Preface

Two of the most important and useful inequalities in the theory of differential equations are the Hardy inequality

$$\left|\frac{p-n}{p}\right|^p \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p}\,dx \leq \int_{\mathbb{R}^n} |\nabla u(x)|^p dx\; (n \in \mathbb{N},\, 1 < p < \infty)$$

and that of Rellich:

$$\left(\frac{n(p-1)(n-2p)}{p^2}\right)^p \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{2p}}\,dx \leq \int_{\mathbb{R}^n} |\Delta u(x)|^p\, dx (n \in \mathbb{N},\, n > 2p),$$

each inequality holding for an appropriate class of functions u. Details of the long history and wide applications of the Hardy inequality are given in [15] and [110] (see also [108]); note that the special case $p = 2$ of it can be regarded as a mathematical representation of Heisenberg's uncertainty principle. As for the Rellich inequality, we refer to [15] for background information and observe that the case $p = 2$ is again distinguished in that it has implications for the self-adjointness problem for Schrödinger operators with singular potentials.

Motivated by the demands of various applications (see, for example, Chapter 1 of [40]), fractional versions of these results have been obtained in recent times. For example, when $s \in (0, 1)$, a fractional analogue of the Hardy inequality takes the form

$$c(n, s, p) \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{ps}}\,dx \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy$$

for all u in a class of functions depending on whether $ps - n$ is positive or negative. There are important applications for versions of the classical and fractional inequalities in which integration occurs over an open subset Ω of $\mathbb{R}^n$; these present interesting and challenging problems involving the geometry of Ω. The Rellich inequality has also enjoyed substantial development in the last few years.

The natural setting for these later inequalities is that of fractional Sobolev spaces which, after their introduction in the 1950s by Aronszajn, Gagliardo and Slobodeckij, have found applications in a vast number of questions involving differential equations and nonlocal effects: see, for example, the references given in [142]. Details of the historical background are given in [171]. Our objective in this book is to present an introduction to such spaces and to go on to establish inequalities such as those mentioned above.

Chapter 1 is devoted to topics that are mainly quite familiar, and are given here for the convenience of the reader and also to establish some standard notation. There follows a brief account of classical Sobolev spaces, including some of the basic embedding theorems. In Chapter 3 fractional Sobolev spaces on an open subset Ω of $\mathbb{R}^n$ are introduced: if $s \in (0, 1)$ and $p \in [1, \infty)$, such a space is

$$W_p^s(\Omega) := \left\{ u \in L_p(\Omega) \colon\ (x, y) \longmapsto \frac{|u(x) - u(y)|}{|x - y|^{\frac{n}{p}+s}} \in L_p(\Omega \times \Omega) \right\},$$

and is endowed with the norm

$$\|u\|_{s,p,\Omega} := \left(\int_\Omega |u|^p\, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, dx\, dy \right)^{1/p}.$$

The term

$$[u]_{s,p,\Omega} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, dx\, dy \right)^{1/p}$$

featured here is the so-called Gagliardo seminorm; it plays the role occupied by the L_p norm of the gradient in the classical first-order Sobolev space. Note that the condition $s \in (0, 1)$ is essential if triviality is to be avoided, for when $s \geq 1$ and Ω are connected, then by Proposition 2 of [33] the only measurable functions such that

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, dx\, dy < \infty$$

are constants. When the underlying space Ω is bounded and has sufficiently smooth boundary these fractional spaces can be identified with certain Besov spaces. This enables such matters as embedding theorems to be taken over from the theory of Besov spaces. However, the loss of control of constants (in embedding inequalities, for example) and the conditions imposed on the boundary make it desirable to supplement the general impression given by the Besov space approach by arguments based on the concrete definition of the fractional spaces. These spaces not only have similarities to those of classical Sobolev type but also exhibit differences: for example, the Poincaré inequality holds for all bounded open subsets of $\mathbb{R}^n$ in the fractional case but not in the classical situation. Details of the behaviour of the Gagliardo seminorm as $s \to 1-$ and $s \to 0+$ (see [24], [33], [34] and [134]) are presented together with an indication of an alternative approach given by Milman et al. ([136], [107]) using interpolation theory. For additional illustration of the role that interpolation can play in fractional spaces see [31]. The chapter concludes with a brief discussion of fractional powers of the Laplacian (in connection with which we mention [112]) and associated eigenvalue problems. Chapter 4 provides a brief look at eigenvalue problems set in fractional spaces: we describe some of the most fundamental results relating to the fractional Laplacian and fractional p-Laplacian in the hope that those unfamiliar with this material will find it as fascinating as we do.

Chapter 5 presents a survey of results on Hardy inequalities in the context of classical Sobolev spaces typified by the inequality

$$\int_\Omega |\nabla u(x)|^p dx \geq C(p, \Omega) \int_\Omega \frac{|u(x)|^p}{\delta(x)^p}\, dx, \;\; u \in C_0^1(\Omega),$$

where Ω is an open set in $\mathbb{R}^n, n \geq 1,\; p \in (0, \infty),\; \delta(x) = \inf\{|x-y| : y \in \mathbb{R}^n \setminus \Omega\}$, and $C(p, \Omega)$ is a positive constant which depends on p and Ω but not on u. Properties of the distance function δ depend on the geometry of Ω and its boundary, and these are important features of the inequality. Hardy inequalities have always attracted a good deal of interest, but the volume of high-quality work in this area does seem to have grown dramatically in this century. The literature is now so enormous as to make the selection of results to include in this book difficult; the choice is inevitably personal and some significant results are bound to have been omitted. Although it is not possible in these pages to give proofs of all the results that we do mention, precise references are provided when proofs are not. We also present the inequality of Laptev and Weidl on $\mathbb{R}^2 \setminus \{0\}$ in which the gradient ∇ of Hardy's inequality is replaced by the magnetic gradient $\nabla + \nu\mathbf{A}$, where $\mathbf{A}$ is a magnetic potential of Aharonov–Bohm type. Discrete versions of the resulting inequality are also discussed. The Hardy theme is continued in Chapter 6 in the fractional setting.

Finally, in Chapter 7 the focus is on the Rellich inequality. To set the scene, results obtained in classical Sobolev spaces are surveyed and then some recent developments of a fractional nature are presented.

Chapters are divided into sections and sections are sometimes divided into subsections. Theorems, Corollaries, Lemmas, Propositions, Remarks and equations are numbered consecutively. At the end of the book there are author, subject and notation indexes.