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# ON THE LOW FREQUENCY ASYMPTOTICS FOR THE 2-D ELECTROMAGNETIC TRANSMISSION PROBLEM

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**Abstract**

We examine the transmission problem in a two-dimensional domain, which consists of two different homogeneous media. We use boundary integral equation methods on the Maxwell equations governing the two media and we study the behaviour of the solution as the two different wave numbers tend to zero. We prove that as the boundary data of the general transmission problem converge uniformly to the boundary data of the corresponding electrostatic transmission problem, the general solution converges uniformly to the electrostatic one, provided we consider compact subsets of the domains.

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## 1. Introduction

Low frequency boundary-value problems for acoustics, electromagnetism and linear elasticity have already been considered by many researchers. The case of the three-dimensional problem is presented in the books by Dassios and Kleinman [5] and Colton and Kress [4]. Transmission boundary-value problems in three dimensions have been considered by Kress and Roach [7] in acoustics, and Wilde [10] in electromagnetics who proved the uniqueness of the solution.

The essential characteristic of the two-dimensional case, as compared to the three-dimensional one, is that the fundamental solution, which is the Hankel function of first kind and order zero, tends to infinity as the wave number $k$ tends to zero. The low frequency behaviour of the solution for the exterior boundary-value problem, in

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two dimensions, is presented in the works of Werner [9] and Kress [6]. These authors considered a suitable combination of a single and double layer potential and proved that the solution of the Helmholtz equation tends to the solution of the Laplace equation as $k$ tends to zero. This idea has been adopted by the present authors in an exterior boundary-value problem for the vector Helmholtz equation [2]. The transmission problem for the Helmholtz equation in two dimensions is studied in [1]. There, the case of different wave numbers is investigated, and the uniqueness of the solution is proved under the assumption that the relevant parameters satisfy a suitable condition.

This paper is organised as follows. In the next section we recall some basic facts about the two-dimensional electromagnetic problem. Then, in Section 3, we focus on the electromagnetic transmission problem in $\mathbb{R}^2$ for two different regions, the unbounded which is lossless and the bounded one, and establish the relevant formulation. Finally, in the last section, we prove that the solution of the general transmission boundary-value problem converges uniformly to the solution of the corresponding electrostatic transmission problem as the relevant wave numbers of the two domains both tend to zero. This is true provided that the boundary data of the general problem converge uniformly to the boundary data of the corresponding electrostatic transmission problem, and we have considered compact subsets of the domains.

## 2. Basics of the two-dimensional electromagnetic problem

Let $D_i$ be a bounded open region in $\mathbb{R}^2$. We denote the exterior domain by $D_e := \mathbb{R}^2 \backslash \overline{D}_i$, which is connected, and the boundary by $\partial D$, belonging to the class $C^2$. Let $C^{0,a}(\partial D)$, $0 < a \leq 1$, be the space of uniformly Hölder continuous functions defined on $\partial D$ and the unit normal vector to the boundary, $\boldsymbol{\nu}$, directed into the exterior $D_e$. We also define the normed subspaces [4]

$$\mathscr{T}^{0,\alpha}(\partial D) = \left\{\boldsymbol{\alpha} : \partial D \to \boldsymbol{C}^2 \mid \boldsymbol{\alpha} \cdot \boldsymbol{\nu} = 0,\ \boldsymbol{\alpha} \in C^{0,\alpha}(\partial D)\right\}$$

of uniformly Hölder continuous tangential fields and

$$\mathscr{S}^{0,\alpha}(\partial D) = \left\{\boldsymbol{\alpha} \in \mathscr{T}^{0,\alpha}(\partial D) \mid \operatorname{Div} \boldsymbol{\alpha} \in C^{0,\alpha}(\partial D)\right\}$$

with Hölder continuous tangential surface divergence fields, where $\operatorname{Div} \alpha$ is the surface divergence of a continuous tangential field, as defined in [4, page 60], and norms

$$\|\boldsymbol{\alpha}\|_{\mathscr{T}^{0,\alpha}} = \|\boldsymbol{\alpha}\|_{C^{0,\alpha}} \quad \text{and} \quad \|\boldsymbol{\alpha}\|_{\mathscr{S}^{0,\alpha}} = \|\boldsymbol{\alpha}\|_{C^{0,\alpha}} + \|\nabla \cdot \boldsymbol{\alpha}\|_{C^{0,\alpha}}.$$

We consider electromagnetic wave propagation in a homogeneous isotropic medium in $\mathbb{R}^2$ with angular frequency $\omega > 0$, which will be described by the electric and magnetic fields

$$\boldsymbol{E}(\boldsymbol{r}, t) = \left(\varepsilon + \frac{i\sigma}{\omega}\right)^{-1/2} \boldsymbol{E}(\boldsymbol{r})e^{-i\omega t} \quad \text{and} \quad \boldsymbol{H}(\boldsymbol{r}, t) = \mu^{-1/2}\boldsymbol{H}(\boldsymbol{r})e^{-i\omega t},$$

where $\sigma$ is the electric conductivity, while the electric permittivity $\varepsilon$ and the magnetic permeability $\mu$ are real positive constants.

The time-dependent Maxwell's equations

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}, t) + \mu\frac{\partial}{\partial t}\boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{0} \quad \text{and}$$
$$\nabla \times \boldsymbol{H}(\boldsymbol{r}, t) - \varepsilon\frac{\partial}{\partial t}\boldsymbol{E}(\boldsymbol{r}, t) = \sigma\boldsymbol{E}(\boldsymbol{r}, t)$$

lead us to the time-reduced Maxwell's equations

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) - ik\boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{0} \quad \text{and} \quad \nabla \times \boldsymbol{H}(\boldsymbol{r}) + ik\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{0}, \tag{2.1}$$

where the wave number $k$ is given by $k^2 = \varepsilon\mu\omega^2 + i\mu\sigma\omega$ and we choose the sign of the wave number such that $\operatorname{Im} k \geq 0$.

If $\boldsymbol{E}$ and $\boldsymbol{H}$ satisfy (2.1), then it has been proved in [4] that they also satisfy the vector Helmholtz equations

$$\Delta\boldsymbol{E}(\boldsymbol{r}) + k^2\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{0} \quad \text{and} \quad \Delta\boldsymbol{H}(\boldsymbol{r}) + k^2\boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{0},$$

and that they are divergence free, that is,

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = 0 \quad \text{and} \quad \nabla \cdot \boldsymbol{H}(\boldsymbol{r}) = 0.$$

Since $\boldsymbol{E}$, $\boldsymbol{H}$ satisfy the same vector Helmholtz equation, with the same wave number, respectively, in what follows we study the asymptotic behaviour of $\boldsymbol{E}$, as $k, k_i \to 0$.

## 3. The two-dimensional electromagnetic transmission problem

In order to distinguish the constitutive parameters and the field quantities in the two different media $D_e$ and $D_i$, we introduce subscripts $e$ or $i$ respectively. Since the medium in the unbounded domain $D_e$ is assumed lossless ($\sigma_e = 0$), we omit the subscript in the wave number $k_e = k > 0$,

$$k^2 = \omega^2\varepsilon_e\mu_e.$$

By contrast, in $D_i$ the wave number $k_i$ is given by $k_i^2 = \varepsilon_i\mu_i\omega^2 + i\mu_i\sigma_i\omega$, with $\sigma_i \geq 0$.

Then the vector Helmholtz equations describing the situation in the two domains $D_e$ and $D_i$ appear as

$$\Delta \boldsymbol{E}_k(\boldsymbol{r}) + k^2 \boldsymbol{E}_k(\boldsymbol{r}) = \boldsymbol{0}, \quad \text{in } D_e \quad \text{and} \tag{3.1}$$

$$\Delta \boldsymbol{F}_{k_i}(\boldsymbol{r}) + k_i^2 \boldsymbol{F}_{k_i}(\boldsymbol{r}) = \boldsymbol{0}, \quad \text{in } D_i. \tag{3.2}$$

In order to study the transmission problem, the following boundary conditions must be satisfied [8]:

$$\boldsymbol{\nu}(\boldsymbol{r}) \times \boldsymbol{E}_k(\boldsymbol{r}) - \boldsymbol{\nu}(\boldsymbol{r}) \times \boldsymbol{F}_{k_i}(\boldsymbol{r}) = \boldsymbol{c}_k(\boldsymbol{r}) \quad \text{and} \tag{3.3}$$

$$\frac{1}{\mu_e}\boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_{\boldsymbol{r}} \times \boldsymbol{E}_k(\boldsymbol{r}) - \frac{1}{\mu_i}\boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_{\boldsymbol{r}} \times \boldsymbol{F}_{k_i}(\boldsymbol{r}) = \boldsymbol{d}_k(\boldsymbol{r}), \tag{3.4}$$

where $\boldsymbol{r} \in \partial D$, while $\boldsymbol{c}_k, \boldsymbol{d}_k \in \mathscr{S}^{0,\alpha}$ are given tangential fields. Moreover, the electric field in $D_e$ must satisfy the Silver-Müller radiation condition:

$$\boldsymbol{E}_k(\boldsymbol{r}) \times \frac{\boldsymbol{r}}{|\boldsymbol{r}|} - ik\boldsymbol{E}_k(\boldsymbol{r}) = o\left(\frac{1}{\sqrt{|\boldsymbol{r}|}}\right), \quad |\boldsymbol{r}| \to +\infty \tag{3.5}$$

uniformly over all directions $\boldsymbol{r}/|\boldsymbol{r}|$. The index $k$ denotes the dependence on the wave number $k > 0$, since $\sigma_e = 0$ for a perfect dielectric.

When $k = k_i = 0$ we have the corresponding electrostatic transmission problem for Maxwell's equations, that is, to find a solution $\boldsymbol{E}_0 \in C^2(D_e) \cap C(\overline{D}_e)$ and $\boldsymbol{F}_0 \in C^2(D_i) \cap C(\overline{D}_i)$ of

$$\nabla \cdot \boldsymbol{E}_0(\boldsymbol{r}) = 0, \quad \nabla \times \boldsymbol{E}_0(\boldsymbol{r}) = \boldsymbol{0} \quad \text{in } D_e \quad \text{and} \tag{3.6}$$

$$\nabla \cdot \boldsymbol{F}_0(\boldsymbol{r}) = 0, \quad \nabla \times \boldsymbol{F}_0(\boldsymbol{r}) = \boldsymbol{0} \quad \text{in } D_i \tag{3.7}$$

satisfying the boundary conditions

$$\boldsymbol{\nu}(\boldsymbol{r}) \times \boldsymbol{E}_0(\boldsymbol{r}) - \boldsymbol{\nu}(\boldsymbol{r}) \times \boldsymbol{F}_0(\boldsymbol{r}) = \boldsymbol{c}_0(\boldsymbol{r}) \quad \text{and} \tag{3.8}$$

$$\frac{1}{\mu_e}\boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_{\boldsymbol{r}} \times \boldsymbol{E}_0(\boldsymbol{r}) - \frac{1}{\mu_i}\boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_{\boldsymbol{r}} \times \boldsymbol{F}_0(\boldsymbol{r}) = \boldsymbol{d}_0(\boldsymbol{r}) \tag{3.9}$$

where $\boldsymbol{c}_0, \boldsymbol{d}_0 \in \mathscr{S}^{0,\alpha}$ are given tangential fields and at infinity

$$\boldsymbol{E}_0(\boldsymbol{r}) = O(1), \quad |\boldsymbol{r}| \to +\infty \tag{3.10}$$

uniformly over all directions $\boldsymbol{r}/|\boldsymbol{r}|$.

## 4. Low frequency asymptotics

Let

$$\Phi_k(\boldsymbol{r}, \boldsymbol{r}') = \frac{i}{4} H_0^{(1)}(k|\boldsymbol{r} - \boldsymbol{r}'|), \quad \boldsymbol{r} \neq \boldsymbol{r}', \ k \neq 0$$

and

$$\Phi_0(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{2\pi} \ln \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}, \quad \boldsymbol{r} \neq \boldsymbol{r}'$$

denote the fundamental solutions to the Helmholtz and the Laplace equation for the two-dimensional case, respectively. Using the asymptotic behaviour of Hankel's function $H_0^{(1)}$ of order zero and of first kind

$$H_0^{(1)}(z) = \frac{2i}{\pi} \left[ \ln \frac{|z|}{2} + c - \frac{\pi i}{2} \right] + O\left( |z|^2 \ln \frac{1}{|z|} \right) \quad |z| \to 0$$

and

$$\frac{d}{dz} H_0^{(1)}(z) = \frac{2i}{\pi z} + O\left( |z| \ln \frac{1}{|z|} \right), \quad |z| \to 0,$$

we have

$$\Phi_k(\boldsymbol{r}, \boldsymbol{r}') = \Phi_0(\boldsymbol{r}, \boldsymbol{r}') - \frac{1}{2\pi}(\ln k - \gamma) + O(k^2 \ln k), \quad k \to 0 \tag{4.1}$$

and

$$\nabla_r \Phi_k(\boldsymbol{r}, \boldsymbol{r}') = \nabla_r \Phi_0(\boldsymbol{r}, \boldsymbol{r}') + O(k^2 \ln k), \quad k \to 0, \tag{4.2}$$

where $\gamma = \ln 2 - c + \pi i/2$ is a constant and $c = 0.5772\ldots$ is the Euler constant.

Since the Hankel function $H_0^{(1)}$ tends to infinity as $k \to 0$, we consider the following solution to the transmission electromagnetic problem described in Equations (3.1)–(3.5), as done in the case of the exterior problem [2]:

$$\begin{aligned} \boldsymbol{E}_k(\boldsymbol{r}) = \mu_e \nabla_r \times \int_{\partial D} \Phi_k(\boldsymbol{r}, \boldsymbol{r}') \boldsymbol{\alpha}_k(\boldsymbol{r}') \, ds(\boldsymbol{r}') \\ + \left(1 - \frac{2\pi}{\ln k}\right) \int_{\partial D} \Phi_k(\boldsymbol{r}, \boldsymbol{r}') \boldsymbol{b}_k(\boldsymbol{r}') \, ds(\boldsymbol{r}') \\ - \frac{1}{|\partial D|} \int_{\partial D} \Phi_k(\boldsymbol{r}, \boldsymbol{r}'') \, ds(\boldsymbol{r}'') \int_{\partial D} \boldsymbol{b}_k(\boldsymbol{r}') \, ds(\boldsymbol{r}'), \quad \boldsymbol{r} \in D_e. \end{aligned} \tag{4.3}$$

Here $\boldsymbol{\alpha}_k, \boldsymbol{b}_k \in \mathscr{S}^{0,\alpha}$ are continuous vector tangential density functions and $|\partial D|$ denotes the arclength of $\partial D$. This solution has to satisfy the vector Helmholtz equation and tends to the solution of the corresponding transmission problem of the potential theoretical case $k = 0$, which also must satisfy Maxwell's equation. We see

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that the field (4.3) satisfies the equation (3.1) and the radiation condition (3.5). As $\boldsymbol{r}$ tends to the boundary we can use the jump relations for vector fields [4] to obtain

$$\begin{aligned}
&\boldsymbol{\nu}(\boldsymbol{r}) \times \boldsymbol{E}_k^+(\boldsymbol{r}) \\
&\quad = \mu_e \boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_r \times \int_{\partial D} \Phi_k(\boldsymbol{r}, \boldsymbol{r}')\boldsymbol{\alpha}_k(\boldsymbol{r}')\, ds(\boldsymbol{r}') + \frac{1}{2}\mu_e \boldsymbol{\alpha}_k(\boldsymbol{r}) \\
&\qquad + \left(1 - \frac{2\pi}{\ln k}\right) \boldsymbol{\nu}(\boldsymbol{r}) \times \int_{\partial D} \Phi_k(\boldsymbol{r}, \boldsymbol{r}')\boldsymbol{b}_k(\boldsymbol{r}')\, ds(\boldsymbol{r}') \\
&\qquad - \frac{1}{|\partial D|} \boldsymbol{\nu}(\boldsymbol{r}) \times \int_{\partial D} \Phi_k(\boldsymbol{r}, \boldsymbol{r}'')\, ds(\boldsymbol{r}'') \int_{\partial D} \boldsymbol{b}_k(\boldsymbol{r}')\, ds(\boldsymbol{r}'), \quad \boldsymbol{r} \in \partial D, \qquad (4.4)
\end{aligned}$$

where the superscript (+) indicates that the limit is obtained by approaching the boundary from inside $D_e$.

Similarly

$$\begin{aligned}
\boldsymbol{F}_{k_i}(\boldsymbol{r}) &= \mu_i \nabla_r \times \int_{\partial D} \Phi_{k_i}(\boldsymbol{r}, \boldsymbol{r}')\boldsymbol{\alpha}_k(\boldsymbol{r}')\, ds(\boldsymbol{r}') \\
&\quad + \left(1 - \frac{2\pi}{\ln k_i}\right) \int_{\partial D} \Phi_{k_i}(\boldsymbol{r}, \boldsymbol{r}')\boldsymbol{b}_k(\boldsymbol{r}')\, ds(\boldsymbol{r}') \\
&\quad - \frac{1}{|\partial D|} \int_{\partial D} \Phi_{k_i}(\boldsymbol{r}, \boldsymbol{r}'')\, ds(\boldsymbol{r}'') \int_{\partial D} \boldsymbol{b}_k(\boldsymbol{r}')\, ds(\boldsymbol{r}'), \quad \boldsymbol{r} \in D_i, \qquad (4.5)
\end{aligned}$$

with continuous vector tangential density functions $\boldsymbol{\alpha}_k$, $\boldsymbol{b}_k$. The field (4.5) satisfies (3.2). By the jump relations for vector fields [4], as $\boldsymbol{r}$ tends to the boundary, we have

$$\begin{aligned}
&\boldsymbol{\nu}(\boldsymbol{r}) \times \boldsymbol{F}_{k_i}^-(\boldsymbol{r}) \\
&\quad = \mu_i \boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_r \times \int_{\partial D} \Phi_{k_i}(\boldsymbol{r}, \boldsymbol{r}')\boldsymbol{\alpha}_k(\boldsymbol{r}')\, ds(\boldsymbol{r}') - \frac{1}{2}\mu_i \boldsymbol{\alpha}_k(\boldsymbol{r}) \\
&\qquad + \left(1 - \frac{2\pi}{\ln k_i}\right) \boldsymbol{\nu}(\boldsymbol{r}) \times \int_{\partial D} \Phi_{k_i}(\boldsymbol{r}, \boldsymbol{r}')\boldsymbol{b}_k(\boldsymbol{r}')\, ds(\boldsymbol{r}') \\
&\qquad - \frac{1}{|\partial D|} \boldsymbol{\nu}(\boldsymbol{r}) \times \int_{\partial D} \Phi_{k_i}(\boldsymbol{r}, \boldsymbol{r}'')\, ds(\boldsymbol{r}'') \int_{\partial D} \boldsymbol{b}_k(\boldsymbol{r}')\, ds(\boldsymbol{r}'), \quad \boldsymbol{r} \in \partial D, \qquad (4.6)
\end{aligned}$$

where the superscript (−) indicates that the limit is obtained by approaching the boundary from inside $D_i$.

Substituting (4.4) and (4.6) into (3.3), we obtain on the boundary

$$(\mu_i + \mu_e)\boldsymbol{\alpha}_k + L_{11}^k \boldsymbol{\alpha}_k + L_{12}^k \boldsymbol{b}_k = 2\boldsymbol{c}_k, \quad k > 0, \ \operatorname{Im} k_i \geq 0, \qquad (4.7)$$

where

$$L_{11}^k \boldsymbol{\alpha}_k(\boldsymbol{r}) = 2\boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_r \times \int_{\partial D} \left[\mu_e \Phi_k(\boldsymbol{r}, \boldsymbol{r}') - \mu_i \Phi_{k_i}(\boldsymbol{r}, \boldsymbol{r}')\right] \boldsymbol{\alpha}_k(\boldsymbol{r}')\, ds(\boldsymbol{r}')$$

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and

$$L_{12}^k b_k(r)$$
$$= 2\nu(r) \times \int_{\partial D} \left[\left(1 - \frac{2\pi}{\ln k}\right)\Phi_k(r, r') - \left(1 - \frac{2\pi}{\ln k_i}\right)\Phi_{k_i}(r, r')\right] b_k(r')\, ds(r')$$
$$- \frac{2}{|\partial D|}\nu(r) \times \int_{\partial D} \left[\Phi_k(r, r'') - \Phi_{k_i}(r, r'')\right] ds(r'') \int_{\partial D} b_k(r')\, ds(r').$$

Applying now the jump relation in (4.3), we find as $r$ tends to the boundary that

$$\begin{aligned}\nu(r) \times \nabla_r \times E_k^+(r) = {} & \mu_e \nu(r) \times \nabla_r \times \nabla_r \times \int_{\partial D} \Phi_k(r, r')\alpha_k(r')\, ds(r') \\ & + \left(1 - \frac{2\pi}{\ln k}\right)\nu(r) \times \nabla_r \times \int_{\partial D} \Phi_k(r, r') b_k(r')\, ds(r') \\ & + \left(1 - \frac{2\pi}{\ln k}\right) b_k(r) \\ & - \frac{1}{|\partial D|}\nu(r) \times \nabla_r \times \int_{\partial D} \Phi_k(r, r'')\, ds(r'') \int_{\partial D} b_k(r')\, ds(r') \\ & - \frac{1}{2|\partial D|}\int_{\partial D} b_k(r)\, ds(r), \quad r \in \partial D, \end{aligned} \tag{4.8}$$

while applying the jump relation to (4.5) once again, we find

$$\begin{aligned}\nu(r) \times \nabla_r \times F_{k_i}^-(r) = {} & \mu_i \nu(r) \times \nabla_r \times \nabla_r \times \int_{\partial D} \Phi_{k_i}(r, r')\alpha_k(r')\, ds(r') \\ & + \left(1 - \frac{2\pi}{\ln k_i}\right)\nu(r) \times \nabla_r \times \int_{\partial D} \Phi_{k_i}(r, r') b_k(r')\, ds(r') \\ & - \frac{1}{2}\left(1 - \frac{2\pi}{\ln k_i}\right) b_k(r) \\ & - \frac{1}{|\partial D|}\nu(r) \times \nabla_r \times \int_{\partial D} \Phi_{k_i}(r, r'')\, ds(r'') \int_{\partial D} b_k(r')\, ds(r') \\ & + \frac{1}{2|\partial D|}\int_{\partial D} b_k(r)\, ds(r), \quad r \in \partial D. \end{aligned} \tag{4.9}$$

Substituting (4.8) and (4.9) into (3.4), we obtain on the boundary

$$\left(\frac{1}{\mu_e}\left(1 - \frac{2\pi}{\ln k}\right) + \frac{1}{\mu_i}\left(1 - \frac{2\pi}{\ln k_i}\right)\right) b_k + L_{21}^k \alpha_k + L_{22}^k b_k = 2d_k, \tag{4.10}$$

$k > 0$, $\operatorname{Im} k_i \geq 0$, where

$$L_{21}^k \alpha_k(r) = 2\nu(r) \times \nabla_r \times \nabla_r \times \int_{\partial D} \left[\mu_e \Phi_k(r, r') - \mu_i \Phi_{k_i}(r, r')\right]\alpha_k(r')\, ds(r')$$

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and

$$L_{22}^k \boldsymbol{b}_k(\boldsymbol{r}) = 2\boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_r \times \int_{\partial D} \left[ \frac{1}{\mu_e}\left(1 - \frac{2\pi}{\ln k}\right) \Phi_k(\boldsymbol{r}, \boldsymbol{r}') - \frac{1}{\mu_i}\left(1 - \frac{2\pi}{\ln k_i}\right) \Phi_{k_i}(\boldsymbol{r}, \boldsymbol{r}') \right] \boldsymbol{b}_k(\boldsymbol{r}')\, ds(\boldsymbol{r}')$$
$$- \frac{2}{|\partial D|} \left[ \boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_r \times \int_{\partial D} \left[ \frac{1}{\mu_e}\Phi_k(\boldsymbol{r}, \boldsymbol{r}'') - \frac{1}{\mu_i}\Phi_{k_i}(\boldsymbol{r}, \boldsymbol{r}'') \right] ds(\boldsymbol{r}'') - \left(\frac{1}{\mu_e} + \frac{1}{\mu_i}\right) \right] \int_{\partial D} \boldsymbol{b}_k(\boldsymbol{r}')\, ds(\boldsymbol{r}').$$

We now introduce the operators $\boldsymbol{J}_k$ and $\boldsymbol{L}_k$ defined as

$$\boldsymbol{J}_k = \begin{bmatrix} (\mu_e + \mu_i) I & 0 \\ 0 & \left( \frac{1}{\mu_e}\left(1 - \frac{2\pi}{\ln k}\right) + \frac{1}{\mu_i}\left(1 - \frac{2\pi}{\ln k_i}\right) \right) I \end{bmatrix} \quad \text{and}$$
$$\boldsymbol{L}_k = \begin{bmatrix} -L_{11}^k & -L_{12}^k \\ -L_{21}^k & -L_{22}^k \end{bmatrix}$$

and the integral equations (4.7) and (4.10) can be written in a compact form as

$$(\boldsymbol{J}_k - \boldsymbol{L}_k)\boldsymbol{X}_k = 2\boldsymbol{B}_k,$$

where

$$\boldsymbol{X}_k = \begin{bmatrix} \boldsymbol{\alpha}_k(\boldsymbol{r}) \\ \boldsymbol{b}_k(\boldsymbol{r}) \end{bmatrix} \quad \text{and} \quad \boldsymbol{B}_k = \begin{bmatrix} \boldsymbol{c}_k(\boldsymbol{r}) \\ \boldsymbol{d}_k(\boldsymbol{r}) \end{bmatrix}.$$

We follow the same idea in order to find the corresponding integral equations for $k = 0$. Taking the limit of the field $\boldsymbol{E}_k$ given by (4.3), as $k \to 0$ and using (4.1) and (4.2) we obtain

$$\boldsymbol{E}_0(\boldsymbol{r}) = \mu_e \nabla_r \times \int_{\partial D} \Phi_0(\boldsymbol{r}, \boldsymbol{r}')\boldsymbol{\alpha}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}') + \int_{\partial D} [\Phi_0(\boldsymbol{r}, \boldsymbol{r}') + 1]\boldsymbol{b}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}')$$
$$- \frac{1}{|\partial D|} \int_{\partial D} \Phi_0(\boldsymbol{r}, \boldsymbol{r}'')\, ds(\boldsymbol{r}'') \int_{\partial D} \boldsymbol{b}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}'), \quad \boldsymbol{r} \in D_e, \tag{4.11}$$

where $\boldsymbol{\alpha}_0, \boldsymbol{b}_0 \in \mathscr{S}^{0,\alpha}$ are continuous tangential density functions. We use (4.11) and the jump relation as $\boldsymbol{r}$ tends to the boundary, so we have

$$\boldsymbol{\nu}(\boldsymbol{r}) \times \boldsymbol{E}_0^+(\boldsymbol{r}) = \mu_e \boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_r \times \int_{\partial D} \Phi_0(\boldsymbol{r}, \boldsymbol{r}')\boldsymbol{\alpha}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}') + \frac{1}{2}\mu_e \boldsymbol{\alpha}_0(\boldsymbol{r})$$
$$+ \boldsymbol{\nu}(\boldsymbol{r}) \times \int_{\partial D} [\Phi_0(\boldsymbol{r}, \boldsymbol{r}') + 1]\boldsymbol{b}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}')$$

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$$- \frac{1}{|\partial D|} \boldsymbol{\nu}(\boldsymbol{r}) \times \int_{\partial D} \Phi_0(\boldsymbol{r}, \boldsymbol{r}'')\, ds(\boldsymbol{r}'') \int_{\partial D} \boldsymbol{b}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}'), \quad \boldsymbol{r} \in \partial D. \tag{4.12}$$

Similarly, taking the limit of the field $\boldsymbol{F}_{k_i}$ as $k_i \to 0$ and using (4.1) and (4.2) for $k_i$, we obtain

$$\begin{aligned} \boldsymbol{F}_0(\boldsymbol{r}) = {} & \mu_i \nabla_r \times \int_{\partial D} \Phi_0(\boldsymbol{r}, \boldsymbol{r}')\boldsymbol{\alpha}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}') \\ & + \int_{\partial D} [\Phi_0(\boldsymbol{r}, \boldsymbol{r}') + 1]\boldsymbol{b}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}') \\ & - \frac{1}{|\partial D|} \int_{\partial D} \Phi_0(\boldsymbol{r}, \boldsymbol{r}'')\, ds(\boldsymbol{r}'') \int_{\partial D} \boldsymbol{b}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}'), \quad \boldsymbol{r} \in D_i, \end{aligned} \tag{4.13}$$

where $\boldsymbol{\alpha}_0, \boldsymbol{b}_0 \in \mathscr{S}^{0,\alpha}$ are continuous tangential density functions. We use (4.13) and the jump relation as $\boldsymbol{r}$ tends to the boundary, so we take

$$\begin{aligned} \boldsymbol{\nu}(\boldsymbol{r}) \times \boldsymbol{F}_0^-(\boldsymbol{r}) = {} & \mu_i \boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_r \times \int_{\partial D} \Phi_0(\boldsymbol{r}, \boldsymbol{r}')\boldsymbol{\alpha}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}') - \frac{1}{2}\mu_i \boldsymbol{\alpha}_0(\boldsymbol{r}) \\ & + \boldsymbol{\nu}(\boldsymbol{r}) \times \int_{\partial D} [\Phi_0(\boldsymbol{r}, \boldsymbol{r}') + 1]\boldsymbol{b}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}') \\ & - \frac{1}{|\partial D|} \boldsymbol{\nu}(\boldsymbol{r}) \times \int_{\partial D} \Phi_0(\boldsymbol{r}, \boldsymbol{r}'')\, ds(\boldsymbol{r}'') \int_{\partial D} \boldsymbol{b}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}'), \quad \boldsymbol{r} \in \partial D. \end{aligned} \tag{4.14}$$

Substituting (4.12) and (4.14) in the boundary condition (3.8), we obtain

$$(\mu_e - \mu_i)\boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_r \times \int_{\partial D} \Phi_0(\boldsymbol{r}, \boldsymbol{r}')\boldsymbol{\alpha}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}') + \frac{1}{2}(\mu_e + \mu_i)\boldsymbol{\alpha}_0(\boldsymbol{r}) = \boldsymbol{c}_0(\boldsymbol{r}).$$

If we set

$$L_{11}^0 \boldsymbol{\alpha}_0(\boldsymbol{r}) = 2(\mu_e - \mu_i)\boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_r \times \int_{\partial D} \Phi_0(\boldsymbol{r}, \boldsymbol{r}')\boldsymbol{\alpha}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}'),$$

we have

$$L_{11}^0 \boldsymbol{\alpha}_0(\boldsymbol{r}) + (\mu_e + \mu_i)\boldsymbol{\alpha}_0(\boldsymbol{r}) = 2\boldsymbol{c}_0(\boldsymbol{r}). \tag{4.15}$$

Now in (4.11), we use the jump relation as $\boldsymbol{r}$ tends to the boundary, so we take

$$\begin{aligned} & \boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_r \times \boldsymbol{E}_0^+(\boldsymbol{r}) \\ & \quad = \mu_e \boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_r \times \nabla_r \times \int_{\partial D} \Phi_0(\boldsymbol{r}, \boldsymbol{r}')\boldsymbol{\alpha}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}') + \frac{1}{2}\boldsymbol{b}_0(\boldsymbol{r}) \end{aligned}$$

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$$+ \boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_r \times \int_{\partial D} [\Phi_0(\boldsymbol{r}, \boldsymbol{r}') + 1]\boldsymbol{b}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}')$$
$$- \frac{1}{|\partial D|}\boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_r \times \int_{\partial D} \Phi_0(\boldsymbol{r}, \boldsymbol{r}'')\, ds(\boldsymbol{r}'') \int_{\partial D} \boldsymbol{b}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}')$$
$$- \frac{2}{|\partial D|}\int_{\partial D} \boldsymbol{b}_0(\boldsymbol{r})\, ds(\boldsymbol{r}), \quad \boldsymbol{r} \in \partial D. \tag{4.16}$$

Similarly,

$$\boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_r \times \boldsymbol{F}_0^-(\boldsymbol{r})$$
$$= \mu_i \boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_r \times \nabla_r \times \int_{\partial D} \Phi_0(\boldsymbol{r}, \boldsymbol{r}')\boldsymbol{\alpha}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}') - \frac{1}{2}\boldsymbol{b}_0(\boldsymbol{r})$$
$$+ \boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_r \times \int_{\partial D} [\Phi_0(\boldsymbol{r}, \boldsymbol{r}') + 1]\boldsymbol{b}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}')$$
$$- \frac{1}{|\partial D|}\boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_r \times \int_{\partial D} \Phi_0(\boldsymbol{r}, \boldsymbol{r}'')\, ds(\boldsymbol{r}'') \int_{\partial D} \boldsymbol{b}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}')$$
$$+ \frac{2}{|\partial D|}\int_{\partial D} \boldsymbol{b}_0(\boldsymbol{r})\, ds(\boldsymbol{r}), \quad \boldsymbol{r} \in \partial D. \tag{4.17}$$

Substituting (4.16) and (4.17) in the boundary condition (3.9), we obtain

$$L_{22}^0 \boldsymbol{b}_0(\boldsymbol{r}) + \left(\frac{1}{\mu_e} + \frac{1}{\mu_i}\right)\boldsymbol{b}_0(\boldsymbol{r}) = 2\boldsymbol{d}_0(\boldsymbol{r}), \tag{4.18}$$

where

$$L_{22}^0 \boldsymbol{b}_0(\boldsymbol{r})$$
$$= 2\left(\frac{1}{\mu_e} - \frac{1}{\mu_i}\right)\boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_r \times \int_{\partial D} \Phi_0(\boldsymbol{r}, \boldsymbol{r}')\boldsymbol{b}_0(\boldsymbol{r})\, ds(\boldsymbol{r}')$$
$$- \left(\frac{1}{\mu_e} - \frac{1}{\mu_i}\right)\frac{2}{|\partial D|}\boldsymbol{\nu}(\boldsymbol{r}) \times \nabla_r \times \int_{\partial D} \Phi_0(\boldsymbol{r}, \boldsymbol{r}'')\, ds(\boldsymbol{r}'') \int_{\partial D} \boldsymbol{b}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}')$$
$$- \left(\frac{1}{\mu_e} + \frac{1}{\mu_i}\right)\frac{1}{|\partial D|}\int_{\partial D} \boldsymbol{b}_0(\boldsymbol{r})\, ds(\boldsymbol{r}).$$

We set

$$\boldsymbol{J}_0 = \begin{bmatrix} (\mu_e + \mu_i)I & 0 \\ 0 & (1/\mu_e + 1/\mu_i)I \end{bmatrix} \quad \text{and} \quad \boldsymbol{L}_0 = \begin{bmatrix} -L_{11}^0 & 0 \\ 0 & -L_{22}^0 \end{bmatrix}$$

and the integral equations (4.15) and (4.18) become

$$(\boldsymbol{J}_0 - \boldsymbol{L}_0)\boldsymbol{X}_0 = 2\boldsymbol{B}_0,$$

where

$$\boldsymbol{X}_0 = \begin{bmatrix} \boldsymbol{\alpha}_0(\boldsymbol{r}) \\ \boldsymbol{b}_0(\boldsymbol{r}) \end{bmatrix} \quad \text{and} \quad \boldsymbol{B}_0 = \begin{bmatrix} \boldsymbol{c}_0(\boldsymbol{r}) \\ \boldsymbol{d}_0(\boldsymbol{r}) \end{bmatrix}.$$

THEOREM 4.1. *The integral operator $J_0 - L_0$ is injective.*

PROOF. Suppose a solution $X_0$ of $(J_0 - L_0)X_0 = \mathbf{0}$ exists. With this solution we construct the harmonic field $E_0$ by (4.11) and $F_0$ by (4.13), which are solutions of the vector Helmholtz equations and satisfy the homogeneous boundary conditions. The harmonic field $E_0$ satisfies the radiation condition.

We extend the harmonic field $E_0$ in $D_i$, as in $D_e$ by (4.11), then using the jump relations for vector fields [4], as $r$ tends to the boundary, $r \in D_e$ for $E_0$ and as $r$ tends to the boundary, $r \in D_i$ for $E_0$, we have

$$\nu(r) \times E_0^+(r) - \nu(r) \times E_0^-(r) = \mu_e \alpha_0(r), \quad r \in \partial D.$$

By the uniqueness of the solution of the homogenous transmission problem of the vector Helmholtz equations [3], we obtain

$$-\frac{1}{\mu_e}\nu(r) \times E_0^-(r) = \alpha_0(r), \quad r \in \partial D. \tag{4.19}$$

In a similar way, we extend the harmonic field $F_0$ in $D_e$, as in $D_i$ by (4.13), then using the jump relations for vector fields [4], as $r$ tends to the boundary, $r \in D_i$ for $F_0$ and as $r$ tends to the boundary, $r \in D_e$ for $F_0$, we have

$$\nu(r) \times F_0^+(r) - \nu(r) \times F_0^-(r) = \mu_i \alpha_0(r), \quad r \in \partial D.$$

Then, by the uniqueness of the solution of the homogenous transmission problem of the vector Helmholtz equations, we obtain

$$\frac{1}{\mu_i}\nu(r) \times F_0^+(r) = \alpha_0(r), \quad r \in \partial D. \tag{4.20}$$

From (4.19) and (4.20), we take

$$\frac{1}{\mu_i}\nu(r) \times F_0^+(r) + \frac{1}{\mu_e}\nu(r) \times E_0^-(r) = \mathbf{0}, \quad r \in \partial D. \tag{4.21}$$

Now, using (4.16) and (4.17) and taking into account the relation (41) from [2], we have

$$\nu(r) \times \nabla_r \times F_0^+(r) + \nu(r) \times \nabla_r \times E_0^-(r) = \mathbf{0}, \quad r \in \partial D. \tag{4.22}$$

We consider

$$E_0'(r) = \frac{1}{\mu_i} F_0^+(r), \quad r \in D_e \quad \text{and} \quad F_0'(r) = -\frac{1}{\mu_e} E_0^-(r), \quad r \in D_i,$$

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then, from (4.21) and (4.22), the new harmonic fields satisfy the homogenous transmission problem (3.6)–(3.10) and by the uniqueness of the solution of the homogenous transmission problem $\boldsymbol{E}_0'(\boldsymbol{r}) = \boldsymbol{F}_0'(\boldsymbol{r}) = \boldsymbol{0}$. This means that $\boldsymbol{\alpha}_0(\boldsymbol{r}) = \boldsymbol{b}_0(\boldsymbol{r}) = \boldsymbol{0}$, on $\partial D$. Therefore the operator $\boldsymbol{J}_0 - \boldsymbol{L}_0$ is injective and by Riesz's theory $(\boldsymbol{J}_0 - \boldsymbol{L}_0)^{-1}$ exists and is bounded. □

Let $A(\boldsymbol{x}) = \left[\alpha_{mn}(\boldsymbol{x})\right]_{2\times 2}$ be a $2 \times 2$ matrix, where $\alpha_{mn}(\boldsymbol{x})$ are functions on a region $D$ in $\mathbb{R}^2$. We define the norm $\|A\|_{\infty,D}$ by the equality

$$\|A\|_{\infty,D} = \max\left\{\sum_{n=1}^{2}\|\alpha_{mn}\|_{\infty,D} : m = 1, 2\right\}, \tag{4.23}$$

where $\|\alpha_{mn}\|_{\infty,D}$ is the sup-norm of $\alpha_{mn}$ on $D$.

LEMMA 4.2. *The integral operators $\boldsymbol{J}_k$, $\boldsymbol{J}_0$, $\boldsymbol{L}_k$ and $\boldsymbol{L}_0$ as they have been defined above satisfy the relation*

$$\|\boldsymbol{J}_k^{-1}\boldsymbol{L}_k - \boldsymbol{J}_0^{-1}\boldsymbol{L}_0\|_{\infty,\partial D} = O\left(\frac{1}{\ln k}\right) + O\left(\frac{1}{\ln|k_i|}\right), \quad k, k_i \to 0. \tag{4.24}$$

PROOF. The integral operators $\boldsymbol{J}_k$ and $\boldsymbol{J}_0$ are invertible, since they are diagonal and $\mu_e, \mu_i > 0$. We have

$$\boldsymbol{J}_k^{-1}\boldsymbol{L}_k - \boldsymbol{J}_0^{-1}\boldsymbol{L}_0 = \boldsymbol{J}_k^{-1}(\boldsymbol{L}_k - \boldsymbol{L}_0) + (\boldsymbol{J}_k^{-1} - \boldsymbol{J}_0^{-1})\boldsymbol{L}_0.$$

Using the asymptotic relations (4.1) and(4.2) for the entries of $\boldsymbol{L}_k - \boldsymbol{L}_0$ we obtain

$$\begin{aligned}
\|L_{11} - L_{11}^0\|_{\infty,\partial D} &= O\left(k^2\ln\frac{1}{k}\right) + O\left(|k_i|^2\ln\frac{1}{|k_i|}\right),\\
\|L_{12} - L_{12}^0\|_{\infty,\partial D} &= O\left(\frac{1}{|\ln k|}\right) + O\left(\frac{1}{|\ln|k_i||}\right),\\
\|L_{21} - L_{21}^0\|_{\infty,\partial D} &= O\left(k^2\ln\frac{1}{k}\right) + O\left(|k_i|^2\ln\frac{1}{|k_i|}\right),\\
\|L_{22} - L_{22}^0\|_{\infty,\partial D} &= O\left(\frac{1}{|\ln k|}\right) + O\left(\frac{1}{|\ln|k_i||}\right),
\end{aligned}$$

as $k, k_i \to 0$. From these last relations and (4.23) we have (4.24). □

We are now in a position to establish the following theorem.

THEOREM 4.3. *The inverse operators $(\boldsymbol{J}_k - \boldsymbol{L}_k)^{-1}$ exist and satisfy*

$$\left\|((\boldsymbol{J}_k - \boldsymbol{L}_k)^{-1} - (\boldsymbol{J}_0 - \boldsymbol{L}_0)^{-1}\right\|_{\infty,\partial D} = O\left(\frac{1}{\ln k}\right) + O\left(\frac{1}{\ln|k_i|}\right), \quad k, k_i \to 0,$$

*for $k$, $k_i$ sufficiently small, namely $0 < k, |k_i| < \kappa < 1$.*

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PROOF. We first prove that the inverse operators $(I - J_k^{-1}L_k)^{-1}$ exist and satisfy for $k, k_i \to 0$

$$\left\|(I - J_k^{-1}J_k)^{-1} - (I - J_0^{-1}L_0)^{-1}\right\|_{\infty,\partial D} = O\left(\frac{1}{\ln k}\right) + O\left(\frac{1}{\ln|k_i|}\right), \qquad (4.25)$$

for $k, k_i$ sufficiently small, namely $0 < k, |k_i| < \kappa < 1$. We write

$$I - J_k^{-1}L_k = (I - J_0^{-1}L_0)\big[I - (I - J_0^{-1}L_0)^{-1}(J_k^{-1}L_k - J_0^{-1}L_0)\big].$$

Therefore $(I - J_k^{-1}L_k)^{-1}$ exist. It follows from (4.24) that there is a $\kappa : 0 < k, |k_i| < \kappa < 1$ such that

$$\left\|(I - J_0^{-1}L_0)^{-1}(J_k^{-1}L_k - J_0^{-1}L_0)\right\|_{\infty,\partial D} \leq q < 1,$$

then, by using the local Neumann expansion as in [9],

$$(I - J_k^{-1}L_k)^{-1} = \sum_{j=0}^{\infty}(-1)^j[(I - J_0^{-1}L_0)^{-1}(J_k^{-1}L_k - J_0^{-1}L_0)]^j(I - J_0^{-1}L_0)^{-1},$$

we obtain that $(I - J_k^{-1}L_k)^{-1}$ exist and satisfy (4.25).

Finally, we have

$$\begin{aligned}(J_k - L_k)^{-1} - (J_0 - L_0)^{-1} \\ = [(I - J_k^{-1}L_k)^{-1} - (I - J_0^{-1}L_0)^{-1}]J_k^{-1} + (I - J_0^{-1}L_0)^{-1}(J_k^{-1} - J_0^{-1}),\end{aligned}$$

and we use the triangle inequality to prove our assumption. □

We now formulate the main result of this paper.

THEOREM 4.4. *The solution $E_k$, $F_{k_i}$ of the transmission electric problem for the vector Helmholtz equations, with boundary data $c_k$, $d_k$, converges uniformly on compact subsets of $\overline{D}_e$, $\overline{D}_i$ to the solution $E_0$, $F_0$, of the transmission electrostatic problem for the vector Helmholtz equations, with boundary data $c_0$, $d_0$, if $c_k \to c_0$, $d_k \to d_0$ uniformly, as $k, k_i \to 0$.*

PROOF. Let the solutions $E_k$, $E_0$ as in (4.3), (4.11) and $F_{k_i}$, $F_0$ as in (4.5), (4.13), respectively. Then the corresponding densities become

$$X_k = 2(J_k - L_k)^{-1}B_k,$$

where $k$, $|k_i|$ are sufficiently small, and

$$X_0 = 2(J_0 - L_0)^{-1}B_0.$$

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From (4.25) and the triangle inequality, we have $\|\boldsymbol{\alpha}_k(\boldsymbol{r}) - \boldsymbol{\alpha}_0(\boldsymbol{r})\|_{\infty,\partial D} \to 0$ and $\|\boldsymbol{b}_k(\boldsymbol{r}) - \boldsymbol{b}_0(\boldsymbol{r})\|_{\infty,\partial D} \to 0$ as $k, k_i \to 0$.

We write the differences

$$\boldsymbol{E}_k - \boldsymbol{E}_0 = \boldsymbol{U}_k^e + \boldsymbol{W}_k^e, \quad \boldsymbol{r} \in D_e, \tag{4.26}$$

and

$$\boldsymbol{F}_{k_i} - \boldsymbol{F}_0 = \boldsymbol{U}_k^i + \boldsymbol{W}_k^i, \quad \boldsymbol{r} \in D_i, \tag{4.27}$$

where

$$\begin{aligned}\boldsymbol{U}_k^p(\boldsymbol{r}) = \mu_p \nabla_r \times \int_{\partial D} & \Phi_0(\boldsymbol{r}, \boldsymbol{r}')\big(\boldsymbol{\alpha}_k(\boldsymbol{r}') - \boldsymbol{\alpha}_0(\boldsymbol{r}')\big)\, ds(\boldsymbol{r}') \\ & + \int_{\partial D} [\Phi_0(\boldsymbol{r}, \boldsymbol{r}') + 1]\big(\boldsymbol{b}_k(\boldsymbol{r}') - \boldsymbol{b}_0(\boldsymbol{r}')\big)\, ds(\boldsymbol{r}') \\ & - \frac{1}{|\partial D|}\int_{\partial D} \Phi_0(\boldsymbol{r}, \boldsymbol{r}'')\, ds(\boldsymbol{r}'') \int_{\partial D} \big(\boldsymbol{b}_k(\boldsymbol{r}') - \boldsymbol{b}_0(\boldsymbol{r}')\big)\, ds(\boldsymbol{r}')\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{W}_k^p(\boldsymbol{r}) = \mu_p \nabla_r \times \int_{\partial D} & \big(\Phi_k(\boldsymbol{r}, \boldsymbol{r}') - \Phi_0(\boldsymbol{r}, \boldsymbol{r}')\big)\boldsymbol{\alpha}_k(\boldsymbol{r}')\, ds(\boldsymbol{r}') \\ & + \int_{\partial D} \left[\left(1 - \frac{2\pi}{\ln k}\right)\Phi_k(\boldsymbol{r}, \boldsymbol{r}') - \Phi_0(\boldsymbol{r}, \boldsymbol{r}') - 1\right] \boldsymbol{b}_k(\boldsymbol{r}')\, ds(\boldsymbol{r}') \\ & - \frac{1}{|\partial D|}\int_{\partial D} \big(\Phi_k(\boldsymbol{r}, \boldsymbol{r}'') - \Phi_0(\boldsymbol{r}, \boldsymbol{r}'')\big)\, ds(\boldsymbol{r}'') \int_{\partial D} \boldsymbol{b}_0(\boldsymbol{r}')\, ds(\boldsymbol{r}'),\end{aligned}$$

with $p = e, i$.

The vector function $\boldsymbol{U}_k^p$ behaves asymptotically, as $|\boldsymbol{r}| \to \infty$

$$\left|\boldsymbol{U}_k^p(\boldsymbol{r}) - \int_{\partial D} \big(\boldsymbol{b}_k(\boldsymbol{r}') - \boldsymbol{b}_0(\boldsymbol{r}')\big)\, ds(\boldsymbol{r}')\right| \le \frac{M_1^p}{|\boldsymbol{r}|}\|\boldsymbol{\alpha}_k - \boldsymbol{\alpha}_0\|_{\infty,\partial D} + M_2^p \|\boldsymbol{b}_k - \boldsymbol{b}_0\|_{\infty,\partial D},$$

with some constants $M_1^p$, $M_2^p$, $p = e, i$. We derive the uniform convergence $\boldsymbol{U}_k^p \to 0$, $p = e, i$ as $k, k_i \to 0$, by using the jump relations.

Using (4.1) and (4.2), for the vector function $\boldsymbol{W}_k^p$, $p = e, i$, we obtain

$$|\boldsymbol{W}_k^p(\boldsymbol{r})| \le \frac{M_3^p}{\ln k_p}\|\boldsymbol{\alpha}_k\|_{\infty,\partial D} + \frac{M_4^p}{\ln k_p}\|\boldsymbol{b}_k\|_{\infty,\partial D}$$

for all $\boldsymbol{r} \in \mathbb{R}^2$ with $|\boldsymbol{r}| \le R$, $R > 0$ where $M_3^p$, $M_4^p$ are constant and depend on $R$, hence $\boldsymbol{W}_k^p \to 0$, as $k, k_i \to 0$.

This final step and Equations (4.26) and (4.27) prove the theorem. □

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