon \in \mathrm{SP}(g)$ *iff* $g|K(\varepsilon)$ *satisfies the specification property* (*the references to the specification property are, for instance,* **[D-G-S]**, **[B$_2$]**)

Proof We use the symbolic representation again $\varepsilon \in \mathrm{SP}(g)$ means that $\phi^{-1}(\varepsilon)$ consists of one sequence and the length of all blocks of the symbols $q-1$ is bounded by a natural number, say k Let us now take $\delta > 0$ and let $l(\delta)$ be chosen so that $\rho(x, y) \leq \delta$ if $x, y \in \Sigma_q^+$ and $x_i = y_i$ for $i = 1, \quad, l(\delta)$

For any $k \geq 2$, any k points $z^{(1)}, \quad, z^{(k)} \in \phi^{-1}(K(\varepsilon))$, any integers $0 \leq a_1 \leq b_1 < a_2 \leq b_2 < \quad < a_k \leq b_k$ with $a_i - b_{i-1} \geq l(\delta) + k + 1$ for $2 \leq i \leq k$ and any integer $p \geq l(\delta) + k + 1 + b_k - a_1$ we will find a periodic point $z \in \phi^{-1}(K(\varepsilon))$ with period p, such that

$$\rho(\sigma^n(z), \sigma^n(z^{(i)})) \leq \delta \qquad \text{for } a_i \leq n \leq b_i, 1 \leq i \leq k$$

Let us first construct a point $\tilde{z} \in \phi^{-1}(K(\varepsilon))$ in the following way

$$\tilde{z}_n = \begin{cases} q-1, & 1 \leq n \leq a_1 \\ z_n^{(i)} = (\sigma^{n-1}(z^{(i)}))_1, & a_i + 1 \leq n \leq b_i + l(\delta), 1 \leq i \leq k \\ q-1, & b_i + l(\delta) + 1 \leq n \leq a_{i+1}, 1 \leq i \leq k (a_{k+1} \overset{\mathrm{df}}{=} a_1 + p) \end{cases}$$

and

$$\tilde{z} = \tilde{z}_1 \quad \tilde{z}_{a_1 - 1}(\tilde{z}_{a_1} \quad \tilde{z}_{a_1 + p})^{\infty}$$

$\tilde{z} \in \phi^{-1}(K(\varepsilon))$, $\sigma^p(\sigma^{a_1}(\tilde{z})) = \sigma^{a_1}(\tilde{z})$ and we only need to find a periodic point z of period p such that $\sigma^{a_1}(z) = \sigma^{a_1}(\tilde{z})$ So, as z we can take the point $\sigma^{mp - a_1}(\sigma^{a_1}(\tilde{z}))$ where $mp - a_1 \geq 0$ Since every factor of a system with the specification property also has the specification property, one part of the proposition is proved

Now let $\varepsilon \notin \mathrm{SP}(g)$ So, we can find an increasing sequence $\{k_n\}_{n=1}^{\infty}$ and $\tilde{\varepsilon} \in \phi^{-1}(\varepsilon)$ such that $\tilde{\varepsilon}_{k_n + i} = q - 1$ for $i = 0, \quad, n$ Let $\delta = 1/(2q)$ (hence $\rho(x, y) \leq \delta \Rightarrow x_1 = y_1$), $z^{(1)} = \tilde{\varepsilon}$, $a_1 = 0$, $b_1 = k_n - 1$, $z^{(2)} \in \sigma^{-a_2}(\tilde{\varepsilon})$ If z is taken to satisfy the specification condition, then $z_i = \tilde{\varepsilon}_i$ for $i = 1, \quad, k_n$ But since $z \in \phi^{-1}(K(\varepsilon))$, this implies that $z_i = q - 1$ for $k_n \leq i \leq k_n + n$ On the other hand $\delta > \rho(\sigma^{a_2} z, \sigma^{a_2} z^{(2)}) = \rho(\sigma^{a_2} z, \tilde{\varepsilon})$ Thus $z_{a_2 + 1} = (\sigma^{a_2} z)_1 = \tilde{\varepsilon}_1 < q - 1$ Since $a_2 > b_1 = k_n - 1$, it gives that $a_2 \geq k_n + n$ and consequently $a_2 - b_1 \geq n + 1$ This completes the proof of the proposition

Applying symbolic representation we also obtain the following

PROPOSITION 6 *The sets* $\mathrm{SP}(g)$ *and* $C(g) \backslash \mathrm{SP}(g)$ *are dense in* $C(g)$

Proof The density of $C(g) \backslash \mathrm{SP}(g)$ is obvious The density of $\mathrm{SP}(g)$ follows from the fact that the set $C(g) \cap \mathrm{Per}(g)$, where $\mathrm{Per}(g)$ is the set of all periodic points of $g \ \ S^1 \to S^1$, is already dense in $C(g)$

COROLLARY 5 *If* $P(\varepsilon, n) = \mathrm{Card}(\{x \in K(\varepsilon) \ \ g^n(x) = x\})$, *then for every* $\varepsilon \in C(g)$

$$\lim_{n \to \infty} \frac{1}{n} \log P(\varepsilon, n) = h_{\mathrm{top}}(g | K(\varepsilon))$$

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Proof As $g|K(\varepsilon)$ is expansive the inequality $h_{\text{top}}(g|K(\varepsilon)) \geq \limsup_{n\to\infty} 1/n \log P(\varepsilon, n)$ follows In order to prove the converse inequality let us consider two cases First, assume that there exists a decreasing sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ of points from SP (g) tending to ε By the specification property, for all $\xi_j, j = 1, 2, \ldots,$ our formula holds Besides, for all $j \geq 1$

$$\lim_{n\to\infty} \frac{1}{n} \log P(\varepsilon_j, n) \leq \liminf_{n\to\infty} \frac{1}{n} \log P(\varepsilon, n)$$

Hence by theorem 4,

$$\liminf_{n\to\infty} \frac{1}{n} P(\varepsilon, n) \geq \lim_{j\to\infty} h_{\text{top}}(g|K(\varepsilon_j)) = h_{\text{top}}(g|K(\varepsilon))$$

Otherwise, because of density of SP (g) in $C(g)$, ε is right-hand isolated in $C(g)$ Since $C(g)$ is closed, there is $\varepsilon < \bar{\varepsilon} \in C(g)$ which is the nearest point to ε Since $\bar{\varepsilon}$ is left-hand isolated, it satisfies the first case Since $\{h_{\text{top}}(g|K(\delta))\}_{\delta \in C(g)} = [0, \log(\deg g)]$ and the function $\delta \mapsto h_{\text{top}}(g|K(\delta))$ is monotone, $h_{\text{top}}(g|K(\bar{\varepsilon})) = h_{\text{top}}(g|K(\varepsilon))$ It implies that

$$\liminf_{n\to\infty} \frac{1}{n} \log P(\varepsilon, n) \geq \lim_{n\to\infty} \frac{1}{n} \log P(\bar{\varepsilon}, n) = h_{\text{top}}(g|K(\bar{\varepsilon})) = h_{\text{top}}(g|K(\varepsilon))$$

The corollary is proved

LEMMA 7 *If* $\delta, \varepsilon \in C(g)$ *and* $h_{\text{top}}(g|K(\varepsilon)) \neq h_{\text{top}}(g|K(\delta))$ *then* $\text{HD}(K(\delta)) \neq \text{HD}(K(\varepsilon))$

Proof We can assume $\delta < \varepsilon$ In view of proposition 6 and the last argument of the previous proof, one can find $a \in \text{SP}(g)$ such that $\delta \leq a < \varepsilon$ and $K(a) \not\supseteq K(\varepsilon)$ If $\text{HD}(K(\delta)) = \text{HD}(K(\varepsilon))$ then $\text{HD}(K(a)) = \text{HD}(K(\varepsilon)) = c$ By the Bowen-Manning formula (see [**U**, th 3], cf also [**B$_3$**], [**McC-M**]), $P(g|K(a), -c\log(|g'||K(a))) = 0$ Since $g|K(a)$ satisfies the specification property and $\log|g'|$ is a Holder continuous function, there exists exactly one equilibrium state μ of $-c\log(|g'|\, K(a))$ (see [**B$_2$**]) But, in view of corollary 2 from [**U**] there is a $g|K(\varepsilon)$-invariant measure ν such that $c = \text{HD}(K(\varepsilon)) = h_\nu / \int \log(|g'||K(\varepsilon))\, d\nu$ Thus $h_\nu + \int -c\log(|g'||K(\varepsilon))\, d\nu = 0$ and since ν is also $g|K(a)$-invariant, it is also an equilibrium state of the function $-c\log(|g'||K(a))$ with respect to the mapping $g|K(a)$ So $\nu = \mu$ and consequently $\mu(K(a)\setminus K(\varepsilon)) = 1 - 1 = 0$, which contradicts to the fact that the equilibrium state of any Holder continuity function with respect to the mapping which satisfies the specification property, is positive on non-empty open sets (see [**B$_2$**]) Thus $\text{HD}(K(\delta)) \neq \text{HD}(K(\varepsilon))$

Due to the comments in the beginning of this section, this lemma immediately implies

COROLLARY 6 $C(g)$ *coincides with the set of those points at which the function* $\varepsilon \mapsto \text{HD}(K(\varepsilon))$ *is not locally constant The sets of those points at which the functions* $\varepsilon \mapsto \text{HD}(K(\varepsilon))$ *and* $\varepsilon \mapsto h_{\text{top}}(g|K(\varepsilon))$ *are not locally right-hand constant, coincide too Call them (following* [**U**]) $C_+(g)$

Now we want to generalize theorem 2 from [U] which refers to the local Hausdorff dimension of $C(g)$ But we also want to include similar results about the sets SP (g) and $C(g)\backslash \mathrm{SP}(g)$ So we need the following slight generalization of definition 2 from [U]

Definition 3 Let (X, d) be a metric space and $A \subset X$ The *local Hausdorff dimension of A at a point* $x \in X$ is defined to be $\lim_{r \to 0} \mathrm{HD}(A \cap B(x, r))$ and denoted by $\mathrm{HD}(A, x)$

THEOREM 5 *For every* $\varepsilon \in C(g)$, $\mathrm{HD}(\mathrm{SP}(g), \varepsilon) = \mathrm{HD}(C(g)\backslash \mathrm{SP}(g), \varepsilon) = \mathrm{HD}(C(g), \varepsilon) = \mathrm{HD}(K(\varepsilon))$

Proof We improve on the idea of the proof of theorem 2 from [U] Since for every $\varepsilon \in C(g)$, $C(g) \cap B(\varepsilon, r) \subset K(\varepsilon - r)$, Theorem 4 implies that

(1) $\mathrm{HD}(\mathrm{SP}(g), \varepsilon), \mathrm{HD}(C(g)\backslash \mathrm{SP}(g), \varepsilon) \le \mathrm{HD}(C(g), \varepsilon) \le \mathrm{HD}(K(\varepsilon))$

Let us now define $\tilde{K}(\varepsilon) = \bigcup_{n=1}^{\infty} K((1-(1/n), \varepsilon)$ By theorem 4

(2) $\mathrm{HD}(\tilde{K}(\varepsilon)) = \mathrm{HD}(K(\varepsilon))$

Consider $\varepsilon \in C_+(g)$ which is not periodic For every $r > 0$ we have

$$\tilde{K}(\varepsilon) = \tilde{K}(\varepsilon + r) \cup \bigcup_{n=0}^{\infty} \tilde{K}(\varepsilon) \cap g^{-n}([\varepsilon, \varepsilon + r))$$

Since $\varepsilon \in C_+(g)$, (2) implies the existence of $m \ge 0$ such that $\mathrm{HD}(\tilde{K}(\varepsilon) \cap g^{-m}([\varepsilon, \varepsilon + r))) = \mathrm{HD}(\tilde{K}(\varepsilon))$ Since $g^m(\tilde{K}(\varepsilon) \cap g^{-m}([\varepsilon, \varepsilon + r))) \subset \tilde{K}(\varepsilon) \cap [\varepsilon, \varepsilon + r)$, one obtains

(3) $\mathrm{HD}(\tilde{K}(\varepsilon) \cap [\varepsilon, \varepsilon + r)) = \mathrm{HD}(\tilde{K}(\varepsilon))$ for $r > 0$

Denote by $n(r) \ge 1$ the minimal number, not less than 1, such that $(\varepsilon - r, \varepsilon) \cap g^{n(r)}((\varepsilon - r, \varepsilon]) \ne \varnothing$ Observe that due to the choice of ε, there exists $\delta > 0$ such that for $i = 1, \quad , n(r) - 1$

(4) $(1 - \delta, \varepsilon) \cap g^i((\varepsilon - r, \varepsilon]) = \varnothing$

Observe also that the set $(\varepsilon - r, \varepsilon] \cap g^{-n(r)}(\varepsilon)$ consists of one point, say ε_1 Since ε is not periodic, $\varepsilon < g^{n(r)}(\varepsilon)$ and consequently $\varepsilon - r < \varepsilon_1 < \varepsilon$ Therefore applying (4)

(5) $[\varepsilon_1, \varepsilon) \cap g^{-n(r)}(\tilde{K}(\varepsilon)) \subset \mathrm{SP}(g) \cap B(\varepsilon, r)$

Since $g^{n(r)}([\varepsilon_1, \varepsilon) \cap g^{-n(r)}(\tilde{K}(\varepsilon)) = g^{n(r)}([\varepsilon_1, \varepsilon)) \cap \tilde{K}(\varepsilon) \subset [\varepsilon, g^{n(r)}(\varepsilon)) \cap \tilde{K}(\varepsilon)$, (5) and (3) imply that $\mathrm{HD}(\mathrm{SP}(g) \cap B(\varepsilon, r)) \ge \mathrm{HD}(\tilde{K}(\varepsilon)) = \mathrm{HD}(K(\varepsilon))$ Hence if $r \to 0$ one gets

(6) $\mathrm{HD}(\mathrm{SP}(g), \varepsilon) \ge \mathrm{HD}(K(\varepsilon))$

So, for the set of ε's considered, the theorem is proved for SP (g) and $C(g)$ But by lemma 4(ii), (iii) from [U], this set is dense in $C(g)$ This completes the proof for the sets SP (g) and $C(g)$

Now let $K_0(\varepsilon) \subset K(\varepsilon)$ be the set of those points whose forward trajectory under g has 0 as an accumulation point The proof for SP (g) shows that to prove our theorem for $C(g)\backslash \mathrm{SP}(g)$ we only need to find a dense subset of $C(g)\backslash \mathrm{Per}(g)$ such that for every ε from it, $\mathrm{HD}(K_0(\varepsilon)) = \mathrm{HD}(K(\varepsilon))$ We claim that as this subset one can take $\mathrm{SP}(g)\backslash \mathrm{Per}(g)$ Indeed, the formula $\mathrm{HD}(\mathrm{SP}(g), \varepsilon) = \mathrm{HD}(K(\varepsilon))$ shows that if we subtract from SP (g) any countable set, it still will be dense in $C(g)$ Let $\varepsilon \in \mathrm{SP}(g)\backslash \mathrm{Per}(g)$ In view of corollary 2 from [U] there is a $g|K(\varepsilon)$-invariant probability measure μ on $K(\varepsilon)$ such that $\mathrm{HD}(K(\varepsilon)) = \mathrm{HD}(\mu)$ - the

Hausdorff dimension of the measure μ. From the proof of lemma 7 we see that such a measure is unique and positive on non-empty open sets of $K(\varepsilon)$. Since for every $\delta>0$, $K(\varepsilon)\backslash K((1-\delta,\varepsilon))$ is a non-empty open set such that

$$(g|K(\varepsilon))^{-1}(K(\varepsilon)\backslash K((1-\delta,\varepsilon)))\subset K(\varepsilon)\backslash K((1-\delta,\varepsilon)),$$

we conclude that $\mu(K(\varepsilon)\backslash K((1-\delta,\varepsilon)))=1$. Hence

$$\mu(K_0(\varepsilon))=\mu\left(\bigcap_{n=1}^{\infty}(K(\varepsilon)\backslash K((1-(1/n),\varepsilon)))\right)=1$$

and thus $\mathrm{HD}(K_0(\varepsilon))\geq \mathrm{HD}(\mu)=\mathrm{HD}(K(\varepsilon))$. This completes the proof of the theorem.

Since $\mathrm{HD}(K(0))=1$, this theorem immediately implies the following.

COROLLARY 7. $\mathrm{HD}(\mathrm{SP}(g))=\mathrm{HD}(C(g)\backslash\mathrm{SP}(g))=\mathrm{HD}(C(g))=1$.

Now we pass to the main theorem of this section. If $h:[0,\infty)\to[0,\infty)$ is an increasing function such that $h(0)=0$, then we will use H_h to denote the Hausdorff measure corresponding to the function h. We will prove the following.

THEOREM 6. *Let $\phi: S^1\to\mathbb{R}$ be a Hölder continuous function and $\varepsilon\in\mathrm{SP}(g)$ be taken so that the function*

$$\phi|K(\varepsilon)+\mathrm{HD}(K(\varepsilon))\log(|g'||K(\varepsilon)):K(\varepsilon)\to\mathbb{R}$$

is not homological to a constant. If μ_ϕ denotes the equilibrium state of ϕ,

$$\phi_c(t)=t^{\mathrm{HD}(\mu_\phi)}\exp\left(c\sqrt{\log(1/t)\log\log\log(1/t)}\right)\qquad \textit{for } c\in(0,\infty),$$

then μ_ϕ is singular with respect to H_{ϕ_c} for $c\leq\sqrt{2\sigma^2(\psi)/\int\log|g'|\,d\mu_\phi}$ and absolutely continuous for $c>\sqrt{2\sigma^2(\psi)/\int\log|g'|\,d\mu_\phi}$. Here $\psi=\phi+\mathrm{HD}(\mu_\phi)\log|g'|$ and $\sigma^2(\psi)$ denotes the middle asymptotic variance of ψ with respect to the probability measure μ_ϕ, i.e.

$$\sigma^2(\psi)=\lim_{n\to\infty}\frac{1}{n}\left(\mathrm{Var}_{\mu_\phi}\left(\sum_{i=0}^{\infty}\psi\circ(g|K(\varepsilon))\right)\right)\neq 0\qquad (\textit{see } [\mathbf{B}_1]).$$

Proof. For $x\in K(\varepsilon)$, $\delta>0$ and an integer $n\geq 0$, let

$$B_n(x,\delta)=\{z\in K(\varepsilon):|g^i(z)-g^i(x)|\leq\delta\quad\text{for } i=0,1,\dots,n-1\}.$$

It follows from [**P-U-Z**] that to prove the theorem, it is sufficient to check the following three conditions.

(a) *The bounded distortion theorem.* For every $\delta>0$ small enough there exists a constant $C>0$ such that for every $n\geq 0$ and every $x\in K(\varepsilon)$

$$\sup\{|(g^n)'(z)|:z\in B_n(x,\delta)\}/\inf\{|(g^n)'(z)|:z\in B_n(x,\delta)\}\leq C_\delta.$$

(b) For every $\delta>0$ small enough there exists a constant $B_\delta>0$ such that for every $n\geq 0$ and every $x\in K(\varepsilon)$

$$B_\delta^{-1}\exp\left(-P(g|K(\varepsilon),(\phi|K(\varepsilon)))n+\sum_{i=0}^{n-1}\phi\circ(g|K(\varepsilon))^i(x)\right)$$

$$\leq\mu_\phi(B_n(x,\delta))\leq B_\delta\exp\left(-P(g|K(\varepsilon),(\phi|K(\varepsilon)))n+\sum_{i=0}^{n-1}\phi\circ(g|K(\varepsilon))^i(x)\right)$$

(c) *The iterated logarithm law* For μ_ϕ-a e $x \in K(\varepsilon)$

$$\limsup \frac{\sum_{i=0}^{\infty} \psi \circ (g\,|\,K(\varepsilon))^i - n \int \psi \, d\mu_\phi}{\sqrt{n \log\log n}} = \sqrt{2\sigma^2(\psi)}$$

The bounded distortion theorem is a well-known fact for every C^2-expanding mapping of the circle Condition (b) follows from the specification property (see [**B**$_2$], [**D-G-S**], [**K-S-S**]) Since the symbolic representation of $g\,|\,K(\varepsilon)$ is just (see [**U**]) the system (X, σ) from § 4 of [**H-K**], the iterated logarithm law follows from this paper, where even the almost sure invariance principle is proved

What is the situation when the function $\phi + \mathrm{HD}\,(K(\varepsilon)) \log(|g'|\,|K(\varepsilon))$ is homological to a constant? Then $\mu_\phi = \mu_{-\mathrm{HD}\,(K(\varepsilon)) \log(|g'|\,|K(\varepsilon))}$ and this measure is equivalent to the Hausdorff measure $H_{t^{\mathrm{HD}\,(K(\varepsilon))}}$ on $K(\varepsilon)$ (see [**B**$_3$], [**P-U-Z**])

In general it is not too easy to check whether a given function is homological to a constant But in our special case we can make a simple observation in this direction Namely, since the fact that $\phi + \mathrm{HD}\,(K(\varepsilon)) \log(|g'|\,|K(\varepsilon))$ is homological to a constant means that there is a Holder continuous function $u \;\; K(\varepsilon) \to \mathbb{R}$ and a constant c such that $\phi + \mathrm{HD}\,(K(\varepsilon)) \log(|g'|\,|K(\varepsilon)) - c = u \circ (g\,|\,K(\varepsilon)) - u$, and for every $\delta \geq \varepsilon$ we have $K(\delta) \subset K(\varepsilon)$, it is obvious that $\phi + \mathrm{HD}\,(K(\delta)) \log(|g'|)$ is also homological to a constant

4 In this section we give some other applications of results proved in §§ 1, 2 Let us start with DE-perturbations The following definition corresponds to definition 3 from [**U**]

Definition 4 We say that a C^2-mapping $\tilde{g} \;\; S^1 \to S^1$ is a DE-*perturbation* obtained from an orientation preserving C^2-expanding mapping $g \;\; S^1 \to S^1$ if the following conditions are satisfied

(a) there exist real numbers $0 < \beta_1, \beta_2 < 1$ such that $\tilde{g}(\beta_1) = \beta_1$, $\tilde{g}(1-\beta_2) = 1-\beta_2$, $\tilde{g}'|[\beta_1, 1-\beta_2] > 1$ and for every $x \in (1-\beta_2, \beta_1)$ $\lim_{n\to\infty} \tilde{g}^n(x) = 0$,

(b) there exist real numbers $\gamma_1 > \beta_1$ and $\gamma_2 > \beta_2$ such that $\tilde{g}|[\gamma_1, 1-\gamma_2] = g|[\gamma_1, 1-\gamma_2]$

It is clear that all the facts proved in [**U**] for one-sided DE-perturbations have corresponding versions for (two-sided) DE-perturbations Using theorem 4 we want to give a stronger version of proposition 3 from [**U**]

THEOREM 7 *If* $\tilde{g}_n \to g$ *in the* C^0-*topology such that* $\gamma_1^{(n)}, \gamma_2^{(n)} \to 0$ *then* $\lim_{n\to\infty} (\mathrm{HD}\,(\Omega(\tilde{g}_n))) = 1$, *where* $\Omega(\tilde{g}_n)$ *denotes the set of all non-wandering points of* $\tilde{g}_n$

Proof Since $\Omega(\tilde{g}_n) \supset K((1-\gamma_2, \gamma_1))$, $\mathrm{HD}\,(\Omega(\tilde{g}_n)) \geq \mathrm{HD}\,(K(1-\gamma_2, \gamma_1))$ and by theorem 4,

$$1 \geq \limsup_{n\to\infty} \mathrm{HD}\,(\Omega(\tilde{g}_n)) \geq \liminf_{n\to\infty} \mathrm{HD}\,(\Omega(\tilde{g}_n)) \geq \mathrm{HD}\,((0,0)) = 1$$

Since $\Omega_0(\tilde{g}) = \Omega(\tilde{g}) \backslash \{0\}$ is a mixing repeller of $\tilde{g}$, the following theorem, related to theorem 6, follows from [**P-U-Z**]

THEOREM 8 *Let ϕ $S^1 \to \mathbb{R}$ be a Holder continuous function such that $\phi|\Omega_0(\tilde{g}) + \mathrm{HD}(\Omega_0(\tilde{g}))\log(|\tilde{g}'||\Omega_0(\tilde{g}))$ is not homological to a constant Then μ_ϕ is singular with respect to H_{ϕ_c} for $c \le \sqrt{2\sigma^2(\psi)/\int \log|\tilde{g}'|\,d\mu_\phi}$ and absolutely continuous for $c > \sqrt{2\sigma^2(\psi)/\int \log|\tilde{g}'|\,d\mu_\phi}$ Here $\psi = \phi + \mathrm{HD}(\mu_\phi)\log|\tilde{g}'|$*

If $\phi + \mathrm{HD}(\Omega_0(\tilde{g}))\log(|\tilde{g}'||\Omega_0(g))$ is homological to a constant then look at the comment after theorem 6

Theorems 7 and 8 are obviously also true in the case of one-sided DE-perturbations

THEOREM 9 *Let ϕ $I \to \mathbb{R}$ be a continuous function defined on a closed interval I If g $I \to I$ is a continuous mapping of Misiurewicz-Szlenk and $P(g, \phi) > \sup(\phi)$, then the pressure function $P(\ ,\phi)$ $C^0(I, I) \to \bar{\mathbb{R}}$ is lower semi-continuous at g $C^0(I, I)$ denotes here the space of all continuous mappings of I into itself with the C^0-topology*

This theorem generalizes a result of Misiurewicz and Szlenk and its proof is a simpler version of the proof of theorem 2 There is an easy example (see figure 2) which shows that the assumption $P(g, \phi) > \sup(\phi)$ cannot be omitted Indeed, let $g([0, \frac{1}{2}]) \subset [0, \frac{1}{2}]$, $g(\frac{1}{2}) = \frac{1}{2}$, $g(1) = 1$ and for every $x \in [\frac{1}{2}, 1)$ $\lim_{n\to\infty} g^n(x) = \frac{1}{2}$, as in figure 2 If we take ϕ $I \to \mathbb{R}$ so that $\phi|[0, \frac{1}{2}] \equiv 0$ and $\phi(1) > h_{\mathrm{top}}(g|[0, \frac{1}{2}])$ then $P(g, \phi) = \phi(1)$, but for every f $I \to I$ for which $f|[0, \frac{1}{2}] = g|[0, \frac{1}{2}]$ and for $x \in [\frac{1}{2}, 1]$ $\lim_{n\to\infty} f^n(x) = \frac{1}{2}$, $P(g, \phi) = h_{\mathrm{top}}(g|[0, \frac{1}{2}])$

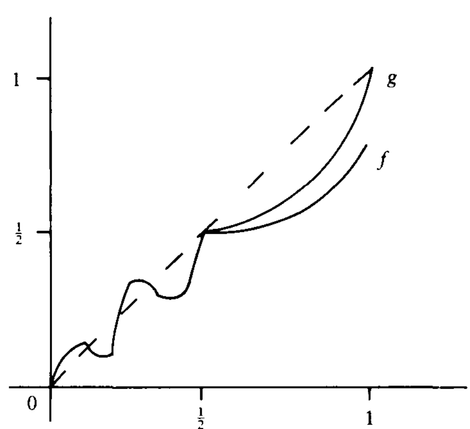

FIGURE 2

Theorem 9 has the following corollary

COROLLARY 8 *If $C^0(I, \mathbb{R}_+)$ denotes the set of all positive continuous functions with topology of uniform convergence and $s'(g, \phi)$ is the special flow defined by a continuous mapping of Misiurewicz-Szlenk g $I \to I$ and positive continuous function ϕ, then the function $C^0(I, I) \times C^0(I, \mathbb{R}_+) \ni (f, \phi) \mapsto h_{\mathrm{top}}(s'(f, \phi))$ is lower semi-continuous at (g, ϕ)*

Proof A theorem of Abramov [A] states that

$$h_{\tilde{\mu}}(s^1) = h_\mu(g) \Big/ \int \phi \, d\mu,$$

where $\tilde{\mu}$ is the s^t-invariant measure induced by a g-invariant measure μ Hence

$$c \stackrel{\text{def}}{=} h_{\text{top}}(s^t(g, \phi)) = \sup_{\mu \in M(g)} h_{\tilde{\mu}}(s^1) = \sup_{\mu \in M(g)} \left(h_\mu(g) \Big/ \int \phi \, d\mu \right)$$

and further

$$0 = \sup_{\mu \in M(g)} \left(h_\mu(g) - c \int \phi \, d\mu \right) \Big/ \int \phi \, d\mu = \sup_{\mu \in M(g)} \left(h_\mu(g) - c \int \phi \, d\mu \right) = P(g, -c\phi)$$

i e $h_{\text{top}}(s^t(g, \phi))$ is the unique solution of the equation $P(g, -t\phi) = 0$ This permits us to make use of theorem 9 and proceed as in the proof of theorem 3

We want to finish this section with a remark which refers to the family $K((s, t))$ for an expanding mapping g of degree 2 Namely, if we pass to the unit interval as was done in [U], we will get the family of mappings which are topologically conjugate to those which arise as the Poincaré maps of Lorenz attractors (see [W] for example)

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