Bitangents of plane quartics

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The topic dealt with in this paper arose out of a study of theta
functions and the moduli of curves. It concerns itself with
configurations of lines in the plane and when they can be
bitangents to a quartic. The techniques used are classical.

1. Bitangents

In the paper [6] we showed that points of V_3 with no zero
coordinates belong to the strong closure of the locus of non-hyperelliptic
curves of genus three. Our aim here is to prove that each such point of
V_3 always corresponds to a non-hyperelliptic curve. The method of
associating a curve to such a point of V_3 was well defined in [6], but it
was not shown that the curve is necessarily non-singular.

We recall that the associated plane quartic had been studied by
Riemann and that he had written down formulae for the 28 bitangents in
the case that the curve is non-singular. Cayley, in [4], p. 74, showed
that in the reverse direction these 28 lines are always bitangents to
that quartic curve. In order to prove that the quartic is non-singular it
will suffice to demonstrate that these 28 bitangents are distinct, and
this shall be our approach.

We begin with a point of V_3 with no zero coordinates and associate
to it the moduli $\alpha_i, \alpha_i', \alpha_i''$. Then take the seven lines

$$t_1 = 0 \ , \quad t_2 = 0 \ , \quad t_3 = 0 \ , \quad t_1 + t_2 + t_3 = 0 \ ,$$

$$\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_3 = 0 \ , \quad \alpha_1' t_1 + \alpha_2' t_2 + \alpha_3' t_3 = 0 \ , \quad \alpha_1'' t_1 + \alpha_2'' t_2 + \alpha_3'' t_3 = 0 \ .$$

It is proved in [5], p. 4.25, that these seven lines are distinct. Dualising, look at the seven points $0_1, \ldots, 0_7$ dual to these lines. Then the projective model of the net of cubics through these seven points will give a double plane (see [8], Chapter VII). Call the jacobian curve of the net J^6 and B^4 its quartic image. The construction of Aronhold (see [1] or [7]) determines the quartic we are investigating, which we shall call Q^4. It can be shown that Q^4 and B^4 are both birationally equivalent to J^6 and hence are projectively equivalent.

Therefore we only need to show that the 28 bitangents of B^4 are distinct. The bitangents of B^4 are the images of cubics in the plane containing J^6. These cubics are either of the form

(a)   the line $0_i 0_j$ and the conic through the remaining 5 points, or

(b)   the unique cubic of the net with a double point at 0_i .

We shall call these (ij) and $(i8)$, respectively.

Using this labelling of the bitangents, there are several general cases to look at. Firstly, if either $(12) = (34)$ or $(56) = (78)$, we can conclude that $0_5, 0_6, 0_7$ are collinear. This will contradict the fact that the original point P of V_3 has no zero coordinates. Secondly, if $(67) = (78)$ we conclude that $0_1, 0_2, \ldots, 0_6$ lie on a conic. This contradicts the formulae in Weber [9] which express the moduli α_i'' in terms of the α, α'. Finally, if $(16) = (17)$ or $(68) = (78)$, we conclude that $0_6 = 0_7$ but, as stated above, our seven points are distinct. This completes the proof that the point P corresponds to a non-hyperelliptic curve.

2. An alternative approach

The proof that the associated curve is non-singular can be approached in another way. The Plücker formulae show that if a quartic has one ordinary double point, then there are only 16 proper bitangents. The other 12 coincide in six pairs as lines passing through the double point and touching the curve elsewhere. If there are more, or more complicated, singularities then there are further coincidences. In this section we will prove that the quartic generated by a point of V_3 (with no zero coordinates) can not even have a single ordinary double point. We will restrict the exposition to a mere outline of the ideas involved.

We use the well-known correspondence between the plane quartic with its 28 bitangents and the cubic surface with its 27 lines. In addition, some results of Cayley will be used (see [2], p. 372; [3], p. 316). Fixing a bitangent l_0 , its stabiliser in $S_6(2)$ is the same as the group of the 27 lines. Moreover, two lines on the surface intersect if and only if the sum of their characteristics with that of l_0 is odd.

When a cubic surface acquires a node in a degeneration, 12 of the 27 lines coincide in pairs. Cayley ([2], [3]) has proved that these 12 lines form a double-sixer. It can also be proved that if c is any even characteristic, then two lines l_i, l_j are a pair of corresponding lines in the double-sixer belonging to c if and only if char l_i + char $l_j = c$ - char l_0 . This implies that the six pairs of degenerate bitangents passing through the ordinary double point have the property that for each pair, the sum of characteristics of the lines is the same. Hence we need to take each non-zero characteristic (even or odd) and write it in six ways as a sum of two odd characteristics (this always being possible in precisely six ways). In each case we need only take one pair of lines and check that the lines are distinct. Many cases drop out by symmetry and, in the end, the desired result can be achieved without too much hard work.

References

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