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THE EIGENVALUES AND SINGULAR VALUES OF MATRIX SUMS AND PRODUCTS. VII [1]

BY
R. C. THOMPSON AND S. THERIANOS

1. **Introduction.** This paper is a continuation of the two series, Eigenvalues of Sums I, II, III [2], [3], [4] and Singular Values of Products I, II, III [5], [6], [7]. Let A, B, $S=A+B$ be n-square Hermitian matrices with eigenvalues

$$\alpha_1 \geq \cdots \geq \alpha_n, \qquad \beta_1 \geq \cdots \geq \beta_n, \qquad \sigma_1 \geq \cdots \geq \sigma_n, \tag{1}$$

respectively. A number of inequalities of the form

$$\sigma_{k_1}+\cdots+\sigma_{k_r} \leq \alpha_{i_1}+\cdots+\alpha_{i_r}+\beta_{j_1}+\cdots+\beta_{j_r} \tag{2}$$

are known [1], [2], [3] linking the eigenvalues of A and B to those of S. Here the integers $i_1, \ldots, i_r, j_1, \ldots, j_r, k_1, \ldots, k_r$ satisfy

$$\begin{aligned} &1 \leq i_1 < \cdots < i_r \leq n,\\ &1 \leq j_1 < \cdots < j_r \leq n,\\ &1 \leq k_1 < \cdots < k_r \leq n. \end{aligned} \tag{3}$$

In order for (2) to be valid for all Hermitian A, B, additional conditions on $i_1, \ldots, i_r, j_1, \ldots, j_r, k_1, \ldots, k_r$ have to be imposed. The exact nature of these conditions is at the present time an unsolved problem. One might hope to deduce these conditions from a consideration of the simplest case, that in which A and B are diagonal. However, Zwahlen [8] has defeated this hope by observing that if $n=18$ the inequality

$$\sigma_2+\sigma_9+\sigma_{18} \leq \alpha_1+\alpha_6+\alpha_{11}+\beta_1+\beta_6+\beta_{11} \tag{4}$$

is valid for all diagonal Hermitian A and B, and is not valid for all Hermitian A and B. A natural problem therefore is: characterize those values of n for which this phenomenon occurs. This we shall do.

Now let A and B be arbitrary (i.e., not necessarily Hermitian) n-square matrices, and let

$$a_1 \geq \cdots \geq a_n, \qquad b_1 \geq \cdots \geq b_n, \qquad p_1 \geq \cdots \geq p_n \tag{5}$$

be the singular values of A, B, $P=AB$ respectively. For each known inequality of the form (2) linking the eigenvalues of sums of Hermitian matrices, there is a

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known inequality [5], [7] of the form

$$p_{k_1}\cdots p_{k_r} \le a_{i_1}\cdots a_{i_r} b_{j_1}\cdots b_{j_r} \tag{6}$$

linking the singular values of matrix products; and conversely. It is therefore worthwhile to ask whether multiplicative inequalities exist which are valid in diagonal situations but are otherwise invalid. We shall answer this question below in the affirmative and determine those n for which this can happen.

2. The main results.

Notation. A capital letter (e.g., B) will denote a matrix, a corresponding lower case Latin or Greek letter (e.g., b or β) will denote its singular values or eigenvalues, respectively.

THEOREM 1. *Let A, B, $S=A+B$ be n-square Hermitian matrices with eigenvalues* (1). *If $n\ge 6$, the inequality*

$$\sigma_2+\sigma_3+\sigma_6 \le \alpha_1+\alpha_3+\alpha_5+\beta_1+\beta_3+\beta_5 \tag{7}$$

is valid whenever A, B are diagonal Hermitian matrices and is not valid for all Hermitian A, B. For $n\le 5$, each inequality of the form (2) *which is valid for all Hermitian diagonal A, B is also valid for all Hermitian A, B.*

THEOREM 2. *Let A, B be arbitrary n-square matrices and let $P=AB$. Let the singular values of A, B, P be* (5). *If $n\ge 6$ the inequality*

$$p_2p_3p_6 \le a_1a_3a_5b_1b_3b_5 \tag{8}$$

is valid whenever A, B are diagonal, but is not valid for all A, B. For $n\le 5$, each inequality of the form (6) *which is valid for all diagonal A, B is valid for all A, B.*

Proof of Theorem 1. We first show that if inequality (7) is valid for all 6×6 Hermitian diagonal matrices A, B, then it is valid for all $n\times n$ Hermitian diagonal matrices A, B when $n\ge 6$. Let A, B be n-square diagonal Hermitian matrices and let $\tilde{A}$, $\tilde{B}$, $\tilde{S}=\tilde{A}+\tilde{B}$ be the principal 6×6 submatrices of A, B, $S=A+B$, respectively, formed by choosing the six rows and columns passing through the diagonal elements σ_1, σ_2, σ_3, σ_4, σ_5, σ_6 of S. Let $\tilde{\alpha}_1\ge\cdots\ge\tilde{\alpha}_6$, $\tilde{\beta}_1\ge\cdots\ge\tilde{\beta}_6$ be the eigenvalues of $\tilde{A}$, $\tilde{B}$ respectively. By a well-known property of principal submatrices of Hermitian matrices, we have

$$\tilde{\alpha}_t \le \alpha_t, \qquad \tilde{\beta}_t \le \beta_t, \qquad t=1,\ldots,6. \tag{9}$$

From the correctness of (7) for 6-square matrices and (9) we get

$$\begin{aligned}\sigma_2+\sigma_3+\sigma_6 &\le \tilde{\alpha}_1+\tilde{\alpha}_3+\tilde{\alpha}_5+\tilde{\beta}_1+\tilde{\beta}_3+\tilde{\beta}_5\\ &\le \alpha_1+\alpha_3+\alpha_5+\beta_1+\beta_3+\beta_5.\end{aligned}$$

Thus if (7) is valid for 6×6 diagonal Hermitian matrices then it is valid for $n\times n$ diagonal Hermitian matrices for all $n\ge 6$.

Now let A, B, $S=A+B$ be 6×6 real diagonal matrices. We shall show that the hypothesis

$$\sigma_2+\sigma_3+\sigma_6 > \alpha_1+\alpha_3+\alpha_5+\beta_1+\beta_3+\beta_5 \tag{10}$$

leads to a contradiction. For suppose real diagonal matrices A, B, $S=A+B$ satisfy (10). If we increase α_2, α_4, α_6, β_2, β_4, β_6 so as to achieve

$$\alpha_1=\alpha_2,\ \alpha_3=\alpha_4,\ \alpha_5=\alpha_6,\ \beta_1=\beta_2,\ \beta_3=\beta_4,\ \beta_5=\beta_6 \tag{11}$$

then the right-hand side of (10) is unchanged and the left-hand side does not decrease. (Compare Zwahlen [8, p. 100].) So we may assume (11) is valid. If we had $\sigma_6=\alpha_5+\beta_5$ then (10) would become

$$\sigma_2+\sigma_3 > \alpha_1+\alpha_3+\beta_1+\beta_3. \tag{12}$$

However, the inequality $\sigma_2+\sigma_3\leq\alpha_1+\alpha_3+\beta_1+\beta_3$ is known to be valid [1, Theorem 8]. Therefore

$$\sigma_6 \neq \alpha_5+\beta_5. \tag{13}$$

Let $\alpha=\alpha_1=\alpha_2$, $\alpha'=\alpha_3=\alpha_4$, $\alpha''=\alpha_5=\alpha_6$, $\beta=\beta_1=\beta_2$, $\beta'=\beta_3=\beta_4$, $\beta''=\beta_5=\beta_6$. Without loss of generality we may take $A=\operatorname{diag}(\alpha, \alpha, \alpha', \alpha', \alpha'', \alpha'')$. Then without loss of generality and because of (13), we may assume B to be one of the following ten matrices:

$$\begin{array}{ll}
\operatorname{diag}(\beta'', \beta'', \beta', \beta', \beta, \beta), & \operatorname{diag}(\beta'', \beta'', \beta', \beta, \beta', \beta),\\
\operatorname{diag}(\beta'', \beta'', \beta, \beta, \beta', \beta'), & \operatorname{diag}(\beta'', \beta', \beta'', \beta', \beta, \beta),\\
\operatorname{diag}(\beta'', \beta', \beta'', \beta, \beta', \beta), & \operatorname{diag}(\beta'', \beta, \beta'', \beta', \beta', \beta),\\
\operatorname{diag}(\beta'', \beta, \beta'', \beta, \beta', \beta'), & \operatorname{diag}(\beta', \beta', \beta'', \beta'', \beta, \beta),\\
\operatorname{diag}(\beta', \beta, \beta'', \beta'', \beta', \beta), & \operatorname{diag}(\beta, \beta, \beta'', \beta'', \beta', \beta').
\end{array}$$

In each of these cases the discussion is further subdivided into subcases according to the possible different orderings of the diagonal elements of $S=A+B$. Straightforward computation in every subcase yields a contradiction to (10).

Therefore (7) is established for all Hermitian diagonal $n\times n$ matrices with $n\geq 6$.

Now let A, B, be the following $n\times n$ matrices ($n\geq 6$) and let $S=A+B$. (These matrices are adapted from matrices in Zwahlen [8, p. 101].)

$$A=\begin{bmatrix} 10 & 0 & 0 & 0\\ 0 & 10 & 0 & 0\\ 0 & 0 & .9 & .3\\ 0 & 0 & .3 & .1\end{bmatrix}\dotplus \operatorname{diag}(1, 0)\dotplus 0_{n-6},$$

$$B=\begin{bmatrix} 9 & 0 & 0 & 3\\ 0 & 9.1 & 2.7 & 0\\ 0 & 2.7 & 1.9 & 0\\ 3 & 0 & 0 & 1\end{bmatrix}\dotplus \operatorname{diag}(0, 1)\dotplus 0_{n-6}.$$

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(Here 0_x is the x-square zero matrix.) Then $\alpha_1=\alpha_2=\beta_1=\beta_2=10$, $\alpha_3=\alpha_4=\beta_3=\beta_4=1$, $\alpha_t=\beta_t=0$ for all $t\quad 5$. Let $f(\lambda)$ be the characteristic polynomial of the leading 4×4 block in S, and let $\sigma_1'\geq\sigma_2'\geq\sigma_3'\geq\sigma_4'$ be the roots of $f(\lambda)$. Since

$$f(0) > 0,\quad f(1) < 0,\quad f(19) > 0,\quad f(19.5) < 0,\quad f(20) > 0,$$

we have $20>\sigma_1'\geq\sigma_2'\geq\sigma_3'>1>\sigma_4'>0$. Thus $\sigma_1=\sigma_1'$, $\sigma_2=\sigma_2'$, $\sigma_3=\sigma_3'$, $\sigma_4=\sigma_5=1$, $\sigma_6=\sigma_4'$, $\sigma_t=0$ for $t>6$. By traces we get $\sigma_2'+\sigma_3'+\sigma_4'=42-\sigma_1'$. The inequality (7) now amounts to

$$\sigma_1'\geq 20,$$

which is false. Hence (7) is invalid for this choice of A, B.

Thus we have proved that (7) is valid in the diagonal case and not valid in general.

Let $n\leq 5$. To complete the proof, suppose (2) is valid for all diagonal $n\times n$ matrices. We wish to show that (2) is valid for all $n\times n$ matrices. This is trivial if $r=n$. If $r=3$ or 4, the trace condition in combination with (2) produces an inequality which when applied to $-S=(-A)+(-B)$ yields an inequality equivalent to (2) and which has the form of (2) with $r\leq 2$. This equivalence is valid in each of the two cases: (i) diagonal Hermitian matrices; (ii) arbitrary Hermitian matrices. However, it is known [1, Theorems 7, 8 and p. 238], [8, Theorems 1, 2] that any inequality of the form (2) with $r=1$ or $r=2$ which holds for all $n\times n$ diagonal Hermitian matrices with nonnegative entries in fact holds for all $n\times n$ Hermitian matrices. One may see this by noticing that from the validity of

$$\sigma_{k_1}\leq\alpha_{i_1}+\beta_{j_1}\quad\text{or of}\quad\sigma_{k_1}+\sigma_{k_2}\leq\alpha_{i_1}+\alpha_{i_2}+\beta_{j_1}+\beta_{j_2}\tag{14}$$

(i.e. (2) when $r=1$ or 2) for all real diagonal matrices it follows that

$$\begin{array}{lll} i_1+j_1\leq k_1+1 & \text{or} & i_1+j_1\leq k_1+1,\\ & & i_1+j_2\leq k_2+1,\quad i_2+j_1\leq k_2+1,\\ & & i_1+i_2+j_1+j_2\leq k_1+k_2+3. \end{array}\tag{15}$$

(See the examples $G1$ and $G2$ in [8]. For later use, notice that if we add scalar matrices or subtract scalar matrices from the matrices in $G1$, $G2$, we obtain positive definite or negative definite diagonal matrices which can be used to deduce (15) from (14).) However, it is known that (14) is valid for all Hermitian $S=A+B$ if (15) is satisfied. Thus if $n\leq 5$ and if (2) is valid for all diagonal n-square Hermitian matrices A, B, $S=A+B$, then (2) is valid for arbitrary n-square Hermitian A, B, $S=A+B$. (This implication is valid even if (2) is only assumed to be true for positive definite diagonal matrices or only for negative definite diagonal matrices.)

This completes the proof of Theorem 1.

Before proving Theorem 2, we prove

THEOREM 3. *Let integers* $i_1, \ldots, i_r, j_1, \ldots, j_r, k_1, \ldots, k_r$ *satisfy* (3). *Then each of the following assertions* (*i*), (*ii*) *implies the other:*

(i) *Let* A, B, $P=AB$ *denote n-square matrices with singular values* $a_1 \geq \cdots \geq a_n$, $b_1 \geq \cdots \geq b_n$, $p_1 \geq \cdots \geq p_n$ *respectively. Then the inequality*

$$p_{k_1} \cdots p_{k_r} \leq a_{i_1} \cdots a_{i_r} b_{j_1} \cdots b_{j_r} \tag{6}$$

is valid for all A, B.

(ii) *Let* A *and* B *denote positive semidefinite matrices with eigenvalues* $\alpha_1 \geq \cdots \geq \alpha_n$, $\beta_1 \geq \cdots \geq \beta_n$. *Let* $P=AB$ *and let the eigenvalues of* P *be* $\pi_1 \geq \cdots \geq \pi_n$. *Then the inequality*

$$\pi_{k_1} \cdots \pi_{k_r} \leq \alpha_{i_1} \cdots \alpha_{i_r} \beta_{j_1} \cdots \beta_{j_r} \tag{16}$$

is valid for all positive semidefinite A, B.

Proof. (i)⇒(ii). Let A and B be positive semidefinite. Let $A=X^*X$ and $B=YY^*$. The eigenvalues of $P=AB=X^*XYY^*$ are the same as the eigenvalues of $XYY^*X^*=(XY)(XY)^*$; that is, the singular values of XY are the square roots of the eigenvalues of P. Also the singular values of X (respectively, Y) are the square roots of the eigenvalues of A (respectively, B). Applying (i) to X, Y, XY, we obtain (ii).

(ii)⇒(i). Let A, B, $P=AB$ be n-square matrices with singular values (5). The singular values of P are the square roots of the eigenvalues of $PP^*=ABB^*A^*$. Thus the squares of the singular values of P are the eigenvalues of the product $(A^*A)(BB^*)$ of positive semidefinite matrices. Applying (ii) to A^*A, BB^*, $(A^*A)(BB^*)$ we get (i).

Proof of Theorem 2. If $M=\operatorname{diag}(m_1, \ldots, m_n)$ is a diagonal matrix with positive diagonal elements, let $\log M=\operatorname{diag}(\log m_1, \ldots, \log m_n)$. For any diagonal matrix M let

$$|M| = \operatorname{diag}(|m_1|, \ldots, |m_n|).$$

We now prove that (8) holds for the singular values of A, B, $P=AB$ when A, B are diagonal. First assume A and B nonsingular. From $P=AB$ we get $|P|=|A|\,|B|$, hence $\log|P|=\log|A|+\log|B|$. Applying Theorem 1 to the diagonal matrices $\log|A|$, $\log|B|$, $\log|P|$ we get

$$\log p_2+\log p_3+\log p_6 \leq \log a_1+\log a_3+\log a_5+\log b_1+\log b_3+\log b_5.$$

This gives $p_2p_3p_6 \leq a_1a_3a_5b_1b_3b_5$. Thus (8) holds. The general case of (8) for (perhaps singular) diagonal matrices follows from the nonsingular case by continuity. Thus (8) is valid for diagonal matrices.

We next demonstrate the existence of matrices for which (8) is false. Let $n\geq 6$, and set

$$X=\begin{bmatrix}10 & 0 & 0 & 0\\ 0 & 10 & 0 & 0\\ 0 & 0 & 1.9 & .3\\ 0 & 0 & .3 & 1.1\end{bmatrix}\dotplus \operatorname{diag}(1,2)\dotplus I_{n-6},$$

$$Y=\begin{bmatrix}9.1 & 0 & 0 & 2.7\\ 0 & 9.2 & 2.4 & 0\\ 0 & 2.4 & 2.8 & 0\\ 2.7 & 0 & 0 & 1.9\end{bmatrix}\dotplus \operatorname{diag}(2,1)\dotplus I_{n-6}.$$

Let $Z=XY$. The eigenvalues of X are $10, 10, 2, 2, 1, 1, 1, 1, \ldots, 1$. The eigenvalues of Y are also $10, 10, 2, 2, 1, 1, 1, 1, \ldots, 1$. Let $f(\lambda)$ be the characteristic polynomial of the leading 4×4 block in Z, and let $\rho_1\geq\rho_2\geq\rho_3\geq\rho_4$ be the eigenvalues of this block. By direct computation we find

$$f(1)>0, f(2)<0, f(10)>0, f(93)<0, f(100)>0.$$

Thus

$$100>\rho_1\geq\rho_2>10>\rho_3>2>\rho_4>1.$$

The eigenvalues of Z in decreasing order are therefore $\rho_1, \rho_2, \rho_3, 2, 2, \rho_4, 1, 1, 1, \ldots, 1$. Moreover, by determinants, $\rho_1\rho_2\rho_3\rho_4=10^4 2^2$. The matrices X, Y are positive semidefinite, so let $X=A^*A$, $Y=BB^*$ and let $P=AB$. Then, as in the proof of Theorem 3, the singular values of A, B, P are

$$10^{1/2}, 10^{1/2}, 2^{1/2}, 2^{1/2}, 1, 1, 1, 1, \ldots, 1;$$

$$10^{1/2}, 10^{1/2}, 2^{1/2}, 2^{1/2}, 1, 1, 1, 1, \ldots, 1;$$

$$\rho_1^{1/2}, \rho_2^{1/2}, \rho_3^{1/2}, 2^{1/2}, 2^{1/2}, \rho_4^{1/2}, 1, 1, \ldots, 1.$$

Thus (8) becomes $\rho_2^{1/2}\rho_3^{1/2}\rho_4^{1/2}\leq 20$, which is equivalent to $\rho_1\geq 100$, a contradiction.

Consequently the singular value inequality (8) is not true of all n-square matrices, $n\geq 6$.

Suppose now that $n\leq 5$ and that (6) is valid for the singular values of the product $P=AB$ of all $n\times n$ digonal matrices with positive diagonal entries. We wish to show that (6) must hold for all $n\times n$ A, B, $P=AB$. This is obvious if $r=n$. Let $r<n$. By considering $\det P=\det A\det B$ and by a passage to P^{-1}, A^{-1}, B^{-1}, we obtain an equivalent inequality (6) with $r\leq 2$. This equivalence holds in both cases: (i) A, B diagonal; (ii) arbitrary A, B. Then $\log P=\log A+\log B$. Hence

$$\log p_{k_1}+\cdots+\log p_{k_r}\leq \log a_{i_1}+\cdots+\log a_{i_r}+\log b_{j_1}+\cdots+\log b_{j_r}.$$

Since an arbitrary real diagonal matrix can be represented as $\log A$, from our knowledge of linear inequalities for the eigenvalues of the sum of diagonal matrices, we deduce (15). However, when (15) is satisfied the inequalities

$$p_{k_1}\leq a_{i_1}b_{j_1},\qquad p_{k_1}p_{k_2}\leq a_{i_1}a_{i_2}b_{j_1}b_{j_2}$$

are known [5], [6], [7] to be valid for all matrices (including diagonal matrices). This means: if $n \leq 5$, an inequality (6) valid for diagonal products is valid for all products.

This completes the proof of Theorem 2.

3. **Additional results.** From the proofs of Theorems 1 and 2 it follows that any inequality (2) (or (6)) with $r=1$ or 2 which holds for eigenvalues of sums (or singular values of products) of diagonal matrices must hold for nondiagonal matrices as well. For $r \geq 3$ this is no longer true. We already know this for $r=3$. For $r>3$ we have:

THEOREM 4. *Let $r>3$ and let $n \geq r+7$. Then both the inequality*

$$\sigma_2+\sigma_3+\sigma_6+\sum_{t=1}^{r-3}\sigma_{10+t} \leq \alpha_1+\alpha_3+\alpha_5+\sum_{t=1}^{r-3}\alpha_{5+t}+\beta_1+\beta_3+\beta_5+\sum_{t=1}^{r-3}\beta_{5+t} \tag{17}$$

(for the eigenvalues of the sum $S=A+B$ of n-square Hermitian matrices) and the inequality

$$p_2p_3p_6\prod_{t=1}^{r-3}p_{10+t} \leq a_1a_3a_5b_1b_3b_5\prod_{t=1}^{r-3}a_{5+t}b_{5+t} \tag{18}$$

(for the singular values of the product $P=AB$ of arbitrary n-square matrices) have the following property: validity for all diagonal cases, but nonvalidity in general.

Proof. We first give the proof for sums. We know that

$$\sigma_2+\sigma_3+\sigma_6 \leq \alpha_1+\alpha_3+\alpha_5+\beta_1+\beta_3+\beta_5 \tag{7}$$

is true for all diagonal Hermitian matrices. The inequality [2, Theorem 1]

$$\sum_{t=1}^{r-3}\sigma_{10+t} \leq \sum_{t=1}^{r-3}\alpha_{5+t}+\sum_{t=1}^{r-3}\beta_{5+t} \tag{19}$$

is valid for all Hermitian matrices. Adding (7) and (13) we see that (17) is valid for all diagonal Hermitian matrices. However, the inequality (17) is false for the matrices appearing in the proof of Theorem 1.

The proof in the multiplicative case is exactly the same, except that in place of (19) we use [5, Theorem 1] the inequality

$$\prod_{t=1}^{r-3}p_{10+t} \leq \prod_{t=1}^{r-3}a_{5+t}\prod_{t=1}^{r-3}b_{5+t}.$$

This completes the proof.

4. **Singular values of sums and eigenvalues of products of positive semidefinite matrices.**

THEOREM 5. *Let A, B, $S=A+B$ be (not necessarily Hermitian) n-square matrices with singular values $a_1 \geq \cdots \geq a_n$, $b_1 \geq \cdots \geq b_n$, $s_1 \geq \cdots \geq s_n$. Let $n \geq 6$. Then the inequality*

$$s_2+s_3+s_6 \leq a_1+a_3+a_5+b_1+b_3+b_5 \tag{20}$$

is valid whenever A, B are diagonal, but is not valid in general. When $n \leq 5$, each inequality of the form

$$s_{k_1}+\cdots+s_{k_r} \leq a_{i_1}+\cdots+a_{i_r}+b_{j_1}+\cdots+b_{j_r} \tag{21}$$

which is valid for all diagonal A, B is valid for all A, B.

Proof. For diagonal A, B, $S=A+B$ we have

$$\begin{aligned} s_2(S)+s_3(S)+s_6(S) &= \sigma_2(|S|)+\sigma_3(|S|)+\sigma_6(|S|) \\ &\leq \sigma_2(|A|+|B|)+\sigma_3(|A|+|B|)+\sigma_6(|A|+|B|) \\ &\leq \alpha_1(|A|)+\alpha_3(|A|)+\alpha_5(|A|)+\beta_1(|B|)+\beta_3(|\ \ |)+\beta_5(|B|) \\ &= a_1(A)+a_3(A)+a_5(A)+b_1(B)+b_3(B)+b_5(B). \end{aligned}$$

Thus (20) is valid for the singular values of diagonal matrices. To show that (20) is not valid for the singular values of nondiagonal A, B, $S=A+B$, use the example in the proof of Theorem 1. In the example the matrices are positive semidefinite so that their eigenvalues and singular values coincide.

Let $n \leq 5$. Suppose (21) is valid for the singular values of all $n \times n$ diagonal matrices A, B, $S=A+B$. We are going to show that (2) is valid for the eigenvalues of all $n \times n$ Hermitian matrices A, B, $S=A+B$. First let A, B be diagonal Hermitian matrices. For sufficiently large t the (diagonal) matrices $A+tI$, $B+tI$, $S+2tI$ are positive definite and so their singular values are their eigenvalues. By (21) applied to $S+2tI=(A+tI)+(B+tI)$ we see (2) holds for the eigenvalues of the diagonal Hermitian matrices A, B, $S=A+B$. Hence, by Theorem 1, the inequality (2) holds for the eigenvalues of all $n \times n$ Hermitian matrices A, B, $S=A+B$.

Since (2) is true for $n \times n$ Hermitian matrices, it is true [1, Theorem 4] for the eigenvalues of the sum of $(2n)\times(2n)$ Hermitian matrices. If A, B, $S=A+B$ are (not necessarily Hermitian) matrices, an application of this fact to

$$\begin{bmatrix} 0 & S \\ S^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$$

shows that (21) is true for the singular values of A, B, $S=A+B$.

THEOREM 6. *Let A, B be positive semidefinite n-square matrices with $P=AB$. Let $\alpha_1 \geq \cdots \geq \alpha_n$, $\beta_1 \geq \cdots \geq \beta_n$, $\pi_1 \geq \cdots \geq \pi_n$ be the eigenvalues of A, B, P. Let $n \geq 6$. Then the inequality*

$$\pi_2\pi_3\pi_6 \leq \alpha_1\alpha_3\alpha_5\beta_1\beta_3\beta_5$$

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is valid whenever A, B are diagonal, but is not valid in general. For $n \leq 5$, each inequality of the form

$$\pi_{k_1} \cdots \pi_{k_r} \leq \alpha_{i_1} \cdots \alpha_{i_r} \beta_{j_1} \cdots \beta_{j_r}$$

which is valid for all diagonal A, B is valid for all A, B.

Proof. This was established in the proof of Theorem 2.

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University of California,
Santa Barbara, California