INVARIANT SUBSPACES OF CONTINUOUS FUNCTIONS

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1. Introduction. Since Beurling (**1**), the study of closed invariant subspaces of $H^p(X)$ and $L^p(X)$ on the unit circle $X = \{z:|z| = 1\}$ has been done extensively and culminated into a very fine theory of generalized analytic functions; cf. Hoffman (**4**), Srinivasan (**7**). Here we say that a space E of complex-valued functions on X is *invariant* if $zE \subseteq E$. Little is known, however, about the structure of closed invariant subspaces E of the space $C(X)$ of continouus functions on X. If E happens to be a subspace of the disk algebra A (i.e., the algebra of continuous functions on the closed unit disk which are analytic on the open unit disk), then E becomes a closed ideal of A and we have a very beautiful theorem of Beurling–Rudin (Rudin (**5**) or Hoffman (**3**)), which has been extended by Voichick (**9**) to analytic functions on a Riemann surface. In this paper, we shall give the structure of general closed invariant subspaces of $C(X)$. Our main tools for attacking the problem are the F. and M. Riesz theorem and the invariant subspace theorem for $L^p(X)$ established recently by one of the present authors (Srinivasan (**7**)). As we shall see later and as we can also imagine, the structure of invariant subspaces of $C(X)$ shares some common features with that of invariant subspaces of $L^p(X)$ as well as that of closed ideals of A. Our theorem may be regarded as a generalization of the Beurling–Rudin theorem and, indeed, our argument yields a very simple proof of that theorem; cf. also Srinivasan and Wang (**8**).

We shall state our main theorem (Theorem 2) in the next section. It is shown that any closed simply invariant subspace of $C(X)$ is of the form $qH^\infty(X) \cap Z(K)$, where q is measurable and $|q| = 1$ a.e. (with respect to the Lebesgue measure on X) and $Z(K)$ denotes the space of continuous functions on X which vanish on a closed subset K of Lebesgue measure zero. Sometimes our expression becomes trivial; i.e., it can contain only the zero function. In the third section, we state a necessary and sufficient condition for non-triviality of $qH^\infty(X) \cap Z(K)$. Finally we prove that the expression is unique in a sense specified later.

An extension of our results to continuous functions on certain closed subsets of the complex plane as well as on a Riemann surface will be discussed in another paper.

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interest in this investigation and also to Professors Errett A. Bishop, Henry Helson, and John L. Kelley for many stimulating conversations we had with them.

2. Structure of the invariant subspaces. Let X be the unit circle $\{z: |z| = 1\}$ in the complex plane where z denotes the complex variable and let B be a uniformly closed subspace of $C(X)$. Our first theorem is not new, but we state it for completeness of exposition. We denote by $Z(K)$ the space of functions in $C(X)$ which vanish on a subset K of X.

THEOREM 1. *$zB = B$ if and only if $B = Z(K)$ for some closed subset K of X.*

Proof. Suppose $zB = B$. Then $\bar{z}B = B$, so that B is invariant under multiplication by functions in the algebra generated by $z, \bar{z}$, and 1. By the Stone–Weierstrass theorem this algebra is uniformly dense in $C(X)$. Thus B is a closed ideal of $C(X)$. Hence $B = Z(K)$ for a closed set K in X. The converse is trivial.

Let σ be the normalized Lebesgue measure on X. Then the disk algebra A, defined in the Introduction, can be viewed canonically as a subspace of $L^\infty(d\sigma)$ (or, more precisely, $L^\infty(X, d\sigma)$). The weak* closure of A in $L^\infty(d\sigma)$ is denoted by $H^\infty(d\sigma)$. The norm of $H^\infty(d\sigma)$ is, of course, that of $L^\infty(d\sigma)$. For $H^\infty(d\sigma)$ and other related concepts used in what follows, we shall refer the reader to Hoffman **(3, 4)** and Srinivasan **(7)**. Our main result is the following:

THEOREM 2. *If $zB \subseteq B$ but $zB \neq B$, then $B = qH^\infty(d\sigma) \cap Z(K)$ where $q \in L^\infty(d\sigma)$ with $|q| = 1$ a.e.-σ, and K is a closed set in X with $\sigma(K) = 0$.*

The proof of this theorem will be given later in this section. We shall now consider the space $M(X)$ of Radon measures on X. It is clear that $L^1(d\sigma)$ can be regarded as a subspace of $M(X)$ by the imbedding $f \rightarrow f\sigma$. $H^1(d\sigma)$ and $H_0{}^1(d\sigma)$ are likewise regarded as subspaces of $M(X)$. We shall often use this convention. Let N be a weakly* closed subspace of $M(X)$.

THEOREM 3. *$zN = N$ if and only if $N = M(K)$ for a closed subset K of X.*

Here $M(K)$ denotes the space of Radon measures on X whose supports are contained in K. This theorem is nothing but the dual form of Theorem 1, so we omit the proof.

THEOREM 4. *If $zN \subseteq N$ but $zN \neq N$, then $N = pH_0{}^1(d\sigma) + M(K)$, where $p \in L^\infty(d\sigma)$ with $|p| = 1$* a.e.-σ, *and K is a closed set in X with $\sigma(K) = 0$. Here K is unique and p is unique up to a constant factor of modulus one.*

Now we present a combined proof of Theorems 2 and 4. Let B be a closed subspace of $C(X)$ and let N be the orthogonal complement of B in $M(X)$, i.e., $N = B^\perp$. Then N is weakly* closed and $B = N^\perp$. Clearly any weakly* closed subspace N of $M(X)$ is obtained in this way. It is also easy to see that B is simply invariant (i.e., $zB \subset B$ and $zB \neq B$) if and only if N is simply invariant. We assume that either B or N, and hence both, are simply invariant. Let K be the set of the common zeros of the functions in B. K is closed in X.

Let $\mu \in N$. Then $\mu \perp B$. Since B is invariant, this means that $f\mu \perp z^n$ for any $f \in B$ and $n = 0, 1, 2, \ldots$. By a theorem of F. and M. Riesz, we have, for each fixed $f \in B$, $f\mu = h\sigma$ for some $h \in H_0^1(d\sigma)$. If $\mu = \mu_a + \mu_s$ is the Lebesgue decomposition of μ with respect to σ, where μ_a and μ_s are the absolutely continuous and the singular parts of σ, respectively, then the above result implies that $f\mu_s = 0$, i.e. $\mu_s \in M(K)$. Thus $\mu_s \in N$ and therefore $\mu_a \in N$. Since μ_a is identified with an element in $L^1(d\sigma)$, we have seen that $N \subseteq (L^1(d\sigma) \cap N) + M(K)$. Since the right-hand side is evidently orthogonal to B, we have the converse inclusion and thus $N = (L^1(d\sigma) \cap N) + M(K')$.

It is clear that $L^1(d\sigma) \cap N$ is L^1-closed and invariant. Indeed it is simply invariant; otherwise N would be doubly invariant. Thus a theorem in **(7)** tells us that $L^1(d\sigma) \cap N = pH_0^1(d\sigma)$ for some $p \in L^\infty(d\sigma)$ with $|p| = 1$ a.e.-σ. So we have $N = pH_0^1(d\sigma) + M(K)$. If there exists a $p' \in L^\infty(d\sigma)$ with $|p'| = 1$ a.e.-σ and a closed set K' such that $N = p'H_0^1(d\sigma) + M(K)$, then

$$pH_0^1(d\sigma) + M(K) = p'H_0^1(d\sigma) + M(K').$$

It follows immediately that $K = K'$. As we shall see below, $\sigma(K) = 0$ and therefore $pH_0^1(d\sigma) = p'H_0^1(d\sigma)$. This shows that $p' = cp$ a.e.-σ with a constant factor c of modulus one. This proves Theorem 4.

Now let $f \in B$. Since $f \perp N$, $f \perp pH_0^1(d\sigma)$ and $f \perp M(K)$. The first relation implies that $pf \in H^\infty(d\sigma)$ and thus $f \in \bar{p}H^\infty(d\sigma)$. $f \perp M(K)$ implies $f \in Z(K)$. So $B \subseteq \bar{p}H^\infty(d\sigma) \cap Z(K)$. On the other hand, $\bar{p}H^\infty(d\sigma) \cap Z(K)$ is evidently orthogonal to $N = pH_0^1(d\sigma) + M(K)$, so that it is contained in $N^\perp = B$. Hence $B = \bar{p}H^\infty(d\sigma) \cap Z(K)$.

Finally we shall show that $\sigma(K) = 0$. Suppose otherwise. Take any $f \in \bar{p}H^\infty(d\sigma) \cap Z(K)$. Then $f = \bar{p}h$ with $h \in H^\infty(d\sigma)$. Since $|p| = 1$ a.e.-σ, h must vanish on a set of positive σ-measure. But this implies that h vanishes identically and therefore $B = \{0\}$, contrary to the fact that $zB \neq B$. Hence $\sigma(K) = 0$. In order to establish Theorem 2 we have only to set $q = \bar{p}$.

Remark. If B is a simply invariant closed subspace of $C(X)$ and if q is defined as in the above proof, then $[B]_* = qH^\infty(d\sigma)$, where $[B]_*$ denotes the weak* closure of B in $L^\infty(d\sigma)$. To see this, we regard B as a subspace of $L^\infty(d\sigma)$. Then the weak* closure $[B]_*$ is nothing but the bipolar set B^{00} of B in the duality between $L^\infty(d\sigma)$ and $L^1(d\sigma)$. Since the polar set B^0 of B in $L^1(d\sigma)$ is the orthogonal complement of B in $L^1(d\sigma)$, we get $B^0 = pH_0^1(d\sigma)$ in view of the equation $N = pH_0^1(d\sigma) + M(K)$. It follows immediately that

$$[B]_* = B^{00} = (pH_0^1(d\sigma))^0 = qH^\infty(d\sigma).$$

This also shows that $[B]_*$ is simply invariant in $L^\infty(d\sigma)$. To get the expression for B given in Theorem 2 we may proceed as follows. First we take the weak* closure $[B]_*$ of B in $L^\infty(d\sigma)$. Then $[B]_*$ is simply invariant and therefore, by a theorem in (7), $[B]_* = qH^\infty(d\sigma)$ for some $q \in L^\infty(d\sigma)$ with $|q| = 1$ a.e.-σ, from which it follows that $B = qH^\infty(d\sigma) \cap Z(K)$. We are indebted to the referee for the elegant proof given above.

3. A condition for non-triviality. In the previous section, we obtained a general expression for closed invariant subspaces of $C(X)$. It can happen, however, that for some $q \in L^\infty(d\sigma)$ with $|q| = 1$ a.e.-σ the intersection $qH^\infty(d\sigma) \cap C(X)$ is trivial; i.e. it can contain only the zero function. Now we shall give a condition which ensures the non-triviality of the intersection. First we note the following easy

LEMMA. *Let K be any closed set in X of Lebesgue measure zero. Then $qH^\infty(d\sigma) \cap Z(K)$ is non-trivial if and only if $qH^\infty(d\sigma) \cap C(X)$ is non-trivial.*

So we do not need to take care of $Z(K)$. We have

THEOREM 5. *$qH^\infty(d\sigma) \cap C(X)$ is non-trivial if and only if q has the following factorization:*

$$q = q_1 q_2 q_3, \tag{1}$$

where $|q_j| = 1$ a.e.-σ $(j = 1, 2, 3)$, q_1 is conjugate inner (i.e., $\bar{q}_1$ is an inner function in the sense of Beurling (3, p. 62)), *q_2 is continuous except on a compact set of Lebesgue measure zero, and*

$$q_3(e^{i\theta}) = \exp\left(\frac{i}{2\pi}\int_0^{2\pi}\frac{k(\theta + t) - k(\theta - t)}{2\tan(t/2)}\,dt\right)$$

with a continuous and Lebesgue integrable function k on $[0, 2\pi]$ with values in $[-\infty, \infty)$ such that $k(0) = k(2\pi)$.

Proof. Suppose that $qH^\infty(d\sigma) \cap C(X)$ is non-trivial and take a non-zero function $g \in qH^\infty(d\sigma) \cap C(X)$. Then $g = qf = qf_1 f_2$, where $f = f_1 f_2$ is the factorization of $f \in H^\infty(d\sigma)$ into its inner part f_1 and its outer part f_2. Since g is continuous, so is $|f|(= |g|)$. As $f \in H^\infty(d\sigma) \subseteq H^1(d\sigma)$, $\log|g| = \log|f|$ is integrable. Thus, if we put

$$k(t) = \log|g(e^{it})| = \log|f(e^{it})|,$$

then k is continuous and integrable on $[0, 2\pi]$ with values in $[-\infty, \infty)$ such that $k(0) = k(2\pi)$. Now we know (3, Chap. 5) that

$$f_2(z) = \lambda\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it} + z}{e^{it} - z}\log|f(e^{it})|dt\right)$$
$$= \lambda\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it} + z}{e^{it} - z}k(t)dt\right),$$

and therefore

$$f_2(z) = \lambda\exp\left(\frac{1}{2\pi}\int_0^{2\pi}P(e^{it}, z)k(t)dt\right)\cdot\exp\left(\frac{i}{2\pi}\int_0^{2\pi}Q(e^{it}, z)k(t)dt\right),$$

where λ is a constant of modulus one, P and Q are real functions and satisfy

$$(e^{it} + z)/(e^{it} - z) = P(e^{it}, z) + iQ(e^{it}, z).$$

Passing to radial limits, we have

$$f_2(e^{i\theta}) = \lambda \exp k(\theta) \cdot \exp\left(- \frac{i}{2\pi} \int_0^{2\pi} \frac{k(\theta + t) - k(\theta - t)}{2 \tan(t/2)} dt \right)$$

almost everywhere with respect to σ. Since

$$|g(e^{i\theta})| = |f_2(e^{i\theta})| = \exp k(\theta),$$

we get

$$q = f^{-1}g = \bar{\lambda}\bar{f}_1 \exp(i \arg g) \exp\left(\frac{i}{2\pi} \int_0^{2\pi} \frac{k(\theta + t) - k(\theta - t)}{2 \tan(t/2)} dt\right).$$

So we have only to define q_j ($j = 1, 2, 3$) as follows: $q_1 = \bar{f}_1$, $q_2 = \bar{\lambda} \exp(i \arg g)$, and q_3 is equal to the last factor of the above expression. q_2 is continuous at every point where k is finite. The set of points at which $k = -\infty$ forms a compact set of Lebesgue measure zero because k is continuous with values in $[-\infty, \infty)$ and integrable.

Conversely, suppose that q has a factorization given by (1). Let K be the set of discontinuities of q_2. Then it is compact and of Lebesgue measure zero. By a theorem of Rudin (**3**, p. 81) there exists a non-zero function h_1 in the disk algebra A whose restriction to K is zero. Then $\bar{q}_1 h_1 f_2, f_2$ being equal to

$$\exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} k(t) dt\right),$$

is in $H^\infty(d\sigma)$. It follows from the properties of k that $(q_3 f_2)(e^{i\theta}) = \exp k(\theta)$ is continuous on X. Therefore,

$$q(\bar{q}_1 h_1 f_2) = (q_2 h_1)(q_3 f_2) \in qH^\infty(d\sigma) \cap C(X).$$

It is clear that this function is different from zero. This completes the proof.

Remark. In the proof of Theorem 5, we can take any non-zero function g from $qH^\infty(d\sigma) \cap C(X)$. So our factorization (1) is not unique. We do not know any "standard" factorization of q which is unique.

By the lemma preceding Theorem 5 we have:

COROLLARY. *$qH^\infty(d\sigma) \cap Z(K)$, with $q \in L^\infty(d\sigma)$, $|q| = 1$ a.e.-σ, and $K \subset X$, is non-trivial if and only if q has a factorization of the form* (1) *and the closure of K is of Lebesgue measure zero. If this is the case, then $qH^\infty(d\sigma) \cap Z(K)$ is a uniformly closed simply invariant subspace of $C(X)$.*

The proof is simple and we do not state it here.

4. Uniqueness of the expression.

THEOREM 6. *In the expression $B = qH^\infty(d\sigma) \cap Z(K)$ of a simply invariant closed subspace B given by Theorem 2, the function q is determined uniquely by B up to a constant factor of modulus one.*

Proof. Let q be any function in $L^\infty(d\sigma)$ such that $|q| = 1$ a.e.-σ and $B = qH^\infty(d\sigma) \cap Z(K)$. Then K must be contained in the set K_0 of the common zeros of the functions in B. K_0 is a compact subset of X of Lebesgue measure zero. We define $q_0 \in L^\infty(d\sigma)$, $|q_0| = 1$ a.e.-σ by $[B]_* = q_0 H^\infty(d\sigma)$. Then, as was shown in the remark of Section 2, we have $B = q_0 H^\infty(d\sigma) \cap Z(K_0)$.

Since $B \subseteq qH^\infty(d\sigma)$, we have $q_0 H^\infty(d\sigma) = [B]_* \subseteq qH^\infty(d\sigma)$ because $qH^\infty(d\sigma)$ is weakly* closed in $L^\infty(d\sigma)$. This means that $\bar{q}q_0 \in H^\infty(d\sigma)$. If we put $p = \bar{q}q_0$, then p is an inner function. We shall show that p is a constant function.

Since $q_0 H^\infty(d\sigma) \cap Z(K_0)$ is non-trivial, the corollary to Theorem 5 says that q_0 admits a factorization of the form (1):

$$q_0 = q_1 q_2 q_3,$$

where $|q_j| = 1$ a.e.-σ $(j = 1, 2, 3)$, $\bar{q}_1$ is inner, q_2 is continuous except on a compact set K' in X of Lebesgue measure zero, and

$$q_3(e^{i\theta}) = \exp\left(\frac{i}{2\pi} \int_0^{2\pi} \frac{k(\theta + t) - k(\theta - t)}{2 \tan(t/2)} dt\right)$$

with a continuous and Lebesgue integrable function k with values in $[-\infty, \infty)$ such that $k(0) = k(2\pi)$. So we have $q = \bar{p}q_0 = \bar{p}q_1 q_2 q_3$. Since p is inner, $p = p_b p_s$, where p_b is a Blaschke product and p_s is a singular function. First we shall show that

(i) *p_b is a constant function.* Suppose otherwise. Then p_b has at least one zero α in the open unit disk. It is easy to see that there exists a function h in $H^\infty(d\sigma)$ such that qh is in B and h does not vanish at α. As

$$B = q_0 H^\infty(d\sigma) \cap Z(K_0),$$

there exists a function $g \in H^\infty(d\sigma)$ such that $qh = q_0 g$. So $h = pg$. But this is a contradiction because pg vanishes at α but h does not. Hence p_b is constant, as was to be proved.

(ii) *p_s is a constant function.* Now we can assume $p = p_s$ by discarding p_b. Suppose, on the contrary, that p_s is not constant; then there exists a positive singular measure μ and a constant λ of modulus one such that

$$p(z) = p_s(z) = \lambda \exp\left(-\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)\right).$$

By the singularity of μ, there exists a compact subset K'' of X of Lebesgue measure zero such that $\mu(K'') > 0$. We set

$$p'(z) = \exp\left(-\frac{1}{2\pi} \int_{K''} \frac{e^{it} + z}{e^{it} - z} d\mu(t)\right)$$

and $p'' = \bar{p}'p$. Both p' and p'' are singular functions and $p = p'p''$. We know (**3**, pp. 68–69) that p' is continuous except on K''. Furthermore, let q_b, q_s be the Blaschke factor and the singular factor of the inner function $\bar{q}_1$, respectively. Let ν be a positive singular measure on X and λ' a constant of modulus one such that

$$q_s(z) = \lambda' \exp\left(-\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,d\nu(t)\right).$$

Define

$$q'(z) = \exp\left(-\frac{1}{2\pi}\int_{K''}\frac{e^{it}+z}{e^{it}-z}\,d\nu(t)\right)$$

and set $q'' = \bar{q}'q_s$. Thus we have $q_s = q'q''$. Of course, it may happen that some of these functions are constant.

On the other hand, it is known—cf. Hoffman (**3**, p. 80) or Rudin (**5**, p. 433)—that, for any compact subset S of X of Lebesgue measure zero, there exists a non-constant outer function which is continuous on the closed unit disk and vanishes on S. Since the union $K_0 \cup K' \cup K''$ is compact and of Lebesgue measure zero, there exists a non-constant outer function h_1 which is continuous on the closed unit disk and vanishes on $K_0 \cup K' \cup K''$. We put

$$f = p''q_b\,q''h_1\,h_2,$$

where

$$h_2(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,k(t)dt\right).$$

Then $f \in H^\infty(d\sigma)$ because each factor of f is in $H^\infty(d\sigma)$. Moreover, qf is continuous on X because

$$qf = (\bar{p}'\bar{q}'q_2h_1)(q_3h_2)$$

and each of two factors on the right-hand side is continuous by the definition of h_1 and h_2. As h_1 vanishes on K_0, qf vanishes on $K \subseteq K_0$; so qf is in $qH^\infty(d\sigma) \cap Z(K)$. Since

$$qH^\infty(d\sigma) \cap Z(K) = q_0\,H^\infty(d\sigma) \cap Z(K_0),$$

there exists $g \in H^\infty(d\sigma)$ such that $qf = q_0\,g$. Therefore $\bar{p}f = g$. Because of the form of f, we have

$$q_b\,q''h_1\,h_2 = p'g.$$

Since both sides are functions in $H^\infty(d\sigma)$, their inner parts must be identical up to a constant factor of modulus one. This implies that p' divides $q_b\,q''$. But this is impossible; indeed, since p' is singular, p' must divide the singular part of $q_b\,q''$, which is q''. Now the supports of measures corresponding to

p' and q'' have no point in common, so that q'' is not divisible by p'. This contradiction shows that p_s must also be a constant function. Hence $q = q_0$ up to a constant factor of modulus one.

Remark. We shall give two examples. (These examples came out of a discussion with Professors Helson and Katznelson.) The closed set K in the expression $B = qH^\infty(d\sigma) \cap Z(K)$ is not necessarily unique. This is shown by the following example.

Example 1. Consider the following function:

$$q(e^{i\theta}) = \begin{cases} 1 & \text{for } 0 < \theta < \pi, \\ -1 & \text{for } \pi < \theta < 2\pi. \end{cases}$$

Then the two-point set $\{1, -1\}$ is the set of the common zeros of the functions in $qH^\infty(d\sigma) \cap C(X)$. In fact, let $h \in H^\infty(d\sigma)$ be such that qh is continuous. If h were not zero at 1, then the real part $\mathrm{Re}(h)$ of h would have a jump at 1; so the conjugate function $\mathrm{Im}(h)$ would be unbounded, contrary to the boundedness of h. Therefore $h(1) = 0$. Likewise, $h(-1) = 0$. Thus

$$qH^\infty(d\sigma) \cap C(X) = qH^\infty(d\sigma) \cap Z(K),$$

where K is any one of the following sets: the empty set, $\{1\}$, $\{-1\}$, $\{1, -1\}$.

If we modify the function in the above example, then we can get a function $q \in L^\infty(d\sigma)$, $|q| = 1$ a.e.-σ, such that $qH^\infty(d\sigma) \cap C(X)$ is trivial.

Example 2. It is well known that there exists a monotonically (real-valued) increasing function $m(\theta)$, defined on $[0, 2\pi]$, whose discontinuities are dense in this interval. Put $q(e^{i\theta}) = \exp(im(\theta))$. Then q has jumps on a dense set. As was stated in Example 1, if qh, $h \in H^\infty(d\sigma)$, is continuous, then h must vanish at each discontinuity of q. So h must vanish identically and consequently $qH^\infty(d\sigma) \cap C(X)$ is trivial. This example also shows that our Theorem 5 is not redundant.

Finally we note that $qH_0^1(d\sigma)$ is weakly* dense in $M(X)$ if and only if $\bar{q}H^\infty(d\sigma) \cap C(X)$ is trivial. If $qH_0^1(d\sigma)$ is not weakly* dense in $M(X)$, then the weak* closure N of $qH_0^1(d\sigma)$ in $M(X)$ is simply invariant and, by Theorems 4 and 6, equal to $qH_0^1(d\sigma) + M(K)$, where K denotes the set of the common zeros of the functions in $\bar{q}H^\infty(d\sigma) \cap C(X)$. By Theorem 5 this is the case if and only if q has a factorization of the following sort: $q = q_1 q_2 q_3$, where $|q_j| = 1$ a.e.-σ $(j = 1, 2, 3)$, q_1 is inner, q_2 is continuous except on a compact set of Lebesgue measure zero, and

$$q_3(e^{i\theta}) = \exp\left(-\frac{i}{2\pi} \int_0^{2\pi} \frac{k(\theta + t) - k(\theta - t)}{2\tan(t/2)}\, dt \right)$$

with a continuous and Lebesgue integrable function k on $[0, 2\pi]$ with values in $[-\infty, \infty)$ such that $k(0) = k(2\pi)$. $qH_0^1(d\sigma)$ is weakly* closed if and only if $\bar{q}H^\infty(d\sigma) \cap C(X)$ has no common zeros.

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