COMPOSITIO MATHEMATICA

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Compositio Math. **150** (2014), 1549–1578.

doi:10.1112/S0010437X14007295

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY

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Matrix factorizations via Koszul duality

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Abstract

In this paper we prove a version of curved Koszul duality for $\mathbb{Z}/2\mathbb{Z}$-graded curved coalgebras and their cobar differential graded algebras. A curved version of the homological perturbation lemma is also obtained as a useful technical tool for studying curved (co)algebras and precomplexes. The results of Koszul duality can be applied to study the category of matrix factorizations $\mathsf{MF}(R, W)$. We show how Dyckerhoff's generating results fit into the framework of curved Koszul duality theory. This enables us to clarify the relationship between the Borel–Moore Hochschild homology of curved (co)algebras and the ordinary Hochschild homology of the category $\mathsf{MF}(R, W)$. Similar results are also obtained in the orbifold case and in the graded case.

1. Introduction

1.1 Background and motivations

Matrix factorizations of an element W in a commutative ring $R = \mathbb{C}[[x_1, \ldots, x_n]]$ were first introduced by Eisenbud [Eis80] in the study of singularity theory. Recently this theory has received renewed interests largely due to its appearance in Kontsevich's homological mirror symmetry conjecture. Indeed the differential graded (dg) category $\mathsf{MF}(R, W)$ of matrix factorizations is conjecturally mirror to the Fukaya category of a Fano symplectic manifold M.

The following fundamental results concerning the structure of the dg category $\mathsf{MF}(R, W)$ were obtained by Dyckerhoff [Dyc11] under the assumption that W has isolated singularities:

(i) the homotopy category $[\mathsf{MF}(R, W)]$ is classically generated by a single object k^{stab};

(ii) the dg algebra $A := \mathsf{End}_{\mathsf{MF}(R,W)}(k^{\mathsf{stab}})$ realizes $\mathsf{MF}(R, W)$ as the dg category of perfect dg modules over A;

(iii) we have $HH_*(\mathsf{MF}(R, W)) \cong \mathsf{Jac}(W)[\dim R]$.

Dyckerhoff's computation of the Hochschild homology was indirect, which uses the fact that $\mathsf{MF}(R, W)$ is a compact smooth Calabi–Yau category to reduce the computation to that of Hochschild cohomology. The computation of Hochschild cohomology in turn relies on Toen's interpretation of it as natural transformations from the identity functor to itself.

There is a different approach to understanding the category $\mathsf{MF}(R, W)$ initiated in [Seg13] and [CT13]. More precisely the data (R, W) naturally give rise to a curved algebra which we will denote by R_W. The category $\mathsf{MF}(R, W)$ can be interpreted as the category of perfect modules over this curved algebra R_W. Through this perspective Căldăraru and the author [CT13] introduced the notion of Borel–Moore Hochschild homology of a curved algebra, and proved that[1]

$$HH_*^{\mathsf{BM}}(R_W) \cong \mathsf{Jac}(W)[\dim R].$$

Received 14 September 2012, accepted in final form 16 January 2014, published online 17 July 2014.
2010 Mathematics Subject Classification 18G35, 14B05, 16E40 (primary).
Keywords: Koszul duality, matrix factorizations, Hochschild homology.

[1] This isomorphism and the isomorphism below hold for arbitrary smooth commutative ring R and W with isolated singularities.

Moreover, in [Seg13] it was shown that

$$HH_*^{\mathsf{BM}}(\mathsf{MF}(R,W)) \cong HH_*^{\mathsf{BM}}(R_W).$$

The main advantage of this approach is we have an explicit complex: the Borel–Moore Hochschild chain complex of R_W. Thus, it would be desirable to relate $HH_*(\mathsf{MF}(R,W))$ with $HH_*^{\mathsf{BM}}(R_W)$ or $HH_*^{\mathsf{BM}}(\mathsf{MF}(R,W))$, which would also yield an easier way of computing $HH_*(\mathsf{MF}(R,W))$. Clarifying this relationship is the main motivation for the current paper. In the following we explain the main ideas, and give a section-wise summary of our main results.

1.2 Curved Koszul duality over $\mathbb{Z}/2\mathbb{Z}$

The main idea of relating the two types of Hochschild homology is to use Koszul duality theory. In fact, Dyckerhoff's results mentioned above already suggest such a link.

For applications to matrix factorizations we need a version of curved Koszul duality theory in the $\mathbb{Z}/2\mathbb{Z}$-graded situation. This theory was developed by Positselski in [Pos11] in great generality where various types of non-standard derived categories were introduced in order to obtain the desired results. In § 2 we give a more direct proof of the version of Koszul duality which is enough for the applications we have in mind. The proofs we give rely on a curved version of homological perturbation lemma (see Appendix A) which is of independent interest. More precisely the main result we obtain in Koszul duality theory is the following theorem.

Theorem 1.1. *Let B_M be a coaugumented curved coalgebra, and let ΩB_M be its cobar dg algebra. Then there is a quasi-equivalence*

$$\mathsf{Tw}(B_M) \cong \mathsf{Tw}(\Omega B_M)$$

of dg categories of twisted complexes. If, furthermore, the coalgebra B is conilpotent, then the dg algebra ΩB_M itself is a compact generator for the homotopy category $[\mathsf{Tw}(\Omega B_M)]$.

Remark. The twisted complexes used in this theorem are not the standard ones in the sense that we allow possibly infinite rank ones and, moreover, we do not assume the upper-triangular condition. Owing to these two non-standard conventions, the second part of the above theorem is not at all obvious. We also remark that these modifications are necessary for the purpose of doing Koszul duality.

1.3 Applications to $\mathsf{MF}(R,W)$

In §§ 3 and 4 we apply Koszul duality theory to study $\mathsf{MF}(R,W)$. For this observe that the commutative ring R is the dual algebra of the symmetric coalgebra C generated by variables $y_1 := x_1^\vee, \ldots, y_n := x_n^\vee$, and the element $W \in R$ is the dual of a linear map $M : C \to \mathbb{C}$. Moreover, matrix factorizations of (R,W) can be identified with twisted complexes over the curved coalgebra C_M which are of finite rank. This simple observation allows us to apply Theorem 1.1 in the situation C_M and ΩC_M to obtain understanding of $\mathsf{MF}(R,W)$. We summarize our main results in the following theorem.

Theorem 1.2. *Assume that W has isolated singularities. Then we have the following:*

(i) *Dyckerhoff's generator k^{stab} arises from Koszul duality;*[2]

(ii) *the dg algebra $A := \mathsf{End}_{\mathsf{MF}(R,W)}(k^{\mathsf{stab}})$ is quasi-isomorphic to the cobar dg algebra ΩC_M;*

(iii) *there is a canonical isomorphism $HH_*(\mathsf{MF}(R,W)) \cong [HH_*^{\mathsf{BM}}(R_W)]^\vee$.*

[2] For a more precise statement we refer to § 3.

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Remark. It is an interesting puzzle to understand the appearance of dualization in the above isomorphism between the $HH_*^{\mathsf{BM}}(R_W)$ and $HH_*(\mathsf{MF}(R,W))$. This might be explained by a relationship between Koszul duality and a natural pairing (generalized Mukai pairing) on the Hochschild homology.

1.4 Applications to $\mathsf{MF}_G(R,W)$

In § 5 we generalize the above results and Dyckerhoff's generation result to the orbifold case. The main results are summarized in the following theorem.

Theorem 1.3. *Assume that W has isolated singularities, and G a finite abelian group acting on R which fixes W. Then we have:*

(i) *the homotopy category $[\mathsf{MF}_G(R,W)]$ of equivariant matrix factorizations is classically generated by*

$$\{k^{\mathsf{stab}} \otimes \mathbb{C}_\chi \mid \chi \text{ is a character for the group } G\}$$

where $\mathbb{C}_\chi$ denotes the one-dimensional representation associated to the character χ;

(ii) *the smash product dg algebra $\Omega(C_M)\sharp G$ realizes $\mathsf{MF}_G(R,W)$ as the dg category of perfect dg modules over $\Omega(C_M)\sharp G$;*

(iii) *for the Hochschild homology we have*

$$HH_*(\mathsf{MF}_G(R,W)) \cong [HH_*^{\mathsf{BM}}(R_W\sharp G)]^\vee.$$

Remark. In [CT13] the vector space $HH_*^{\mathsf{BM}}(R_W\sharp G)$ was explicitly computed as

$$HH_*^{\mathsf{BM}}(R_W\sharp G) = \left(\bigoplus_{g\in G} HH_*^{\mathsf{BM}}(R_W|_g)\right)^G,$$

where $R_W|_g$ denotes the curved algebra associated to the LG model on the g-fixed points of $\mathsf{Spec}(R)$.

1.5 Applications to $\mathsf{MF}^{\mathsf{gr}}(S,W)$

In § 6 we study graded matrix factorizations where similar results are obtained. In the graded case we consider $S := \mathbb{C}[x_1,\ldots,x_n]$ the polynomial ring in n variables endowed with its standard grading. Let $W \in S$ be a homogeneous polynomial of degree d. Denote $G := \mathbb{Z}/d\mathbb{Z}$ which acts on S by ring homomorphism generated by

$$\zeta(x_j) := \zeta \cdot x_j.$$

Since W is of degree d this action preserves W. In this situation we can consider the dg category $\mathsf{MF}^{\mathsf{gr}}(S,W)$ of graded matrix factorizations (see § 6 for its definition).

Theorem 1.4. *Assume that W has isolated singularities. Then we have:*

(i) *the homotopy category $[\mathsf{MF}^{\mathsf{gr}}(S,W)]$ is classically generated by*

$$k^{\mathsf{stab}}(d-1), k^{\mathsf{stab}}(d-2), \ldots, k^{\mathsf{stab}}$$

where the shifts in the parentheses are polynomial degree shifts of graded S-modules;

(ii) *there is a $\mathbb{Z}$-graded[3] smash product algebra $\Omega(C_M)\sharp G$ realizing $\mathsf{MF}^{\mathsf{gr}}(S,W)$ as the dg category of perfect dg modules;*

[3] This $\mathbb{Z}$-graded is not the standard polynomial grading, see § 6 for its precise definition.

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(iii) *for the Hochschild homology we have*

$$HH_*(\mathsf{MF}^{\mathsf{gr}}(S,W)) \cong [HH_*^{\mathsf{BM}}(S_W \sharp G)]^\vee$$

where the operation $\vee$ *denotes graded dualization.*

2. Koszul duality for the (co)bar constructions

In this section we recall the cobar construction and prove a version of Koszul duality between $\mathbb{Z}/2\mathbb{Z}$-graded curved coalgebras and their cobar algebras. Then we prove some useful properties concerning the derived categories of cobar algebras. The results proved in this section are essentially due to Positselski [Pos11], although we give more direct proofs here.

Throughout this section we work with $\mathbb{Z}/2\mathbb{Z}$-graded objects over a base field k. For instance, a graded vector space means a $\mathbb{Z}/2\mathbb{Z}$-graded (or super) vector space; a complex means a super vector space endowed with an odd endomorphism that squares to zero; and a dg category means a category enriched over $\mathbb{Z}/2\mathbb{Z}$-graded complexes. Linear algebra operations such as tensor product or homomorphism between vector spaces are all taken over k unless otherwise stated. Finally all algebras and coalgebras are assumed to be unital.

2.1 Curved algebras

A curved algebra structure on a super vector space A is an associative algebra structure on A together with an even central element $W \in A$.

Example 2.1. An example of a curved algebra that will be of primary interest in this paper. Let V be a finite-dimensional vector space over a field k. Consider the commutative algebra $R := \widehat{\mathsf{sym}(V^\vee)}$ together with a choice of an element W in it. Since R is commutative any element in it is automatically central.

2.2 Twisted complexes over A_W: matrix factorizations

We can define the category $\mathsf{Tw}(A_W)$ of twisted complexes over a curved algebra A_W. The objects of this category are pairs (E,Q) where E is a $\mathbb{Z}/2\mathbb{Z}$-graded free A-module and Q is an odd A-linear map such that $Q^2 = W\,\mathsf{id}$. The morphism space between two objects (E,Q) and (F,P) consists of all A-linear maps from E to F. As such, the Hom space inherits a differential defined by $D(\varphi) = P \circ \varphi - (-1)^{|\varphi|}\varphi \circ Q$. One easily checks that D squares to zero as $W\,\mathsf{id}$ is in the center of matrix algebras.

This differential makes the category $\mathsf{Tw}(A_W)$ into a dg category. Note that here we allow possibly infinite rank modules in the construction of $\mathsf{Tw}(A_W)$. We denote by $\mathsf{Tw}^b(A_W)$ the full subcategory of $\mathsf{Tw}(A_W)$ consisting of twisted complexes that are of finite rank. The category $\mathsf{Tw}^b(A_W)$ ($\mathsf{Tw}(A_W)$ respectively) is sometimes also referred to as the category of matrix factorizations $\mathsf{MF}(A,W)$ ($\mathsf{MF}^\infty(A,W)$ respectively).

As the category $\mathsf{Tw}(A_W)$ has a dg structure we can define the notion of homotopy between morphisms and objects. More precisely, we say two morphisms f and g are homotopic if $f-g$ is exact. We say two objects E and F are homotopy equivalent if there are morphisms $f: E \to F$ and $g: F \to E$ such that $f \circ g$ is homotopic to id_F and $g \circ f$ is homotopic to id_E.

2.3 Curved coalgebras

Dualizing the definition for curved algebras we arrive at the definition for curved coalgebras. Namely curved coalgebra structure on a vector space B is a $\mathbb{Z}/2\mathbb{Z}$-graded coassociative coalgebra

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structure on B together with an even map $M : B \to k$ such that the composition

$$B \xrightarrow{\Delta} B \otimes B \xrightarrow{M \otimes \mathsf{id} - \mathsf{id} \otimes M} B$$

is zero. Using Sweedler's notation for the coproduct Δ, the above is equivalent to

$$M(x^{(1)})x^{(2)} - x^{(1)}M(x^{(2)}) = 0$$

for all $x \in B$. As before we denote a curved coalgebra by B_M.

Example 2.2. As a dual example of Example 2.1 we consider $C := \mathsf{sym}(V)$ to be the vector space of symmetric tensors on V. Again we consider C as a super vector space concentrated in the even part. There is a natural coalgebra structure on $C = \mathsf{sym}(V)$ defined by

$$\Delta(v_1 \cdots v_n) := \sum_{p \geqslant 0, q \geqslant 0, p+q=n, \sigma \in S(p,q)} (v_{\sigma(1)} \cdots v_{\sigma(p)}) \otimes (v_{\sigma(p+1)} \cdots v_{\sigma(n)})$$

where $S(p,q)$ is the subgroup of the symmetric group $S(n)$ consisting of (p,q)-shuffles. It follows from definition that the dual algebra[4] of C is the commutative algebra R in Example 2.1. The curvature term is any linear map $M : C \to k$.

2.4 Basics of comodules

We recall some useful properties of cofree comodules. First of all for purposes of this paper we only consider cofree comodules of the form $B \otimes V$ for some k-vector space V (possibly infinite-dimensional). Moreover, in the abelian category $\mathscr{A}$ of B-comodules, cofree comodules are injective objects and hence is closed under direct product in $\mathscr{A}$. For example, we have $\prod(B \otimes V_i) \cong B \otimes (\prod V_i)$ where the product on the left-hand side is *taken in* $\mathscr{A}$. A special property for $\mathscr{A}$ is that the class of injective objects is also closed under direct sum in $\mathscr{A}$. Explicitly, we have $\coprod(B \otimes V_i) \cong B \otimes (\coprod V_i)$.

2.5 Twisted complexes over B_M: matrix cofactorizations

Given a curved coalgebra B_M we can construct a category $\mathsf{Tw}(B_M)$ of twisted complexes. The objects are pairs (E, Q) with E a cofree B-comodule and Q an odd comodule map on E such that the dual of the matrix factorization identity holds,

$$Q^2(x) = M(b^{(1)})x^{(2)}.$$

Here we write the coaction map to be $\rho(x) = b^{(1)} \otimes x^{(2)}$ for $x \in E$, $b^{(1)} \in B$ and $x^{(2)} \in E$. The Hom spaces and differentials on Hom spaces are defined in a similar way as for matrix factorizations. Objects in $\mathsf{Tw}(B_M)$ will be called matrix cofactorizations. There is also a dg structure on $\mathsf{Tw}(B_M)$. The full subcategory of $\mathsf{Tw}(B_M)$ consisting of matrix cofactorizations that are of finite rank over B will be denoted by $\mathsf{Tw}^b(B_M)$.

There is a simple relation between the two dg categories $\mathsf{Tw}^b(R_W)$ and $\mathsf{Tw}^b(C_M)$, made precise in the following lemma.

Lemma 2.3. *Let B_M be a curved coalgebra, and let A_W be its dual curved algebra. Define a functor $D : \mathsf{Tw}^b(B_M)^{\mathsf{op}} \to \mathsf{Tw}^b(A_W)$ by formula*

$$(E, Q) \overset{D}{\mapsto} (E^\vee, Q^\vee)$$

[4] Note that the dual of a coalgebra is always an algebra, but not *vice versa* due to infinite dimensionality.

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on objects and for any morphism $f \in \mathsf{Hom}_{\mathsf{Tw}(C_M)}((E,Q),(F,P))$

$$D(f) := f^\vee : (F^\vee, P^\vee) \to (E^\vee, Q^\vee).$$

Then D *is an equivalence between* $\mathsf{Tw}^b(B_M)^{\mathsf{op}}$ *and* $\mathsf{Tw}^b(A_W)$.

Proof. Observe that a map $h : B \to B$ of B-comodules is uniquely determined by its composition with the counit map. Conversely any k-linear map $\alpha : B \to k$ defines a map of B-comodules by

$$B \to B \otimes B \xrightarrow{\alpha \otimes \mathsf{id}} k \otimes B = B.$$

This defines an isomorphism between $\mathsf{Hom}_B(B,B)$ and $\mathsf{Hom}_k(B,k) = A$. More generally for two cofree C-comodules $E_1 = B \otimes V_1$ and $E_2 = B \otimes V_2$ with V_1 and V_2 finite-dimensional vector spaces over k we have

$$\begin{aligned}\mathsf{Hom}_B(E_1,E_2) &= \mathsf{Hom}_B(B \otimes V_1, B \otimes V_2)\\ &\cong \mathsf{Hom}_B(B,B) \otimes \mathsf{Hom}_k(V_1,V_2) \cong A \otimes \mathsf{Hom}(V_1,V_2).\end{aligned}$$

For the Hom space between DE_2 and DE_1, we have

$$\begin{aligned}\mathsf{Hom}_A((B \otimes V_2)^\vee, (B \otimes V_1)^\vee) &= \mathsf{Hom}_A(A \otimes V_2^\vee, A \otimes V_1^\vee)\\ &= A \otimes \mathsf{Hom}(V_2^\vee, V_1^\vee) = A \otimes \mathsf{Hom}(V_1,V_2),\end{aligned}$$

where the first and the last equality follow from V_1 and V_2 being finite dimensional. Thus, we have verified that the functor D is an equivalence. A direct computation shows that it also preserves the differential and hence the lemma is proved. □

2.6 The cobar construction

Let B_M be a curved coalgebra. A k-linear map $\eta : k \to B$ is called a coaugumentation of B_M if:

(i) η splits the counit map;

(ii) η is a map of coalgebras;

(iii) $M \circ \eta = 0$.

Denote by B^+ the cokernel of η which can be identified with the kernel of the counit through the splitting in condition (i). Then condition (ii) implies that B^+ is a quotient coalgebra of B and condition (iii) implies that $M : B \to k$ factor through B^+.

Given a coaugumented curved coalgebra B_M, we can construct a dg algebra ΩB_M, known as its cobar construction. Explicitly as an associative algebra ΩB_M is the free tensor algebra generated by $B^+[-1]$ which is simply

$$\mathbb{T}B^+[-1] = \bigoplus_{k=0}^{\infty} (B^+[-1])^{\otimes k}.$$

The differential d is a derivation on ΩB_M determined by the following two components

$$B^+ \hookrightarrow B \to B \otimes B \to B^+ \otimes B^+;$$
$$B^+ \hookrightarrow B \xrightarrow{M} k.$$

Example 2.4. Let us work out the cobar construction of the curved coalgebra C_M defined in Example 2.2. There is a natural coaugmentation on C: the inclusion of scalars. In order for

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it to be compatible with curvature, we assume that M vanishes on the scalar part of C. This coaugmentation induces, in particular, a direct sum decomposition $C \cong C^+ \oplus k$. The cobar algebra ΩC_M is the free tensor algebra generated by $\mathsf{sym}(V)^+[-1]$ with differential given by the sum of two components which we denote by d^+ and d^-. These maps act on monomials $f_1|\cdots|f_k$ by

$$d^+(f_1|f_2|\cdots|f_k) = \sum_{i=1}^{k}(-1)^{i-1}f_1|\cdots|\Delta(f_i)|\cdots|f_k,$$
$$d^-(f_1|f_2|\cdots|f_k) = \sum_{i=1}^{k}(-1)^{i-1}M(f_i)f_1|\cdots|\widehat{f_i}|\cdots|f_k,$$

where Δ is the coproduct on C^+ induced from that of C.

2.7 Twisting cochains

For a curved coalgebra B_M and a unital dg algebra A, one can construct a curved dg algebra structure on the space of k-linear maps $\mathsf{Hom}(B, A)$. It is defined by the following formulas:

(i) curvature: $W(B,A)$: $B \xrightarrow{M} k \xrightarrow{\text{unit}} A$;

(ii) differential: $(d\varphi)(x) = d(\varphi(x))$;

(iii) product: $(\varphi * \psi)(x) = (-1)^{|x^{(1)}||\psi|}\varphi(x^{(1)})\psi(x^{(2)})$.

A twisting cochain from B to A is an odd element $\tau \in \mathsf{Hom}(B, A)$ such that

$$\tau * \tau + d\tau + W(B, A) = 0.$$

There is a natural twisting cochain $\tau_{B_M} : B \to \Omega B_M$ defined by the composition

$$B \to B^+ \xrightarrow{-\mathsf{id}} B^+[1] \hookrightarrow \Omega B_M.$$

2.8 Correspondence of twisted complexes

We can use the twisting cochain τ_{B_M} to define a correspondence between categories of twisted complexes. We work out explicitly this correspondence for a coaugumented curved coalgebra B_M its cobar algebra ΩB_M. The goal is to construct dg functors

$$\Phi : \mathsf{Tw}(B_M) \to \mathsf{Tw}(\Omega B_M)$$
$$\Psi : \mathsf{Tw}(\Omega B_M) \to \mathsf{Tw}(B_M).$$

We begin with the construction of Φ. Let (E, Q) be a matrix cofactorization over B_M. We need to produce a twisted complex $\Phi(E)$ over ΩB_M. As a vector space over k it is simply $\Omega B_M \otimes E$. The left ΩB_M-module structure is induced from that of ΩB_M. The differential on $\Omega B_M \otimes E$ is defined using the natural twisting cochain τ_{B_M}:

$$d(x \otimes e) = dx \otimes e + (-1)^{|x|}x \otimes Qe + (-1)^{|x|}x\tau(y^{(1)}) \otimes e^{(2)},$$

where we have denoted the coaction map $\rho : E \to B \otimes E$ by $\rho(e) = y^{(1)} \otimes e^{(2)}$ for $y^{(1)} \in B$. One checks that $d^2 = 0$ and that it is compatible with the left module structure on $\Phi(E)$. We write $\Phi(E) = \Omega \otimes^\tau E$ where the superscript τ is to indicate that we are using the twisting cochain τ to define the differential on $\Phi(E)$. Note that $\Phi(E)$ is of infinite rank whenever B is of infinite dimension over k. For this reason we need to consider $\mathsf{Tw}(\Omega B_M)$ instead of $\mathsf{Tw}^b(\Omega B_M)$. For a morphism $f:(E,Q)\to(F,P)$ in $\mathsf{Tw}(B_M)$, define $\Phi(f)=\mathsf{id}\otimes f$ from $\Phi(E)$ to $\Phi(F)$. One can check that Φ is a dg functor between dg categories.

In the reverse direction, if (F,d) is a twisted complex over ΩB_M, we need to define a matrix cofactorization $\Psi(F)$ over B_M. As a vector space this is $B\otimes F$. The left B-comodule structure is induced from that of B and the matrix cofactorization map is defined by

$$Q(x\otimes f)=dx\otimes f+(-1)^{|x|}x\otimes df+(-1)^{|x^{(1)}|}x^{(1)}\otimes\tau(x^{(2)})f$$

where $\tau(x^{(2)})f$ is the action of ΩB_M on F. One checks that Q satisfies the matrix cofactorization identity and hence defines a twisted complex (again of infinite rank) over B_M. Similarly the above construction extends to the morphism space and hence defines a dg functor Ψ in the reverse direction.

2.9 Curved cobar duality over $\mathbb{Z}/2\mathbb{Z}$

The two functors Φ and Ψ form an adjoint pair $\Phi\dashv\Psi$, with its counit and unit natural transformations defined as follows. First we construct the unit natural transformation $\eta:\mathsf{id}\to\Psi\Phi$. For any object $(E,Q)\in\mathsf{Tw}(B_M)$, consider the morphism η_E from E to $\Psi\Phi(E)=B\otimes^\tau\Omega B_M\otimes^\tau E$ defined by

$$e\mapsto y^{(1)}\otimes 1\otimes e^{(2)},$$

where $y^{(1)}$ and $e^{(2)}$ are defined by the coaction map $E\to B\otimes E$. Next we construct the counit $\epsilon:\Phi\Psi\to\mathsf{id}$. For an object $F\in\mathsf{Tw}(\Omega B_M)$ we consider the natural map $\epsilon_F:\Phi\Psi(F):=\Omega B_M\otimes^\tau B_M\otimes^\tau F\to F$ defined by

$$x\otimes 1\otimes f\mapsto xf$$

and zero on the other tensors. One verifies directly by definition that the compositions

$$\Phi\xrightarrow{\Phi\eta}\Phi\Psi\Phi\xrightarrow{\epsilon\Phi}\Phi,$$
$$\Psi\xrightarrow{\eta\Psi}\Psi\Phi\Psi\xrightarrow{\Psi\epsilon}\Psi$$

are both identity transformations, proving that $\Phi\dashv\Psi$.

Furthermore, the natural transformations η and ϵ are compatible with the dg structure in the sense that for all $(E,Q)\in\mathsf{Tw}(B_M)$ and $F\in\mathsf{Tw}(\Omega B_M)$, the unit and counit maps η_E, ϵ_F are closed morphisms. Thus, the dg functors Φ and Ψ remain adjoin functors after passing to the homotopy categories. The following theorem proves that in fact Φ and Ψ are homotopy inverse of each other, i.e. they are inverse equivalences on the associated homotopy categories. Hence, the two categories $\mathsf{Tw}(B_M)$ and $\mathsf{Tw}(\Omega B_M)$ are homotopy equivalent.

THEOREM 2.5. *Let the notation be as introduced above. Then for all objects $(E,Q)\in\mathsf{Tw}(B_M)$ and $F\in\mathsf{Tw}(\Omega B_M)$, the unit and counit maps η_E, ϵ_F are homotopy equivalences.*

Proof. We start by showing that η_E is a homotopy equivalence. □

LEMMA 2.6. *Let $f:(E,Q)\to(F,P)$ be a closed morphism in $\mathsf{Tw}(B_M)$. Define the cone of f to be the matrix cofactorization $(E[1]\oplus F,T)$ with T given by the matrix*

$$T=\begin{bmatrix}-Q & 0\\ f & P\end{bmatrix}.$$

Then f is a homotopy equivalence if and only if $\mathsf{cone}(f)$ is contractible.

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Proof. If $\mathsf{cone}(f)$ is contractible, there exists a morphism $H : \mathsf{cone}(f) \to \mathsf{cone}(f)$ such that

$$\mathsf{id} = [T, H].$$

Writing H as a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

after a matrix multiplication, we find that the map b defines a homotopy inverse of f. A similar consideration works for the reversed direction. The lemma is proved. □

By the above lemma, it is enough to prove that $\mathsf{cone}(\eta_E)$ is contractible. The cone $\mathsf{cone}(\eta_E)$ is explicitly given by $E[1] \oplus (B \otimes^\tau \Omega B_M \otimes^\tau E)$ on which acts an operator D which satisfies the matrix cofactorization identity. We next write down the map D on $E[1]$ and $(B \otimes^\tau \Omega B_M \otimes^\tau E)$. On elements of the form $b_0[b_1|\cdots|b_l] \otimes e \in (B \otimes^\tau \Omega B_M \otimes^\tau E)$ the predifferential D acts by

$$\begin{aligned}
D &:= d_\Delta - Q + d_M;\\
d_\Delta(b_0[b_1|\cdots|b_l] \otimes e) &:= b_0^{(1)}[b_0^{(2)}|\cdots|b_l] \otimes e + \sum_{i=1}^{l} (-1)^{i-1} b_0[b_1|\cdots|\Delta(b_i)|\cdots|b_l] \otimes e\\
&\quad + (-1)^l b_0[b_1|\cdots|b_l|b^{(1)}] \otimes e^{(2)};\\
-Q(b_0[b_1|\cdots|b_l] \otimes e) &:= (-1)^{l+1} b_0[b_1|\cdots|b_l] \otimes Qe;\\
d_M(b_0[b_1|\cdots|b_l] \otimes e) &:= \sum_{i=1}^{l} (-1)^{i-1} b_0[b_1|\cdots|M(b_i)|\cdots|b_l] \otimes e.
\end{aligned}$$

On elements in $E[1]$ the predifferential D acts by

$$\begin{aligned}
D &:= d_\Delta - Q + d_M;\\
d_\Delta(e) &:= b^{(1)} \otimes 1 \otimes e^{(2)} \in B \otimes 1 \otimes E;\\
-Q(e) &:= -Q(e);\\
d_M(e) &:= 0.
\end{aligned}$$

We observe that the differential d_Δ is simply the cobar resolution of the B-comodule E

$$E \to B \otimes E \to B \otimes B^+ \otimes E \to \cdots$$

which is exact. Moreover, since the B-comodule E is cofree (hence, injective) there exists a B-linear homotopy H on the cobar resolution above that makes the complex contractible over B. Note that the homotopy reduces the number of B-tensor components by one. This homotopy operator H defines a homotopy retraction data $(0, 0, H)$ between the zero complex and the cobar resolution (see Appendix A for details on homological perturbation technique). We also want to require H to be special, i.e. $H^2 = 0$. This can be achieved by making the following transformation

$$H \mapsto H d_\Delta H.$$

As the maps d_Δ are also B-linear, the special homotopy retraction is also B-linear. To show that $\mathsf{cone}(\eta_E)$ is contractible, we need to show that there exists a B-linear homotopy for D. For this we consider D as obtained from d_Δ by a small perturbation $Q + d_M$. Then apply homological perturbation lemma to obtain the homotopy for D. As mentioned earlier, the map D is not really

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a differential as D^2 is not zero. Thus the ordinary homological perturbation lemma does not apply to this case. However, D satisfies the matrix cofactorization identity by its construction. In this situation a curved version of homological perturbation lemma can still be applied as is explained in Appendix A. To perform perturbation we need the following lemma.

LEMMA 2.7. *The curved perturbation* $\delta := Q + d_M$ *is small. That is, we can define the operator* $(\mathsf{id} - \delta \circ H)^{-1}$ *on* $\mathsf{cone}(\eta_E)$. *In fact, the operator* $\delta \circ H$ *is locally nilpotent on* $\mathsf{cone}(\eta_E)$.

Proof. For a $\mathbb{Z}^{\geqslant 0}$-graded vector space we say an operator on it is locally nilpotent if for any element of bounded degree it is nilpotent. In our case, we consider the space $\mathsf{cone}(\eta_E)$ be graded by the number of B-tensors. Observe that the operator Q preserves the number of B-tensors while d_M reduces the number of B-tensors by one. The homotopy operator also reduces the number of B-tensors by one. Hence, the composition $\delta \circ H$ strictly reduces the number of B-tensors. So it must be locally nilpotent by degree consideration. Since $\delta \circ H$ is a locally nilpotent operator, one can define the operator $(\mathsf{id} - \delta \circ H)^{-1}$ on the direct sum of each graded components which is $\mathsf{cone}(\eta_E)$.[5] The lemma is proved. ☐

Applying the curved homological perturbation lemma (Lemma A.1) over the linear category of B-linear morphisms, we conclude that there exists a homotopy H_1 for the operator D. Hence, $\mathsf{cone}(\eta_E)$ is contractible.

We have finished half of the proof of the theorem. Next we prove the other half, i.e. ϵ_F is a homotopy equivalence. The author learned this argument from Positselski. It is an expanded version of the proof given in [Pos11, § 6.4]. To show that ϵ_F is a homotopy equivalence it suffices to show that the cocone $K := \mathsf{cone}(\epsilon_F)[-1]$ is contractible. This dg module as a vector space is $F[-1] \oplus \Omega B_M \otimes B \otimes F$. Define a finite decreasing filtration on it by

$$F^0K := K \supset F^1K := \Omega(B_M) \otimes B \otimes F \supset F^2K := \Omega(B_M) \otimes B^+ \otimes F \supset F^3K := 0.$$

One checks that the differential on K does not preserve this filtration but sends F^iK to $F^{i-1}K$. Moreover, the induced differential on the associated graded components agrees with the canonical resolution

$$0 \to \Omega B_M \otimes B^+ \otimes F \to \Omega B_M \otimes k \otimes F \to F \to 0$$

which is exact. Then we can define a dg ΩB_M-submodule of K by

$$L := F^2K + dF^2K,$$

where d is the differential on K. It follows from the exactness of the above short exact sequence that both L and K/L are contractible. In general, this does not imply that K is also contractible. However, in our case the dg module K/L is free as ΩB_M-modules, which implies that K admits a direct sum decomposition $L \oplus K/L$ as ΩB_M-modules. Note that this splitting does not necessarily preserve the differential on K, nevertheless it realizes K as the cone of a closed map from $L[-1]$ to K/L, which implies that K itself is also contractible. The proof of Theorem 2.5 is now complete.

2.10 Homological properties of $\mathsf{Tw}(\Omega B_M)$

We first introduce some notation. For a dg category $\mathscr{D}$ we denote by $[\mathscr{D}]$ its homotopy category. Recall that $[\mathscr{D}]$ has the same objects as $\mathscr{D}$, but the morphism spaces between objects are given by

[5] Note that it is important here that here $\mathsf{cone}(\eta_E)$ is a direct sum rather than a direct product.

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the zeroth cohomology of the morphism spaces in $\mathscr{D}$. Our next goal is to have an understanding of the category $[\mathsf{Tw}(\Omega B_M)]$.

It is a well-known fact that for a dg algebra A the category $[\mathsf{Tw}(A)]$ is a triangulated category. However, it *does not* agree with the derived category of A in general. The reason is that the derived category of A is defined by the localization of $[\mathsf{Tw}(A)]$ with respect to the class of acyclic objects (dg modules with zero cohomology) which might not be trivial in $[\mathsf{Tw}(A)]$. Equivalently this is to say that there might exist objects in $\mathsf{Tw}(A)$ that are acyclic while not being contractible. One such example is to take $A = k[x]/x^2$ and $E \in \mathsf{Tw}(A)$ to be

$$\cdots A \to A \xrightarrow{\cdot x} A \to A \cdots$$

where the maps are all given by multiplication by x. Then E is acyclic while it is not contractible.

However, for a coaugumented conilpotent coalgebra B endowed with a curvature term M, we will show that acyclic objects are the same as contractible objects in $\mathsf{Tw}(\Omega B_M)$. Recall that a coaugumented coalgebra B is called conilpotent if B^+, as the quotient coalgebra of B, is the union of the kernels of finite iterated coproducts.

Proposition 2.8. *Let B be a coaugumented conilpotent coalgebra and let F be an object in $\mathsf{Tw}(\Omega B_M)$. Then F is acyclic if and only if F is contractible.*

Proof. It suffices to prove that if F is acyclic, then it is contractible. As F is an acyclic complex there always exists a contracting homotopy for F over the field k. Let H be such a k-linear special homotopy of F. Consider the Koszul dual $\Psi(F) = B \otimes^\tau F$. The B-linear map $\mathsf{id} \otimes H$ defines a special contracting homotopy for the complex $(B \otimes F, \mathsf{id} \otimes d_F)$. The predifferential Q on $\Psi(F)$ is given by

$$Q = \mathsf{id} \otimes d_F + d^\tau$$

where the map d^τ comes from the natural twisting cochain τ associated with the curved coalgebra B_M. We consider $\delta := d^\tau$ as a curved perturbation of $\mathsf{id} \otimes d_F$ and apply the curved homological perturbation lemma as in the proof of the Theorem 2.5. For this we need to prove the curved perturbation δ is small. This is an immediate consequence of the conilpotency condition on C. In fact, the conilpotency condition implies that $\delta \circ (\mathsf{id} \otimes H)$ is a locally nilpotent operator. Thus by the curved homological perturbation lemma (Lemma A.1) the object $\Psi(F)$ is contractible in $\mathsf{Tw}(B_M)$. It follows that the object $\Phi\Psi(F)$ is also contractible. By Theorem 2.5, $\Phi\Psi(F)$ is homotopy equivalent to F and hence F is also contractible. Thus, the proposition is proved. □

2.11 Terminologies about generators

Proposition 2.8 immediately implies that the dg algebra ΩB_M itself is a generator for the triangulated category $[\mathsf{Tw}(\Omega B_M)]$ if B is conilpotent. To make a more precise statement we recall several distinct notions of generators for triangulated categories. We follow the exposition in [BB03]. Let $\mathscr{D}$ be a triangulated category. A set of objects $\mathscr{E} := \{E_i \mid i \in I\}$ is said to classically generate $\mathscr{D}$ if the smallest triangulated subcategory of $\mathscr{D}$ containing $\mathscr{E}$ that is closed under isomorphism and direct summands is equal to $\mathscr{D}$ itself. We say that $\mathscr{D}$ is finitely generated if it is classically generated by one object.

The second notion of generation is defined via the orthogonal category of $\mathscr{E}$. Namely, we say that $\mathscr{E}$ weakly generates $\mathscr{D}$ if the right orthogonal $\mathscr{E}^\perp$ is trivial. (The right orthogonal $\mathscr{E}^\perp$ is by definition the full subcategory of $\mathscr{D}$ consisting of objects A such that $\mathsf{Hom}_{\mathscr{D}}(E_i[n], A) = 0$ for all i and all n.) It is clear that classical generators are also weak generators. But the converse is not

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true in general, often we will drop the adverb 'weak' and say that $\mathscr{E}$ generates $\mathscr{D}$ if $\mathscr{E}$ weakly generates it.

If furthermore the category $\mathscr{D}$ admits arbitrary direct sums one can define the notion of compactness for objects. In such a category an object E in $\mathscr{D}$ is said to be compact if the functor $\mathsf{Hom}_{\mathscr{D}}(E,-)$ commutes with direct sums. Denote by $\mathscr{D}^c$ the full subcategory consisting of compact objects. We say that $\mathscr{D}$ is compactly generated if $\mathscr{D}^c$ generates $\mathscr{D}$. We need the following result by Ravenel and Neeman [Nee92].

Theorem 2.9. *Assume that a triangulated category $\mathscr{D}$ admitting arbitrary coproduct is compactly generated. Then a set of compact objects classically generates $\mathscr{D}^c$ if and only if it generates $\mathscr{D}$.*

Corollary 2.10. *Let the notation and assumptions be the same as in Proposition 2.8. Then the dg-module $\Omega(B_M)$ is a compact generator for the category $[\mathsf{Tw}(\Omega B_M)]$. Moreover, it classically generates the compact subcategory $[\mathsf{Tw}(\Omega B_M)]^c$.*

Proof. It is clear that the object ΩB_M is compact. Moreover, if $F \in [\mathsf{Tw}(\Omega B_M)]$ is right orthogonal to ΩB_M, it implies that the object F is acyclic. Then it follows from Proposition 2.8 that F is, in fact, contractible hence becomes zero in $[\mathsf{Tw}(\Omega B_M)]$. The last assertion follows from Theorem 2.9. □

3. Generators for $\mathsf{MF}(R, W)$

In this section we work with the curved coalgebra C_M and its dual curved algebra R_W as introduced in Examples 2.1 and 2.2. As symmetric coalgebras with their canonical coaugmentations are conilpotent coalgebras, all the results in the previous section hold for C_M. We prove that the image of the cobar algebra ΩC_M itself under the Koszul duality functor lies in $\mathsf{Tw}^b(C_M)$. Hence its k-linear dual makes sense and defines a matrix factorization in $\mathsf{Tw}^b(R_W) = \mathsf{MF}(R, W)$. Then we identify it with Dyckerhoff's k^{stab}. Corollary 2.10 and Proposition 3.1 then implies a homological interpretation for k^{stab} to classically generate $\mathsf{MF}(R, W)$. This homological interpretation is used in §§ 5 and 6 to produce classical generators for the derived categories of equivariant or graded matrix factorizations.

3.1 Compact generator for $[\mathsf{Tw}\, C_M]$

We begin to construct a compact generator for the homotopy category of $\mathsf{Tw}(C_M)$. Note that it is clear that in both the category $\mathsf{Tw}(C_M)$ and $\mathsf{Tw}(\Omega C_M)$ arbitrary coproducts exist and, hence, one can talk about compactness of objects in these categories. By Theorem 2.5 the two dg categories $\mathsf{Tw}(C_M)$ and $\mathsf{Tw}(\Omega C_M)$ are homotopy equivalent via the dg functors Φ and Ψ that preserve coproducts. Hence, Φ and Ψ send compact generators to compact generators. By Corollary 2.10 the object ΩC_M is a compact generator for the homotopy category $[\mathsf{Tw}(\Omega C_M)]$ as symmetric coalgebras are conilpotent. It follows that the matrix cofactorization $\Psi(\Omega C_M)$ is a compact generator for the homotopy category of $\mathsf{Tw}(C_M)$.

Proposition 3.1. *The homotopy category of $\mathsf{Tw}(C_M)$ is compactly generated by $\Psi(\Omega C_M)$. Moreover, $\Psi(\Omega C_M)$ is homotopy equivalent to an object in $\mathsf{Tw}^b(C_M)$.*

Proof. Previous discussions have already proved the first assertion. We only need to prove the second assertion. The idea is again to use the curved homological perturbation Lemma A.1.

By definition, the predifferential Q on $\Psi(\Omega C_M) := C \otimes^\tau \Omega C_M$ can be split into three parts defined by

$$\begin{aligned} d^+(x \otimes y) &:= x \otimes d^+(y); \\ d^-(x \otimes y) &:= x \otimes d^-(y); \\ d^\tau(x \otimes y) &:= x^{(1)} \otimes \tau(x^{(2)})y; \\ Q &:= d^+ + d^- + d^\tau. \end{aligned}$$

Consider the sum $\delta := d^- + d^\tau$ as a curved perturbation for the operator d^+. We can choose a k-linear special homotopy H (always exists over a field) between $(\wedge^*(V), 0)$ and $(\Omega C_M, d^+)$ such that H deceases the tensor degree (as d^+ increases it). Then extend it to a special homotopy between

$$(C \otimes \wedge^*(V), 0) \cong (C \otimes \Omega C_M, d^+)$$

by putting id on the C part. To see that the curved perturbation $\delta := d^- + d^\tau$ is small, note that d^- reduces the number of tensor components, d^τ reduces the degree of the C part and H reduces the number of tensor components. This allows us to apply the curved homological perturbation lemma (Lemma A.1) to $\Psi(\Omega C_M)$, which implies that $\Psi(\Omega C_M)$ is homotopy equivalent to a matrix cofactorization on $C \otimes \wedge^*(V)$. Thus, the proposition is proved. This matrix cofactorization obtained via perturbation will still be denoted by $\Psi(\Omega C_M)$. □

3.2 Relationship with Dyckerhoff's generator k^{stab}

In [Dyc11, § 2.3] Dyckerhoff defined a matrix factorization on $R \otimes \wedge^*(V^\vee)$ which he denoted by k^{stab}. The space k^{stab} is a super space with parity determined by the exterior degree. The matrix factorization on k^{stab} is defined by choosing a basis $x_1, \ldots, x_n$ of $V^\vee$, and write W in the form $\sum_{i=1}^n x_i W_i$. Denote the dual basis for V by $y_1, \ldots, y_n$. Then the matrix map $Q^\vee$ is defined by

$$Q^\vee(f \otimes \alpha) := x_i f \otimes \lrcorner y_i \alpha + W_i f \otimes x_i \wedge \alpha$$

where $\lrcorner y_i$ denotes the contraction operator and repeated indices are implicitly summed. The goal here is to compare Dyckerhoff's k^{stab} with $D\Psi(\Omega C_M)$ produced by Koszul duality (where D is the dualizing functor between cofactorizations and factorizations). The following proposition proves that they are homotopy equivalent objects. This can be viewed as a generalization of the classical fact that $D\Psi(\Omega(C))$ is homotopy equivalent to the Koszul complex of the residue field k.

Proposition 3.2. *With the notation introduced above we have a homotopy equivalence*

$$D(\Psi(\Omega C_M)) \cong k^{\mathsf{stab}}$$

between objects in $\mathsf{MF}(R, W)$.

Proof. Since the functor D is an equivalence of categories, we denote by $E := (E, Q)$ the matrix cofactorization whose dual is k^{stab}. As Ψ is a homotopy inverse to Φ, it is enough to prove that

$$\Phi \circ \Psi(\Omega C_M) \cong \Phi(E).$$

As shown in the proof of Theorem 2.5 the counit of the adjunction map

$$\Phi \circ \Psi(\Omega C_M) \xrightarrow{\epsilon_{\Omega C_M}} \Omega C_M$$

is a homotopy equivalence. Hence, it suffices to show that $\Phi(E)$ and ΩC_M is homotopy equivalent. The object $\Phi(E)$ as a vector space is given by $\Omega C_M \otimes C \otimes \wedge^*(V)$. Define a linear map α from ΩC_M to $\Phi(E)$ by

$$[f_1|\cdots|f_k] \mapsto [f_1|\cdots|f_k] \otimes 1 \otimes 1$$

where the middle 1 is the image of the coaugmentation map of $1 \in k$. The last 1 is the unit in $\wedge^*(V)$. The map α clear respects the left $\Omega(C_H)$-module structure. Moreover, it is a map of complexes as Q vanishes on $1 \otimes 1$ ($Q^\vee$ increase the polynomial degree on C, Q must decrease the degree). We use homological perturbation to show that α is a homotopy equivalence. Again we split the differential D on $\Phi(E)$ into several parts and use homological perturbation lemma. Explicitly for an element $a \otimes f \otimes y \in \Omega C_M \otimes C \otimes \wedge^*(V)$, the map D is the sum of the following four parts:

$$\begin{aligned} d_\Omega(a \otimes f \otimes y) &:= d_\Omega(a) \otimes f \otimes y; \\ d^\tau(a \otimes f \otimes y) &:= a\tau(f^{(1)}) \otimes f^{(2)} \otimes y; \\ Q^+(a \otimes f \otimes y) &:= a \otimes \frac{\partial f}{\partial y_i} \otimes y_i \wedge y; \\ Q^-(a \otimes f \otimes y) &:= a \otimes D_i(f) \otimes \lrcorner x_i \end{aligned}$$

where y_i as before is a basis for the vector space V. The map D_i is defined by

$$C \to C \otimes C \xrightarrow{D(W_i \otimes \mathsf{id})} k \otimes C = C.$$

The map Q^+ is simply the Koszul differential on $C \otimes \wedge^*(V)$. We consider the differential $d := d_\Omega + Q^+$ on the underlying vector space of $\Phi(E)$ and the other part $\delta := d^\tau + Q^-$ as perturbations of d. One can easily write down a special homotopy H for the Koszul differential Q^+ and extend it by id on ΩC_M to give a homotopy retraction data between ΩC_M and $(\Omega C_M \otimes C \otimes \wedge^*(V), d)$. The fact that the perturbation δ is small follows from the conilpotency property of C and that the curvature M vanishes on scalar and linear terms. Moreover, observe that both H and δ are ΩC_M-linear and

$$\delta \circ \alpha = 0,$$

which implies that the perturbed inclusion is still α and the perturbed differential is still d_Ω on ΩC_M by formulas in Appendix A. Hence, the proposition is proved. □

Remark. It follows from this proposition that the endomorphism dg algebra $\mathsf{End}(k^{\mathsf{stab}})$ is homotopy equivalent to ΩC_M. One can easily prove that the homology of ΩC_M is $\wedge^*(V)$ assuming that W vanishes on scalars and linear terms. The minimal model A_∞ algebras on $\mathsf{End}(k^{\mathsf{stab}})$ has been studied by Dyckerhoff and here we could use ΩC_M to obtain similar results.

3.3 Discussion on generating results

Recall that the dualizing operator D is an equivalence

$$D : \mathsf{Tw}^b(C_M) \to \mathsf{MF}(R, W)^{\mathsf{op}}.$$

Moreover, it was observed in [Dyc11, § 4.3] that there is another equivalence

$$\vee : \mathsf{MF}(R, W)^{\mathsf{op}} \to \mathsf{MF}(R, -W)$$

defined by $E^\vee := \mathsf{Hom}_R^{\mathbb{Z}/2\mathbb{Z}}(E, R)$. Moreover, since the Koszul complex k^{stab} is self-dual, we have $(k^{\mathsf{stab}})^\vee = k^{\mathsf{stab}}[\dim R]$ in $\mathsf{MF}(R, -W)$.

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Proposition 3.3. *The object k^{stab} weakly generates $[\mathsf{MF}(R,W)]$.*

Proof. The equivalence $\vee \circ D : \mathsf{Tw}^b(C_{-M}) \to \mathsf{MF}(R,W)$ suggests to use $-M$ as the curvature term. By Theorem 2.5, we know that $\Psi(\Omega C_{-M})$ weakly generates $\mathsf{Tw}(C_{-M})$. By Proposition 3.1, $\Psi(\Omega C_{-M})$ also weakly generates $\mathsf{Tw}^b(C_{-M})$. Hence, we conclude that $(D\Psi(\Omega C_{-M}))^\vee \cong (k^{\mathsf{stab}})^\vee = k^{\mathsf{stab}}[\dim R]$ (the isomorphism is by Proposition 3.2) weakly generates $[\mathsf{MF}(R,W)]$, which is equivalent to k^{stab} weakly generates. The proposition is proved. □

However, we do not know how to prove that k^{stab} classically generates $[\mathsf{MF}(R,W)]$ using homological methods. The problem here is that the subcategory $\mathsf{Tw}^b(C_{-M})$ might not be compact in $\mathsf{Tw}(C_{-M})$. Indeed we show that this is equivalent to the condition that the object k^{stab} classically generates $[\mathsf{MF}(R,W)]$. We need the following theorem (which can be found in [Nee92]) that characterizes compact objects.

Theorem 3.4. *Let $\mathscr{D}$ be a triangulated category with arbitrary coproduct. Moreover, assume that $\mathscr{D}$ is compactly generated by a set of compact objects $\mathscr{E}$. Then an object of $\mathscr{D}$ is compact if and only if it is a direct summand of an iterated extension of copies of objects of $\mathscr{E}$ shifted in both directions.*

Proposition 3.5. *The full subcategory $[\mathsf{Tw}^b(C_{-M})]$ of $[\mathsf{Tw}(C_{-M})]$ is compact if and only if k^{stab} classically generates $[\mathsf{MF}(R,W)]$.*

Proof. Assume that $[\mathsf{Tw}^b(C_{-M})]$ is a compact subcategory of $[\mathsf{Tw}(C_{-M})]$, i.e. every object of $[\mathsf{Tw}^b(C_{-M})]$ is compact, then it follows from Theorem 3.4 that the object $\Psi(\Omega C_{-M})$ in $[\mathsf{Tw}^b(C_{-M})]$ classically generates $[\mathsf{Tw}^b(C_{-M})]$ as it is a compact generator for $[\mathsf{Tw}(C_{-M})]$. Applying the equivalence functor $\vee \circ D$ implies that k^{stab} classically generates $[\mathsf{MF}(R,W)]$.

Conversely, if k^{stab} classically generates $[\mathsf{MF}(R,W)]$, using the equivalence $\vee \circ D$ we conclude that objects in $[\mathsf{Tw}^b(C_{-M})]$ can be obtained from $\Psi(\Omega C_{-M})$ by taking direct factors of iterated extensions and shifts. By Theorem 3.4 this implies that objects in $[\mathsf{Tw}^b(C_{-M})]$ are compact in $[\mathsf{Tw}(C_{-M})]$ as $\Psi(\Omega C_{-M})$ is a compact generator.

We will now show that the homological smoothness of the dg algebra ΩC_{-M} implies that the object k^{stab} classically generates $[\mathsf{MF}(R,W)]$. Recall that a dg algebra A is called homologically smooth if A considered as an $A \otimes A$-bimodule is a perfect object, i.e. it is a direct factor of finite rank free $A \otimes A$ dg module. □

Proposition 3.6. *If the dg algebra ΩC_{-M} is homologically smooth, then the full subcategory $[\mathsf{Tw}^b(C_{-M})]$ of $[\mathsf{Tw}(C_{-M})]$ is compact.*

Proof. A matrix cofactorization structure on $C \otimes V$ is equivalent to a ΩC_{-M} dg module structure on V. Hence, it suffices to show that any finite-dimensional dg ΩC_{-M}-module is compact in $\mathsf{Tw}(\Omega C_{-M})$. Homological smoothness implies the existence of resolution of diagonal by a perfect complex of $\Omega C_{-M} \otimes \Omega C_{-M}$-bimodules. Via integral transform it produces a resolution for any finite-dimensional dg module by a perfect complex of ΩC_{-M}-modules. Thus, the proposition is proved. □

The following proposition summarizes our discussion on generators.

Proposition 3.7. *The following are equivalent:*

(i) *ΩC_{-M} is homologically smooth;*
(ii) *$[\mathsf{Tw}^b(C_{-M})]$ is compact in $[\mathsf{Tw}(C_{-M})]$;*

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(iii) k^{stab} *classically generates* $\mathsf{MF}(R, W)$*;*
(iv) 0 *is an isolated singularity of* $W = 0$.

Proof. Indeed, we have seen (i) $\Rightarrow$ (ii) and (ii) $\Leftrightarrow$ (iii). The fact that (iii) $\Leftrightarrow$ (iv) is due to Murfet [KMB11, Appendix], and (iv) $\Rightarrow$ (i) is due to Dyckerhoff [Dyc11, § 7]. □

4. Hochschild invariants

As another application of Theorem 2.5 we show that one can calculate the Hochschild homology of $\mathsf{MF}(R_W)$ using the Borel–Moore Hochschild chain complex of the curved algebra R_W. The latter was introduced and explicitly computed in [CT13, § 4]. We assume that W has isolated singularities throughout this section.

4.1 Reducing to Hochschild homology of ΩC_M

As mentioned in § 2 the dg category $\mathsf{Tw}^b(R_W)$ is isomorphic as a dg category to $\mathsf{Tw}^b(C_M)^{op}$. Since the Hochschild homologies for opposite dg categories are isomorphic, we have

$$HH_*(\mathsf{Tw}^b(R_W)) \cong HH_*(\mathsf{Tw}^b(C_M)).$$

If W has isolated singularities, by Dyckerhoff's generating result and Proposition 3.5 it follows that $[\mathsf{Tw}^b(C_M)]$ is a compact subcategory of $[\mathsf{Tw}(C_M)]$ (see § 3). Thus, we have an inclusion of dg categories

$$\mathsf{Tw}^b(C_M) \hookrightarrow \mathsf{Tw}(C_M)^c.$$

Moreover, Theorem 3.4 implies that every compact object in $\mathsf{Tw}(C_M)$ is a direct factor of an object in $\mathsf{Tw}^b(C_M)$ as $\Psi(\Omega C_M) \in \mathsf{Tw}^b(C_M)$ compactly generates $[\mathsf{Tw}(C_M)]$ by Proposition 3.1. This implies the above inclusion of categories is an equivalence up to factors, which yields

$$HH_*(\mathsf{Tw}^b(C_M)) \cong HH_*(\mathsf{Tw}(C_M)^c)$$

by Keller's result [Kel99]. The right-hand side category $\mathsf{Tw}(C_M)^c$ is homotopy equivalent to the category $\mathsf{Tw}(\Omega C_M)^c$ via the coproduct preserving homotopy equivalences Φ and Ψ. As the Hochschild homology is also homotopy invariant, we conclude that

$$HH_*(\mathsf{Tw}(C_M)^c) \cong HH_*(\mathsf{Tw}(\Omega C_M)^c).$$

Finally the Hochschild homology of $\mathsf{Tw}(\Omega C_M)^c$ can be calculated by that of the dg algebra ΩC_M by Proposition 3.1. Combining all of these isomorphisms we have shown that

$$HH_*(\mathsf{MF}(R, W)) \cong HH_*(\Omega C_M).$$

In the following we relate the latter homology group with the Borel–Moore Hochschild homology of the curved algebra R_W.

4.2 Hochschild homology of C_M

We begin with the classical case where the curvature W is not presented. First we recall the Hochschild homology of a coalgebra C. Let C be a coalgebra with a coaugmentation, form the cobar algebra ΩC. The Hochschild chain complex $C_*(C)$ is by definition given by the complex

$$(\Omega C \otimes^\tau C \otimes^\tau \Omega C) \underset{\Omega C \otimes \Omega C}{\otimes} \Omega C.$$

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Here the superscript τ on tensor symbol is again to denote the twisted tensor product using the natural twisting cochain $\tau : C \to \Omega C$. Observe that $C_*(C)$ is simply $C \otimes \Omega C$ as a vector space, but the differential is twisted by the natural twisting cochain from C to ΩC. To simply the notation we use $C \tilde{\otimes} \Omega C$ to denote the Hochschild complex $C_*(C)$.

The advantage of this definition of the Hochschild complex for coalgebras is that it is quite simple to relate it to the Hochschild complex of its Koszul dual algebra ΩC. Indeed the latter complex is by definition given by

$$(\Omega C) \otimes^{\tau} B\Omega C \otimes^{\tau} \Omega C) \underset{\Omega C \otimes \Omega C}{\otimes} \Omega C.$$

Note that these two complexes only differ by the middle term where twisted tensor products are formed. The fact that they are quasi-isomorphic follows from the following classical lemma, see [LV12] for example.

Lemma 4.1. *Let $C_1 \xrightarrow{\tau_1} A$ be a twisting cochain between a dg coalgebra C_1 and an dg algebra A. Let $C_2 \xrightarrow{\gamma} C_1$ be a quasi-isomorphism of dg coalgebras. Then the composition τ_2*

$$C_2 \to C_1 \to A$$

is also a twisting cochain. Moreover, for any dg A-module F, the map defined by

$$C_2 \otimes^{\tau_2} F \xrightarrow{\gamma \otimes \mathsf{id}} C_1 \otimes^{\tau_1} F$$

is a quasi-isomorphism.

We apply the lemma to the unit morphism of the adjunction $\Omega \dashv B$

$$\eta_C : C \to B\Omega C$$

and the natural twisting cochain $C \to \mathbb{B}\Omega C \to \Omega C$. The fact the η_C is a quasi-isomorphism is well know for ordinary (dg) algebras (even non-curved A_∞ algebras). We end up with the following quasi-isomorphism between the two Hochschild complexes

$$C_*(C) := C \,\tilde{\otimes}\, \Omega C \xrightarrow{\eta_C \otimes \mathsf{id}} C_*(\Omega C) := B\Omega C \,\tilde{\otimes}\, \Omega C.$$

We can add the curvature term W (or M) into the previous discussion. All of the constructions explained above remain the same as we have already explained the twisting cochain and the twisted tensor products in the curved case in § 2. However, the proof of Lemma 4.1 does not generalize as the coalgebra $B\Omega C_M$ is curved with noncommutative coproduct. Hence, the differential does not square to zero in this case. It is even problematic to talk about the notion of quasi-isomorphism for these coalgebras. Nevertheless the map $\eta_C \otimes \mathsf{id}$ remains a quasi-isomorphism on the associated Hochschild complexes. This is proved in the following proposition.

Proposition 4.2. *The map $\eta_C \otimes \mathsf{id}$ is a quasi-isomorphism between the chain complexes $C_*(C_M)$ and $C_*(\Omega C_M)$.*

Proof. Observe the existence of a $\mathbb{Z}$-grading on the space $C_*(C_M)$ by the number of C tensor components. Define the following $\mathbb{Z}$-grading on the space $B\Omega C_M \otimes \Omega C_M$ by

$$\mathsf{deg}(f_1 \otimes \cdots \otimes f_k) := k \text{ for an element in } \Omega C_M;$$
$$\mathsf{deg}([\alpha_1|\cdots|\alpha_n] \otimes \beta) := \mathsf{deg}(\alpha_1) + \cdots + \mathsf{deg}(\alpha_n) + \mathsf{deg}(\beta) - n.$$

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Then one breaks the Hochschild differentials into two parts. The first part is simply the differential when the curvature is not presented. The second part is the differential defined by the curvature term M. For simplicity, we denote them by d^+ and d^-, respectively. (We will not bother to distinguish them on the two complexes as we will specify the complex when making statements.) Observe that the first differential increases the degrees defined above by one and the second differential decreases the degree by one. Hence, we have a morphism of mixed complexes

$$\eta_C \otimes \mathsf{id} : (C_M \tilde{\otimes} \Omega C_M, d^+, d^-) \to (B\Omega C_M \tilde{\otimes} \Omega C_M, d^+, d^-).$$

Through the associated bi-complex of these mixed complexes (for the construction of a bi-complex associated to a mixed complex, see for example [CT13, § 4.7]), we can conclude that the $\eta_C \otimes \mathsf{id}$ is a quasi-isomorphism as it is so on the E^1-page. The proof is complete. □

Remark. In the proof it is important that we are dealing with direct sum complexes and d^+ is degree increasing, because only in this case the spectral sequences under consideration starts with the differential d^+.

4.3 Relating to Borel–Moore Hochschild complex

To relate to the Borel–Moore Hochschild homology we dualize the Hochschild complex $C_*(C_M)$. There is a natural chain map from the Borel–Moore Hochschild chain complex $C_*^{\mathsf{BM}}(R_W)$ of R_W to $C_*(C_M)^\vee$ defined by

$$R \otimes R^+ \cdots R^+ \otimes R^+ \hookrightarrow (C \otimes C^+ \cdots C^+ \otimes C^+)^\vee.$$

This map is in fact a map between mixed complexes whose associated double complexes are isomorphic on the E^1-page. This fact follows from the classical Hochschild–Konstant–Rosenberg (HKR) theorem. Strictly speaking the HKR theorem applies only to the left-hand side, i.e. for the algebra R. However, for the right-hand side, the Hochschild complex of the coalgebra C, it suffice to observe that the Hochschild chain complex $C_*(C)$ is actually double graded by the tensor degree and the polynomial degree. Moreover, its graded k-linear dual agrees with the Hochschild chain complex of the symmetric algebra $\mathsf{sym}(V^\vee)$ to which we can apply HKR theorem. We summarize the main results obtained in the following theorem.

THEOREM 4.3. *We have the following isomorphisms:*

$$HH_*(\mathsf{MF}(R, W)) \cong HH_*(\mathsf{Tw}^b(C_M)) \cong HH_*(C_M);$$
$$HH_*^{\mathsf{BM}}(R_W) \cong HH_*(C_M)^\vee.$$

Remark. When W has isolated singularities, the vector space $HH_*(C_M)$ is finite dimensional. Moreover, on $HH_*(\mathsf{Tw}^b(R_W))$ there exists a natural non-degenerate pairing that identifies it with its dual space.

5. Equivariant matrix factorizations

In this section we study the orbifold version of Theorem 2.5 and its applications to categories of equivariant matrix factorizations. Throughout the section we work over the ground field $k = \mathbb{C}$ as we need to consider characters of groups.

5.1 Equivariant Koszul duality

Let $C := \mathbf{S}(V)$ to be the symmetric coalgebra over a vector space V and let $M : C \to k$ be a linear map on C that vanishes on scalar and linear terms. Consider a finite abelian group G

acting on C via coalgebra morphisms and that the action preserves the linear map M, i.e. the composition

$$C \xrightarrow{g} C \xrightarrow{M} k$$

is equal to M for any element $g \in G$. Given such data we would like to consider the dg category of equivariant twisted complexes over the curved coalgebra C_M. The objects are pairs (E, Q) where E is a cofree C-comodule with a G-action of the form

$$E := \bigoplus_i C \otimes \mathbb{C}_{\chi_i}.$$

Here $\mathbb{C}_{\chi_i}$ denotes the one-dimensional G-representation associated to a given character χ_i and we allow indices to repeat in the direct sum above. The linear map Q is a G-equivariant C-comodule morphism on E. Moreover, Q satisfies the matrix cofactorization identity. The morphism spaces between objects would be G-equivariant C-comodule maps. We denote this category by $\mathsf{Tw}([C_M/G])$ to mimic the orbifold notation. As before we denote by $\mathsf{Tw}^b([C_M/G])$ the full subcategory consisting of finite-rank objects. Since the cobar construction is functorial, we also have a G-action on the cobar algebra ΩC_M. Thus, the category $\mathsf{Tw}([\Omega C_M/G])$ can be defined in a similar way.

The Koszul duality functors Φ and Ψ are defined in the same way as before. Namely for an equivariant matrix cofactorization (E, Q) define

$$\Phi(E) := \Omega C_M \otimes^{\tau} E$$

where $\Phi(E)$ inherits the tensor product G-representation. One can check that the functors Φ and Ψ send equivariant objects to equivariant objects and equivariant morphisms to equivariant morphisms. Moreover, the homotopies constructed in the proof of Theorem 2.5 can be made G-equivariant by averaging if necessary. Thus, we have arrived at the following theorem.

Theorem 5.1. *The functors Φ and Ψ restricted to the equivariant categories to give a homotopy equivalence*

$$\mathsf{Tw}([C_M/G]) \cong \mathsf{Tw}([\Omega C_M/G]).$$

5.2 Smash product algebras

To make better use of the above Theorem 5.1, we first need to make a change of category. Namely we will switch from equivariant categories to categories of twisted complexes over a smash product algebra. More precisely since G acts on the curved coalgebra C_M in a way that preserves the curved coalgebra structure, we could form the smash product curved coalgebra $C_M \sharp G$. As a vector space it is $C \otimes k[G]$ and the coproduct is defined by

$$x \otimes g \mapsto \sum_{g_1 g_2 = g} (x^{(1)} \otimes g_1) \otimes (g_1^{-1}(x^{(2)}) \otimes g_2).$$

The curvature of $C_M \sharp G$ is defined by M on the component $C \otimes \mathsf{id}_G$ and zero otherwise. The dg category $\mathsf{Tw}(C_M \sharp G)$ is closely related to the equivariant dg category $\mathsf{Tw}([C_M/G])$. Observe that the smash product coalgebra $C_M \sharp G$ carry natural G-action and $C_M \sharp G$-linear maps are equivalent to C-linear maps that are also G-equivariant. Thus, the category $\mathsf{Tw}(C_M \sharp G)$ is a fully faithful subcategory of $\mathsf{Tw}([C_M/G])$ consists of objects that are free $\mathbb{C}_M \sharp G$-comodules. Conversely every objects of $\mathsf{Tw}([C_M/G])$ is a direct summand of an object in $\mathsf{Tw}(C_M \sharp G)$ through the fully faithful embedding. To see this observe that for any object $(E, Q) \in \mathsf{Tw}([C_M/G])$ form the object

$$g^*(E, Q) := \left(\bigoplus_{g \in G} g^* E, \bigoplus_{g \in G} g^* Q \right).$$

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One easily checks that $g^*(E, Q)$ is an object of $\mathsf{Tw}(C_M \sharp G)$. Such a relation between the two categories are called equivalence up to factors (from [Kel99]). If two categories are equivalent up to factors, then lots of properties of them are the same. For example, (classical) generators of the smaller category are also (classical) generators of the bigger one. It is also proved by Keller [Kel99] that the Hochschild type invariants are isomorphic for these two categories. Observe that Φ and Ψ restrict to a homotopy equivalence

$$\mathsf{Tw}(\Omega C_M \sharp G) \cong \mathsf{Tw}(C_M \sharp G).$$

As a conclusion we summarize the previous discussion in the following commutative diagram.

$$\begin{array}{ccc}
\mathsf{Tw}(\Omega C_M \sharp G) & \xrightarrow{\text{Koszul duality}} & \mathsf{Tw}(C_M \sharp G) \\
\Big\downarrow \text{inclusion} & & \Big\downarrow \text{inclusion} \\
\mathsf{Tw}([\Omega C_M / G]) & \xrightarrow{\text{Koszul duality}} & \mathsf{Tw}([C_M / G])
\end{array}$$

The vertical inclusions are all equivalences up to factors.

5.3 Applications to $\mathsf{MF}_G(R, W)$

The advantage of the smash product construction is that it is clear in this description the object $\Omega C_M \sharp G$ compactly generates the homotopy category of $\mathsf{Tw}(\Omega C_M \sharp G)$. Indeed for an object $F \in \mathsf{Tw}(\Omega C_M \sharp G)$ we have

$$\mathsf{Hom}_{\mathsf{Tw}(\Omega C_M \sharp G)}(\Omega C_M \sharp G, F) = \mathsf{Hom}_{\mathsf{Tw}(\Omega(C_M))}(\Omega C_M, F)$$

through the inclusion mentioned above. By Corollary 2.10 if the latter is acyclic, then the dg-module F is contractible over ΩC_M. Averaging the contracting homotopy yields a contraction over $\Omega C_M \sharp G$. Hence, arguing as in Corollary 2.10 shows that the object $\Omega C_M \sharp G$ compactly generates $[\mathsf{Tw}(\Omega C_M \sharp G)]$. As the categories $\mathsf{Tw}(\Omega C_M \sharp G)$ and $\mathsf{Tw}([\Omega C_M / G])$ are equivalent up to factors, the object $\Omega C_M \sharp G$ (through the inclusion functor) also compactly generates the homotopy category of the latter one.

Applying the Koszul duality functor Ψ yields compact generators for the homotopy category of $\mathsf{Tw}([C_M / G])$. Moreover, one can easily identify the generators by observing that the object $\Omega C_M \sharp G$ when considered as objects in $\mathsf{Tw}([\Omega C_M / G])$ is isomorphic to the direct sum

$$\bigoplus_{\chi} \Omega C_M \otimes \mathbb{C}_\chi$$

over the characters of G. Hence, its image under Ψ is the direct sum

$$\bigoplus_{\chi} \Psi(\Omega C_M) \otimes \mathbb{C}_\chi.$$

Observe that twisting by characters does not change the homology of $\Omega C_M \otimes \mathscr{C}_\chi$ and, hence, Propositions 3.1 and 3.2 still apply which assert that their k-linear duals are (homotopy equivalent to) matrix factorizations of the form

$$\{k^{\mathsf{stab}} \otimes \mathbb{C}_\chi \mid \chi \text{ is a character for the group } G\}.$$

Theorem 5.2. *Let notation be as above and assume that W has isolated singularities. Then the category $[\mathsf{MF}_G(R, W)]$ is classically generated by objects $k^{\mathsf{stab}} \otimes \mathbb{C}_\chi$.*

Proof. It is enough to show that the subcategory $[\mathsf{Tw}^b(C_M\sharp G)]$ is compact in $[\mathsf{Tw}(C_M\sharp G)]$ in view of Proposition 3.5. For this observe that taking cohomology commutes with taking G-invariants and hence for a finite-rank object E we have

$$\begin{aligned}\mathsf{Hom}_{[\mathsf{Tw}(C_M\sharp G)]}\Big(E,\bigoplus E_i\Big) &:= H^0\Big(\mathsf{Hom}_{\mathsf{Tw}(C_M\sharp G)}\Big(E,\bigoplus E_i\Big)\Big)\\ &= H^0\Big(\mathsf{Hom}_{\mathsf{Tw}(C_M)}\Big(E,\bigoplus E_i\Big)\Big)^G\\ &= \Big[\bigoplus H^0(\mathsf{Hom}_{\mathsf{Tw}(C_M)}(E,E_i))\Big]^G\\ &= \bigoplus \mathsf{Hom}_{[\mathsf{Tw}(C_M\sharp G)]}(E,E_i).\end{aligned}$$

Here we have used the fact that E is of finite rank and the group G is finite, which implies that E viewed as an object in $[\mathsf{Tw}(C_M)]$ is compact by Proposition 3.5. The theorem is proved. □

5.4 Equivariant Hochschild homology

The computation of Hochschild homology of $\mathsf{MF}_G(R,W)$ can be done in the same way as in § 4. Again we assume that W has isolated singularities throughout the discussion. We begin with an isomorphism

$$HH_*(\mathsf{MF}_G(R,W)) \cong HH_*(\mathsf{Tw}^b([C_M/G]))$$

as the two dg categories are opposite to each other by the k-linear dual functor D. Since the compact generators $\Psi(\Omega C_M)\otimes\mathbb{C}_\chi$ of $\mathsf{Tw}([C_M/G])$ lies inside $\mathsf{Tw}^b([C_M/G])$ which is compact under the assumption of W having isolated singularities, we have

$$HH_*(\mathsf{Tw}^b([C_M/G])) \cong HH_*(\mathsf{Tw}([C_M/G])^c) \cong HH_*(\mathsf{Tw}(C_M\sharp G)^c).$$

The latter isomorphism follows from the fact that the two categories are equivalence up to factors. Finally we invoke the Koszul duality of the curved coalgebra $C_M\sharp G$ which gives a homotopy equivalence

$$\mathsf{Tw}(C_M\sharp G)^c \cong \mathsf{Tw}(\Omega(C_M\sharp G))^c$$

between dg categories. From this homotopy equivalence and the fact that $\Omega C_M\sharp G$ is a compact generator, we conclude that

$$HH_*(\mathsf{Tw}(C_M\sharp G)^c) \cong HH_*(\mathsf{Tw}(\Omega(C_M\sharp G))^c) \cong HH_*(\Omega(C_M\sharp G)).$$

Combining the above isomorphisms yields the following isomorphism

$$HH_*(\mathsf{MF}_G(R,W)) \cong HH_*(\Omega(C_M\sharp G)).$$

Then the same proof as in § 4 implies the following proposition.

PROPOSITION 5.3. *Let the notation be as above and assume that W has isolated singularities. Then we have the following isomorphisms:*

$$\begin{aligned}HH_*(\mathsf{MF}_G(R,W)) &\cong HH_*(\mathsf{Tw}^b([C_M/G])) \cong HH_*(C_M\sharp G);\\ HH_*^{\mathsf{BM}}(R_W\sharp G) &\cong HH_*(C_M\sharp G)^\vee.\end{aligned}$$

Remark. The homology groups $HH_*^{\mathsf{BM}}(R_W\sharp G)$ are explicitly computed in [CT13, § 6] via certain localization formula for Borel–Moore homology groups.

6. Graded matrix factorizations

In this section, we study the category of graded matrix factorizations via Koszul duality. The main ideas remain the same as in the orbifold case. The results obtained are closely related to the work of Orlov [Orl09] (on the relationship between graded matrix factorizations and derived category of coherent sheaves) and Seidel [Sei] (on the A_∞ category of coherent sheaves on Calabi–Yau hypersurfaces). Throughout the section we work over the ground field $k = \mathbb{C}$.

6.1 Gradings

For a graded commutative ring S and a homogeneous curvature element $W \in R$ of degree d, one can define the dg category of graded matrix factorizations $\mathsf{MF}^{\mathsf{gr}}(R, W)$ (see [CT13, § 2] for a definition). As is explained in [CT13, § 2] this category is closely related to certain orbifold construction. We recall some relevant results below.

The symmetric algebra $S := \mathbf{S}(V^\vee)$ (non-complete) has a $\mathbb{Z}$-grading by the ordinary polynomial degrees. The polynomial degree of a homogeneous element $f \in S$ will be denoted by $|f|$. Consider $G := \mathbb{Z}/d\mathbb{Z}$ acting on S by

$$\hat{i}(f) := \zeta^{i|f|} f$$

for $\zeta := \mathsf{exp}(2\pi\sqrt{-1}/d)$, a dth root of unity. Clearly the G-action on S preserves the curvature element W. This implies that the G-action in fact acts on the curved algebra S_W. We can then form the smash product curved algebra $S_W \sharp G$. One theorem proved in [CT13, § 2.12] was the fact that graded matrix factorizations can be regarded as $\mathbb{Z}$-graded twisted complexes over $S_W \sharp G$. A subtle point there was that $S_W \sharp G$ does not form a $\mathbb{Z}$-graded curved algebra with the usual polynomial grading.

To fix this problem we need to introduce a new $\mathbb{Z}$-grading on $S_W \sharp G$. Note that the underlying vector space of $S_W \sharp G$ is $S \otimes k[G]$. The group algebra $k[G]$ has a special basis indexed by characters of G. Explicitly we denote by χ_i for $i \in [0, d-1]$, the characters of the group G. They act on G by

$$\chi_i(\hat{j}) := (\zeta_d)^{i \cdot j}.$$

Then the elements

$$U_\chi := \frac{1}{|G|} \sum_{g \in G} \chi(g) \sharp g$$

indexed by these characters form an orthogonal idempotent basis for the group algebra $k[G]$. Using this basis we can define a new $\mathbb{Z}$-grading on the vector space $S \otimes k[G]$. The homogeneous elements are of the form

$$f \otimes U_{\chi_j}$$

for some homogeneous polynomial $f \in S$. Define an integer $i \in [0, d-1]$ by

$$i \equiv j - |f| \pmod{d}.$$

Then the new grading of $f \otimes U_{\chi_j}$ is defined by

$$\mathsf{deg}(f \otimes U_{\chi_j}) := \frac{2}{d}(|f| - j + i).$$

We mention some important properties for this new $\mathbb{Z}$-grading on $S_W \sharp G$. First of all as promised the curvature term $W \sharp \mathsf{id}_G$ has degree two with respect to this grading. To see this observe that

$$W \sharp \mathsf{id}_G = \sum_{\chi_j} W \otimes U_{\chi_j}.$$

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Since $|W| = d$ we have $i = j$ and hence

$$\mathsf{deg}(W \otimes U_{\chi_j}) = \frac{2}{d} \cdot |W| = \frac{2}{d} \cdot d = 2.$$

Secondly the category of $\mathbb{Z}$-graded twisted complexes over $S_W \sharp G$ is closely related to the category of graded matrix factorizations. In fact, it was shown in [CT13, 2.12] that they are equivalent up to factors. (There we considered $S_W \sharp G$ as a category, then the twist construction would yields, in fact, an equivalence. Here we prefer to consider $S_W \sharp G$ as a curved algebra.) Namely there is an inclusion

$$\mathsf{Tw}^b_{\mathbb{Z}}(S_W \sharp G) \hookrightarrow \mathsf{MF}^{\mathsf{gr}}(S, W)$$

which is fully faithful and an equivalence up to factors.

6.2 Graded dualization

Next we dualize the $\mathbb{Z}$-graded curved algebra to consider a $\mathbb{Z}$-graded curved coalgebra $C_M \sharp G$ where C is the symmetric coalgebra $\mathbf{S}(V)$. We still denote the polynomial degree for a homogeneous $f \in \mathbf{S}(V)$ by $|f|$. A new $\mathbb{Z}$-grading on $C_M \sharp G$ is defined similarly. Namely homogeneous elements in $C_M \sharp G$ are of the form

$$f \otimes U_{\chi_j}$$

and the degree of it is given by

$$\mathsf{deg}(f \otimes U_{\chi_j}) := -\frac{2}{d}(|f| - j + i)$$

for the same i as in the case of algebras. With respect to this $\mathbb{Z}$-grading the map $M : C \to k$ has degree two. Hence, it forms a $\mathbb{Z}$-graded curved coalgebra. When forming the category $\mathsf{Tw}^b_{\mathbb{Z}}(C_M \sharp G)$ we do not want to allow arbitrary coalgebra maps but only the direct sums of the homogeneous ones. We introduce a notation to deal with such situations. Let E be a vector space with a $\mathbb{C}^*$-action, we denote by E^{gr} the vector space defined by

$$E^{\mathsf{gr}} := \bigoplus E_i,$$

where E_i is the subspace of E on which $\mathbb{C}^*$ acts by λ^i. With this notation we have

$$\mathsf{Hom}_{\mathsf{Tw}_{\mathbb{Z}}(C_M \sharp G)}(-,-) := [\mathsf{Hom}_{\mathsf{Tw}(C_M \sharp G)}(-,-)]^{\mathsf{gr}}.$$

Then the $\mathbb{Z}$-graded k-linear dual operation D defines an equivalence

$$\mathsf{Tw}^b_{\mathbb{Z}}(C_M \sharp G)^{\mathsf{op}} \cong \mathsf{Tw}^b_{\mathbb{Z}}(S_W \sharp G)$$

between dg categories.

6.3 $\mathbb{Z}$-graded curved Koszul duality

The next step is to understand the curved Koszul duality for the $\mathbb{Z}$-graded curved coalgebra $C_M \sharp G$. This is easily accomplished by matching the degrees. For this we define a $\mathbb{Z}$-grading on $\Omega C_M \sharp G$ that matches with the new $\mathbb{Z}$-grading on $C_M \sharp G$. The homogeneous elements in $\Omega C_M \sharp G$ are of the form

$$[f_1 | \cdots | f_k] \otimes U_{\chi_j}$$

for some character χ_j of the group G. Its degree is defined by

$$\mathsf{deg}([f_1 | \cdots | f_k] \otimes U_{\chi_j}) := -\frac{2}{d}\left(\sum_l |f_l| - j + i\right) + k,$$

where the integer $i \in [0, d-1]$ is defined by

$$i \equiv j - \sum_l |f_l| \pmod{d}.$$

Define the $\mathbb{Z}$-graded Koszul duality functors by (the same formula as before)

$$E \in Tw_{\mathbb{Z}}(C_M \sharp G) \overset{\Phi}{\mapsto} \Omega(C) \otimes^\tau E$$
$$F \in Tw_{\mathbb{Z}}(\Omega C_M \sharp G) \overset{\Psi}{\mapsto} C \otimes^\tau F.$$

The degrees on $\Phi(E)$ can be defined by

$$\mathsf{deg}([f_1|\cdots|f_k] \otimes f_0 \otimes U_{\chi_j}) := -\frac{2}{d}\left(\sum_{l=0}^k |f_l| - j + i\right) + k,$$

where the integer $i \in [0, d-1]$ is defined by

$$i \equiv j - \sum_{l=0}^k |f_l| \pmod{d}.$$

Similar one can also define degrees for $\Psi(F)$. With respect to these gradings the twisted differentials on $\Phi(E)$ or $\Psi(F)$ have degree one. Moreover, it is easy to see that Φ and Ψ are homotopy equivalences by observing that the homotopy equivalences used in the proof of Theorem 2.5 respect the new $\mathbb{Z}$-grading (the homotopies are of degree -1).

Theorem 6.1. *The functors Φ and Ψ are homotopy inverses between dg categories*

$$\mathsf{Tw}_{\mathbb{Z}}(\Omega C_M \sharp G) \cong \mathsf{Tw}_{\mathbb{Z}}(C_M \sharp G).$$

6.4 Applications to $\mathsf{MF}^{\mathsf{gr}}(S, W)$

We assume that W has isolated singularities from now on. One can argue in the same way as in the orbifold case that $\Omega C_M \sharp G$ compactly generates $[\mathsf{Tw}_{\mathbb{Z}}(\Omega C_M \sharp G)]$. Through the $\mathbb{Z}$-graded Koszul duality functor, $\Psi(\Omega C_M \sharp G)$ defines a compact generator for $[\mathsf{Tw}_{\mathbb{Z}}(C_M \sharp G)]$. The same proof as in § 3 shows that the object $\Psi(\Omega C_M \sharp G)$, in fact, is homotopy equivalent to an object in $\mathsf{Tw}^b_{\mathbb{Z}}(C_M \sharp G)$. Thus, its k-linear graded dual object in $\mathsf{MF}^{\mathsf{gr}}(S, W)$ makes sense. To identify this object we consider the natural forgetful functor from $\mathsf{Tw}_{\mathbb{Z}}(C_M \sharp G)$ to $\mathsf{Tw}(C_M \sharp G)$. Note that this is well-defined as the new $\mathbb{Z}$-grading on $C_M \sharp G$ is in $2\mathbb{Z}$ and, hence, its reduction modulo two reduces to the purely even grading on the curved coalgebra $C_M \sharp G$. Using the forgetful functor we see that as matrix factorizations the object $D\Psi(\Omega C_M \sharp G)$ is given by

$$\bigoplus_i k^{\mathsf{stab}} \otimes \chi_i.$$

Through the correspondence

$$\mathsf{Tw}^b_{\mathbb{Z}}(R_W \sharp G) \hookrightarrow \mathsf{MF}^{\mathsf{gr}}(S, W)$$

defined in [CT13, § 2], twisting by characters χ_j corresponds to twisting (j) of ordinary graded S-modules. Hence, if we assume any lifting of the $\mathbb{Z}$-grading on k^{stab}, we conclude that the object $D\Psi(\Omega C_M)$ in the category $\mathsf{MF}^{\mathsf{gr}}(S, W)$ given by the direct sum of the objects

$$k^{\mathsf{stab}}(d-1), k^{\mathsf{stab}}(d-2), \ldots, k^{\mathsf{stab}}.$$

Theorem 6.2. *Assume that W has isolated singularities, then the collection of objects $k^{\mathsf{stab}}(d-1), k^{\mathsf{stab}}(d-2), \ldots, k^{\mathsf{stab}}$ classically generates $[\mathsf{MF}^{\mathsf{gr}}(R, W)]$.*

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Proof. The theorem follows from the fact that the category $\mathsf{Tw}^b_{\mathbb{Z}}(C_M\sharp G)$ is compact in $\mathsf{Tw}_{\mathbb{Z}}(C_M\sharp G)$ which follows from the fact that taking cohomology of a differential of $\mathbb{Z}$-degree one (in particular, it is homogeneous) commutes with both taking G-invariants and the operation $- \mapsto -^{\mathsf{gr}}$. □

Remark. In the Calabi–Yau situation, i.e. when $\mathsf{dim}(S) = d = \mathsf{deg}(W)$, the category $[\mathsf{MF}^{\mathsf{gr}}(S,W)]$ is equivalent to the bounded derived category of coherent sheaves $\mathsf{D}^b_{\mathsf{coh}}(X)$ on $X := \mathsf{Proj}\, S/W$. Denote by $i : X \hookrightarrow \mathbf{P}^{d-1} := \mathsf{Proj}\, S$ the natural embedding of X into the projective space. Then the above collection of generators corresponds to the collection

$$i^*\omega_{\mathbf{P}^{d-1}}[d-1], i^*(\wedge^{d-2}\Omega_{\mathbf{P}^{d-1}})[d-2], \ldots, i^*\Omega_{\mathbf{P}^{d-1}}[1], \mathscr{O}_X$$

through a correspondence $\mathsf{D}^b_{\mathsf{coh}}(X) \cong [\mathsf{MF}^{\mathsf{gr}}(S,W)]$. This can be proved by observing that the degree shift in $[\mathsf{MF}^{\mathsf{gr}}(S,W)]$ corresponds to the composition of the homological degree shift functor and the Seidel–Thomas twist functor associated to the spherical object $\mathscr{O}_X$ on $\mathsf{D}^b_{\mathsf{coh}}(X)$, see [BFK12].

Remark. The homology of the dg algebra $\Omega C_M\sharp G$ is easily seen to be $\wedge^*(V)\sharp G$. This latter notation is slightly misleading because we did not mean the smash product algebra. It is simply the smash product vector space. The presence of the curvature term puts A_∞ structure on $\wedge^*(V)\sharp G$ via homotopy transfer property. However, this computation quickly gets complicated. The author has not been able to describe it even in the case of elliptic curves. We mention two closely related results in these directions. In an unpublished notes [Sei], Seidel has obtained the above picture for an A_∞ structure on $\wedge^*(V)\sharp G$ via quite different methods. Explicit calculations for A_∞ structures on elliptic curves have been obtained by Polishchuk in [Pol11], again through other methods. In latter case even the underlying vector space is different.

6.5 Hochschild homology of $\mathsf{MF}^{\mathsf{gr}}(S,W)$

The Hochschild homology of the dg category $\mathsf{MF}^{\mathsf{gr}}(S,W)$ can also be related with the Borel–Moore Hochschild homology of a curved algebra. The proof is the same as the orbifold case except that we use graded k-linear dualizing functor. We omit the proof here. The precise results are stated in the following proposition.

Proposition 6.3. *Let the notation be as above and assume that W has isolated singularities. Then we have the following isomorphisms:*

$$HH_*(\mathsf{MF}^{\mathsf{gr}}(S,W)) \cong HH_*(\mathsf{Tw}^b_{\mathbb{Z}}([C_M/G])) \cong HH_*(C_M\sharp G);$$
$$HH^{\mathsf{BM}}_*(S_W\sharp G) \cong HH_*(C_M\sharp G)^\vee,$$

where the $\vee$ denotes the graded dual operation.

Remark. Again the groups $HH^{\mathsf{BM}}_*(S_W\sharp G)$ has been computed in [CT13, § 6]. What is new here is the existence of a $\mathbb{Z}$-grading on these homology groups. In the Calabi–Yau situation, the dg version of Calabi–Yau/LG correspondence shows that this computation provides an alternative way to compute the Hochschild homology of Calabi–Yau hypersurfaces.

Acknowledgements

I would like to thank my advisor Andrei Căldăraru for his continuous support and valuable discussions as well as for reading the first manuscript of this work. I thank Tony Pantev for his encouragement and Bernhard Keller for answering several questions. I also thank Damien

Calaque for explaining Koszul duality and Paul Seidel for sharing his unpublished notes. Furthermore, I am thankful to Tobias Dyckerhoff and Daniel Pomerleano for pointing out a mistake in an earlier version of the paper and also for making interesting comments. Last but not least I thank Leonid Positselski for answering numerous questions I had in understanding his curved Koszul duality paper. The work is part of the author's thesis which was done in the University of Wisconsin, Madison.

Appendix A. Curved homological perturbation lemma

In this appendix we recall the homological perturbation technique as studied in [Cra04]. Then we prove that the homological perturbation lemma remains true when curvatures are presented. This is useful to study homotopy between precomplexes.

In this section we work with a k-linear abelian category $\mathscr{C}$. Our primary application concerns with $\mathscr{C}$ being the category of B-comodules for a coalgebra B over k.

A.1 Deformation retractions

Let (L, b) and (M, d) be two complexes over $\mathscr{C}$. A deformation retraction between them consists of the following data. There are morphisms

$$i : (L, b) \to (M, d) \quad \text{and} \quad p : (M, d) \to (L, b)$$

such that

$$p \circ i = \mathsf{id}_L.$$

Moreover, there is a homotopy H between $i \circ p$ and id_M, i.e. we have

$$i \circ p = \mathsf{id} + dH + Hd.$$

The triple (i, p, H) is then called a deformation retraction between (L, b) and (M, d). If, in addition, these maps also satisfy

$$Hi = 0, \quad pH = 0 \quad \text{and} \quad H^2 = 0, \tag{A.1}$$

then it is called a special homotopy retraction.

A.2 Perturbations

A perturbation of the complex (M, d) is an odd map $\delta : M \to M$ such that $(d+\delta)^2 = 0$. Following the terminologies in [Cra04], we call δ small if $(\mathsf{id} - \delta H)$ is invertible. For a small perturbation δ, define the operator

$$A := (\mathsf{id} - \delta H)^{-1}\delta$$

and define the perturbed homotopy retraction operators by

$$b_1 := b + pAi, \quad i_1 := i + HAi, \quad p_1 := p + pAH, \quad H_1 := H + HAH. \tag{A.2}$$

The homological perturbation lemma states that the data (i_1, p_1, H_1) defines a new special deformation retraction between the perturbed complexes (L, b_1) and $(M, d + \delta)$. This simple lemma plays an important role in the homotopy theory of algebras.

A.3 Curved homological perturbation lemma

Next we prove a curved version of the homological perturbation lemma. Namely we assume the same initial conditions for i, p, H. But for the perturbation, we do not assume that $(d+\delta)^2 = 0$.

Instead, we assume that

$$(d+\delta)^2 \text{ lies in the center of the algebra } \mathsf{End}(M).$$

We denote this central element by $F := (d+\delta)^2 \in \mathsf{End}(M)$ and call δ a curved perturbation.

The differential $d_1 := d+\delta$ no longer squares to zero but lies in the center of $\mathsf{End}(M)$. Such a pair (M, d_1) is called a precomplex. What curved homological perturbation achieves is the fact one can still obtain a deformation retract between precomplexes by perturbing ordinary complexes. The main result of this appendix is the following lemma.

Lemma A.1 (Curved homological perturbation lemma). *Let (i, p, H) a special homotopy retraction data between complexes (L, b) and (M, d). Let δ be a curved perturbation of (M, d). Then formula A.2 defines a new special homotopy retract between the precomplexes (L, b_1) and (M, d_1) in the following sense:*

(i) (L, b_1) *is a precomplex;*
(ii) $d_1 \circ i_1 = i_1 \circ b_1$ *(i_1 is a map of precomplexes);*
(iii) $b_1 \circ p_1 = p_1 \circ d_1$ *(p_1 is a map of precomplexes);*
(iv) $p_1 \circ i_1 = \mathsf{id}_L$ *and* $i_1 \circ p_1 = \mathsf{id}_M + d_1 H_1 + H_1 d_1$ *(homotopy retract);*
(v) $H_1 \circ i_1 = 0, p_1 \circ H_1 = 0$ *and* $H_1^2 = 0$ *(specialness).*

Proof. The proof is analogous to the proof of the ordinary perturbation lemma in [Cra04]. We basically only need to check the above formulas with a weaker condition that F is in the center (weaker as zero is in the center). We begin with the following lemma. □

Lemma A.2. *We have the following:*

(i) $\delta H A = A H \delta = A - \delta$;
(ii) $(\mathsf{id} - \delta H)^{-1} = \mathsf{id} + AH$ *and* $(\mathsf{id} - H\delta)^{-1} = \mathsf{id} + HA$;
(iii) $AipA + Ad + dA = F + FAH + FHA$.

Proof. The first two equations are direct computations and are the same as in [Cra04]. For the last one, we have

$$\begin{aligned}
AipA + Ad + dA &= A(\mathsf{id} + dH + Hd)A + Ad + dA \\
&= A^2 + AdHA + AHdA + Ad + dA \\
&= A^2 + Ad(HA + \mathsf{id}) + (AH + \mathsf{id})dA \\
&= A^2 + Ad(\mathsf{id} - H\delta)^{-1} + (\mathsf{id} - \delta H)^{-1} dA \\
&= (\mathsf{id} - \delta H)^{-1}[(\mathsf{id} - \delta H)A^2(\mathsf{id} - H\delta) \\
&\quad + (\mathsf{id} - \delta H)Ad + dA(\mathsf{id} - H\delta)](\mathsf{id} - H\delta)^{-1} \\
&= (\mathsf{id} - \delta H)^{-1}[\delta^2 + \delta d + d\delta](\mathsf{id} - H\delta) - 1 \\
&= F(\mathsf{id} - \delta H)^{-1}(\mathsf{id} - H\delta)^{-1} \\
&= F(\mathsf{id} + AH)(\mathsf{id} + HA) \\
&= F + FAH + FHA.
\end{aligned}$$

With these preparations, the proof of Lemma A.1 follows easily as an extension of the case without curvature. Let us first prove part (A):

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$$\begin{aligned}b_1^2 &= (b+pAi)(b+pAi)\\ &= bpAi + pAib + p(AipA)i\\ &= bpAi + pAib + p(F + FAH + FHA - Ad - dA)i\\ &= pFi + pFAHi + pFHAi\\ &= pFi + pFAHi + pHFAi \ (F \text{ is central})\\ &= pFi \ (\text{specialness}).\end{aligned}$$

Thus, b_1^2 is simply the restriction of F on its subspace L (via i and p). Hence, it is in the center of $\mathsf{End}(L)$, which proves that (L, b_1) is a precomplex. For part (B) we have

$$\begin{aligned}i_1b_1 - (d+\delta)i_1 &= (i+HAi)(b+pAi) - (d+\delta)(i+HAi)\\ &= ib + ipAi + HAib + H(AipA)i - di - dHAi - \delta i - \delta HAi\\ &= ipAi + HAib + H(F + FAH + FHA - dA - Ad)i\\ &\quad - dHAi - \delta i - (A-\delta)i\\ &= ipAi - HdAi - dHAi - Ai + HFi + HFAHi + HFHAi\\ &= (ip - Hd - dH - \mathsf{id})Ai + HFi + HFAHi + HFHAi\\ &= FHi + HFAHi + FHHAi\\ &= 0 \ (\text{by specialness and } F \text{ is central}).\end{aligned}$$

Similarly we check that p_1 is a map of precomplexes:

$$\begin{aligned}b_1p_1 - p_1(d+\delta) &= (b+pAi)(p+pAH) - (p+pAH)(d+\delta)\\ &= bpAH + pAip + p(AipA)H - p\delta - pAHd - p(AH\delta)\\ &= bpAH + pAip - p(Ad+dA)H + p(F + FAH + FHA)H\\ &\quad - p\delta - pAHd - p(A-\delta)\\ &= pAip - pAdH - pAHd - pA + pFH + pFAHH + pFHAH\\ &= pFH + pFAHH + pFHAH\\ &= 0 \ (\text{by specialness and } F \text{ is central}).\end{aligned}$$

This proves part (C). For part (D) we have

$$\begin{aligned}p_1i_1 &= (p+pAH)(i+HAi)\\ &= pi + pHAi + pAHi + pAHHAi\\ &= \mathsf{id}\,(\text{by specialness}).\end{aligned}$$

In the reversed direction, we have

$$\begin{aligned}\mathsf{id} + H_1d_1 + d_1H_1 - i_1p_1 &= \mathsf{id} + (H+HAH)(d+\delta)\\ &\quad + (d+\delta)(H+HAH) - (i+HAi)(p+pAH)\\ &= H\delta + HAHd + H(AH\delta) + \delta H + dHAd + (\delta HA)H\\ &\quad - ipAH - HAip - H(AipA)H\\ &= H\delta + HAHd + H(A-\delta) + \delta H + dHAd + (A-\delta)H\\ &\quad - ipAH - HAip + H(Ad+dA)H - H(F + FAH + FHA)H\\ &= HA(Hd + \mathsf{id} + dH - ip) + (dH + \mathsf{id} - ip + Hd)AH\\ &\quad - HFH - HFAHH - HFHAH\\ &= 0 \ (\text{again by specialness and } F \text{ is central}).\end{aligned}$$

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Thus, we have shown that (i_1, p_1, H_1) forms a deformation retraction. It still remains to prove part (E). This is again a computation:

$$\begin{aligned} H_1 \circ i_1 &= (H + HAH)(i + HAi) \\ &= Hi + HHAi + HAHi + HAHHAi \\ &= 0; \\ p_1 \circ H_1 &= (p + pAH)(H + HAH) \\ &= pH + pHAH + pAHH + pAHHAH \\ &= 0; \\ H_1 \circ H_1 &= (H + HAH)(H + HAH) \\ &= HH + HAHH + HHAH + HAHHAH \\ &= 0. \end{aligned}$$

Thus, the lemma is proved. □

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