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HELICOIDAL MINIMAL SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$

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Abstract

It is shown that a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ is invariant under a one-parameter group of screw motions if and only if it lies in the associate family of helicoids. It is also shown that the conjugate surfaces of the parabolic and hyperbolic helicoids in $\mathbb{H}^2 \times \mathbb{R}$ are certain types of catenoids.

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1. Introduction

It is well known that a minimal surface in Euclidean 3-space $\mathbb{R}^3$ can be deformed isometrically to a minimal surface. The isometrically deformed minimal surface is called *associate*. In fact, the isometric deformation is obtained by rotating the shape operator. When the rotation angle is $\pi/2$, the two surfaces are called *conjugate*. A famous example of conjugate minimal surfaces in $\mathbb{R}^3$ is catenoids and helicoids. It can be shown that the associate surfaces of helicoids (hence of catenoids) are invariant under a one-parameter group of screw motions in $\mathbb{R}^3$. Therefore one may ask if a minimal surface invariant under a one-parameter group of screw motions is an associate of a helicoid. Regarding this question, it was shown by H. A. Schwarz that if two (open, simply connected) minimal surfaces in $\mathbb{R}^3$ are isometric, then they are associate (see [8, p. 166]).

In this paper, we will discuss the corresponding question in the product space $\mathbb{H}^2 \times \mathbb{R}$. In fact, associate minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ have been well studied after a series of works [1, 7, 9]. In particular, the existence of the minimal associate family was shown in [2, 4]. It was also shown in [11] that if two minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ invariant under a one-parameter group of elliptic screw motions are isometric, then

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TABLE 1. Conjugate surfaces to the elliptic helicoids in $\mathbb{H}^2 \times \mathbb{R}$.

Eliptic helicoid $EH_\beta, \beta > 1$	Elliptic catenoid EC_α	$\beta^2 - \alpha^2 = 1$
Elliptic helicoid EH_1	Parabolic catenoid PC	
Elliptic helicoid $EH_\beta, \beta < 1$	Hyperbolic catenoid HC_α, $\alpha < 1$	$\beta^2 + \alpha^2 = 1$

they are associate, a partial result similar to that of H. A. Schwarz. Other partial results for minimal surfaces invariant under a one-parameter group of parabolic or hyperbolic screw motions were given in [10]. But these results split the three cases of screw motions and do not show the interrelation between them.

We give in this paper a complete description of the minimal surfaces invariant under screw motions. In fact, both minimal surfaces and the surfaces invariant under a one-parameter group of screw motions in $\mathbb{H}^2 \times \mathbb{R}$ give a solution to the Bonnet problem: it was shown in [3] that if two real analytic surfaces Σ_1, Σ_2 are Bonnet mates, then both Σ_1 and Σ_2 are either minimal surfaces or parts of surfaces invariant under a one-parameter group of screw motions. We also give a complete set of conjugate relations between these invariant minimal surfaces. It is interesting that some of the minimal surfaces invariant under a one-parameter group of elliptic screw motions have some of their associate (and conjugate) surfaces which are invariant under a one-parameter group of parabolic or hyperbolic screw motions. Our proof follows the geometric existence argument of associate surfaces of Daniel [2], which is summarised in Section 2.1.

We show in Theorem 2.6 that a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ is helicoidal if and only if it lies in the associate family of helicoids. A surface is called *helicoidal* if it is invariant under a one-parameter group of screw motions in $\mathbb{H}^2 \times \mathbb{R}$. Screw motions in $\mathbb{H}^2 \times \mathbb{R}$ are rigid motions generated by rotations in $\mathbb{H}^2$ and vertical translations along $\mathbb{R}$. The same theorem for the maximal surfaces in the three-dimensional Lorentz–Minkowski space was proven in [6]. Whereas the Bjöling representation formula is a crucial tool for the proof there, we compute the Killing vector fields explicitly in terms of the conformal parameter of the surface, which gives a helicoidal motion in $\mathbb{H}^2 \times \mathbb{R}$ under which the surface is invariant. As our method is constructive in a sense, it applies to the case of minimal surfaces in $\mathbb{R}^3$ as well.

It was also shown in [2] that, for surfaces in $\mathbb{H}^2 \times \mathbb{R}$, the elliptic helicoids (ruled minimal surfaces invariant under the elliptic screw motion) EH_β are conjugate to the elliptic catenoids EC_α, the parabolic catenoid PC or the hyperbolic catenoids HC_α according to the size of the 'rotating speed' α of the catenoids and the 'pitch' β of the helicoids. Table 1 summarises the result.

Elliptic (parabolic, hyperbolic, respectively) catenoids are minimal surfaces invariant under elliptic (parabolic, hyperbolic, respectively) rotations. As there are two more kinds of minimal helicoids in $\mathbb{H}^2 \times \mathbb{R}$, namely, the parabolic helicoid (ruled minimal surface invariant under the parabolic screw motion) PH and the

Table 2. Conjugate correspondence between helicoids and catenoids.

$\mathrm{EH}_{\beta>1}$	EC_α	$\beta^2 - \alpha^2 = 1$
EH_1	PC	
$\mathrm{EH}_{\beta<1}$	$\mathrm{HC}_{\alpha<1}$	$\beta^2 + \alpha^2 = 1$
PH	HC_1	
HH_β	$\mathrm{HC}_{\alpha>1}$	$\alpha^2 - \beta^2 = 1$

hyperbolic helicoids (ruled minimal surfaces invariant under the hyperbolic screw motion) HH_β [5], it seems natural to ask what the conjugate minimal surfaces of PH and HH_β are. We show in Theorem 3.2 in the last section that they are certain hyperbolic catenoids, to complete the correspondence table as in Table 2.

2. Helicoidal minimal surfaces

In this section, we show that a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ is helicoidal if and only if it lies in the associate family of helicoids. For this purpose, the following theorem in [2] is crucial.

2.1. Daniel's theorem. Let Σ be a simply connected Riemann surface and $X : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ a conformal minimal immersion. Let N be the induced unit normal vector field and S the symmetric operator on Σ induced by the shape operator of $X(\Sigma)$. Let ξ be the vertical unit vector field (corresponding to the factor $\mathbb{R}$) and T be the vector field on Σ such that $dX(T)$ is the projection of ξ onto $T(X(\Sigma))$ and $\nu = \langle N, \xi \rangle$. Let $z_0 \in \Sigma$. Then there exists a unique continuous family $X_\theta, \theta \in \mathbb{R}$, of conformal immersions $X_\theta : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ such that:

(i) $X_\theta(z_0) = X(z_0)$, $(dX_\theta)_{z_0} = (dX)_{z_0}$;
(ii) the metrics induced on Σ by X and X_θ are the same;
(iii) the symmetric operator on Σ induced by the shape operator of $X_\theta(\Sigma)$ is $e^{\theta J} S$, where J is the complex structure on Σ;
(iv) $\xi = dX_\theta(e^{\theta J} T) + \nu N_\theta$, where N_θ is the unit normal to X_θ.

2.2. A conformal parametrisation of helicoidal surfaces. Let Σ be a simply connected Riemann surface and $X : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ be a helicoidal immersion. A natural parametrisation $X(u, v) : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ for such a helicoidal immersion can be introduced by letting the v-curves be the trajectories of the helicoidal motion, that is, orbits, and letting the u-curves be their orthogonal trajectories. Then

$$\langle X_u, X_v \rangle = 0.$$

Since $\|X_v\|$ is constant along the orbit, one can assume, by reparametrising u if necessary, that

$$\|X_v\|^2 = \|X_u\|^2 := \lambda(u)$$

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on Σ. For notational convention, we write

$$X_{uu} := \nabla_{X_u} X_u, \quad X_{vv} := \nabla_{X_v} X_v, \quad X_{uv} := \nabla_{X_v} X_u, \quad X_{vu} := \nabla_{X_u} X_v.$$

Then, since $[X_u, X_v] = 0$, we have $X_{uv} = X_{vu}$. If, furthermore, X is minimal, then

$$X_{uu} + X_{vv} = 0.$$

2.3. Helicoidal minimal surfaces. Now let $X(u, v) : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ be a conformal minimal helicoidal immersion with the above parametrisation. Then the following result holds.

Proposition 2.1. *Both $\langle X_u, \xi \rangle$ and $\langle X_v, \xi \rangle$ are constant functions and $\lambda(u)$ is a convex function.*

Proof. The function $\langle X_v, \xi \rangle$ is a constant function from the choice of the parameter v. Then, since $X(\Sigma)$ is minimal and ξ is a parallel vector field,

$$\begin{aligned}\frac{\partial}{\partial u}\langle X_u, \xi \rangle &= \langle X_{uu}, \xi \rangle = -\langle X_{vv}, \xi \rangle = -\frac{\partial}{\partial v}\langle X_v, \xi \rangle = 0,\\ \frac{\partial}{\partial v}\langle X_u, \xi \rangle &= \langle X_{uv}, \xi \rangle = \langle X_{vu}, \xi \rangle = \frac{\partial}{\partial u}\langle X_v, \xi \rangle = 0,\end{aligned}$$

which shows that $\langle X_u, \xi \rangle$ is a constant function.

Now let R be the curvature tensor of $\mathbb{H}^2 \times \mathbb{R}$. Then

$$\frac{d\lambda(u)}{du} = \frac{\partial}{\partial u}\langle X_v, X_v \rangle = 2\langle X_{vu}, X_v \rangle = 2\langle X_{uv}, X_v \rangle.$$

Let K be the sectional curvature of the section generated by X_u and X_v. Since a sectional curvature of the space $\mathbb{H}^2 \times \mathbb{R}$ is nonpositive and $X(\Sigma)$ is minimal,

$$\begin{aligned}\frac{1}{2}\frac{d^2\lambda}{du^2} &= \frac{\partial}{\partial u}\langle X_{uv}, X_v \rangle\\ &= \langle X_{uvu}, X_v \rangle + \langle X_{uv}, X_{uv} \rangle\\ &= \langle X_{uvu} - X_{uuv}, X_v \rangle + \langle X_{uuv}, X_v \rangle + \langle X_{uv}, X_{uv} \rangle\\ &= R(X_u, X_v, X_u, X_v) + \frac{\partial}{\partial v}\langle X_{uu}, X_v \rangle - \langle X_{uu}, X_{vv} \rangle + \langle X_{uv}, X_{uv} \rangle\\ &= R(X_u, X_v, X_u, X_v) - \frac{\partial}{\partial v}\langle X_{vv}, X_v \rangle + \langle X_{uu}, X_{uu} \rangle + \langle X_{uv}, X_{uv} \rangle\\ &= -K\|X_u \wedge X_v\|^2 - \frac{1}{2}\langle X_v, X_v \rangle_{vv} + \langle X_{uu}, X_{uu} \rangle + \langle X_{uv}, X_{uv} \rangle\\ &= -K\|X_u \wedge X_v\|^2 + \langle X_{uu}, X_{uu} \rangle + \langle X_{uv}, X_{uv} \rangle \geq 0.\end{aligned}$$

This completes the proof. □

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Now let $\partial_u, \partial_v, T \in T\Sigma$ be vector fields satisfying $dX(\partial_u) = X_u$, $dX(\partial_v) = X_v$, and $dX(T)$ is the projection of ξ onto $T(X(\Sigma))$. Let N be the induced unit normal vector field and let $\nu = \langle N, \xi\rangle$. Set $\langle X_u, \xi\rangle = a$ and $\langle X_v, \xi\rangle = b$. Then, since $\|\xi\| = 1$,

$$\begin{aligned}\xi &= \frac{\langle X_u, \xi\rangle}{\|X_u\|^2} X_u + \frac{\langle X_v, \xi\rangle}{\|X_v\|^2} X_v + \langle N, \xi\rangle N \\ &= \frac{a}{\lambda(u)} X_u + \frac{b}{\lambda(u)} X_v + \nu N,\end{aligned}$$

which implies that

$$\begin{aligned}T &= \frac{a}{\lambda(u)} \partial_u + \frac{b}{\lambda(u)} \partial_v, \\ \nu^2 &= \frac{\lambda(u) - (a^2 + b^2)}{\lambda(u)}.\end{aligned}$$

2.4. Associated immersions. Let $X : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ be the conformal helicoidal minimal immersion given in Section 2.3 and let Y be the associated minimal immersion to X corresponding to the angle θ. Then, from (ii) of Section 2.1,

$$\|Y_u\|^2 = \|Y_v\|^2 = \lambda(u).$$

Let $T_\theta \in T\Sigma$ be the vector field such that $dY(T_\theta)$ is the projection of ξ into $T(Y(\Sigma))$, N_θ the induced unit normal vector field and $\nu_\theta = \langle N_\theta, \xi\rangle$. Then, from (iv) of Section 2.1,

$$T_\theta = \frac{a\cos\theta - b\sin\theta}{\lambda(u)} \partial_u + \frac{a\sin\theta + b\cos\theta}{\lambda(u)} \partial_v$$

and

$$\nu_\theta^2 = \frac{\lambda(u) - (a^2 + b^2)}{\lambda(u)}.$$

Then, since $Y_u = (\cos\theta)X_u - (\sin\theta)X_v$, $Y_v = (\sin\theta)X_u + (\cos\theta)X_v$,

$$\begin{aligned}\langle Y_u, \xi\rangle &= a\cos\theta - b\sin\theta, \\ \langle Y_v, \xi\rangle &= a\sin\theta + b\cos\theta.\end{aligned}$$

Let $\pi : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2$ be the horizontal projection. Then the horizontal components $Y_u^H := d\pi(Y_u)$, $Y_v^H := d\pi(Y_v)$ are given by

$$Y_u^H = Y_u - (a\cos\theta - b\sin\theta)\xi, \tag{2.1}$$

$$Y_v^H = Y_v - (a\sin\theta + b\cos\theta)\xi. \tag{2.2}$$

Since ξ is a parallel field,

$$Y_{uu}^H = Y_{uu}, \quad Y_{uv}^H = Y_{uv} = Y_{vu} = Y_{vu}^H, \quad Y_{vv}^H = Y_{vv}. \tag{2.3}$$

Note first that two vector fields Y_u^H, Y_v^H are linearly dependent if and only if $N_\theta \perp \xi$, that is, if and only if $\lambda(u) = a^2 + b^2$. Now suppose that there exists a point u_0 such that $\lambda(u_0) = a^2 + b^2$. Then, since the function $\lambda(u)$ is convex, it follows that either $\lambda(u) = a^2 + b^2$ on an interval I containing u_0 or $\lambda(u) \neq a^2 + b^2$ if $u \neq u_0$.

Proposition 2.2. *Suppose that $\lambda(u) = a^2 + b^2$ on an interval I. Then the surface $Y(\Sigma)$ is a part of a vertical plane.*

Proof. Since λ is constant on I, we have from (2.2) that

$$(a \sin\theta + b\cos\theta)Y_u^H + (a\cos\theta - b\sin\theta)Y_v^H = 0,$$

which gives

$$\begin{aligned}(a \sin\theta + b\cos\theta)Y_{uu}^H + (a\cos\theta - b\sin\theta)Y_{vu}^H &= 0,\\ (a \sin\theta + b\cos\theta)Y_{uv}^H + (a\cos\theta - b\sin\theta)Y_{vv}^H &= 0.\end{aligned}$$

From (2.3),

$$\begin{aligned}(a \sin\theta + b\cos\theta)Y_{uu} + (a\cos\theta - b\sin\theta)Y_{vu} &= 0,\\ (a \sin\theta + b\cos\theta)Y_{uv} + (a\cos\theta - b\sin\theta)Y_{vv} &= 0.\end{aligned}$$

Combining these with the minimality equation $Y_{uu} + Y_{vv} = 0$ leads to

$$Y_{uu} = Y_{vv} = 0, \quad Y_{uv} = 0,$$

which implies that the surface $\{Y(u, v) : u \in I\}$ is totally geodesic and hence is a vertical plane by [5]. Now the elliptic regularity theorem gives that the whole surface $Y(\Sigma)$ is (a part of) a vertical plane. □

Now assume that $Y(\Sigma)$ is not a vertical plane and suppose that $\lambda(u_0) = a^2 + b^2$ and $\lambda(u) \neq a^2 + b^2$ if $u \neq u_0$. Let us set $\Sigma^+ = \{(u, v) \in \Sigma : u > u_0\}$, $\Sigma^0 = \{(u_0, v) \in \Sigma\}$ and $\Sigma^- = \{(u, v) \in \Sigma : u < u_0\}$. For $p \in Y(\Sigma^+)$, there is a neighbourhood U_p, $p \in U_p \subset Y(\Sigma^+)$, such that:

(a) π is a diffeomorphism on U_p;
(b) Y_u^H, Y_v^H are linearly independent on U_p.

Then one can regard Y_u^H, Y_v^H restricted to U_p as vector fields on $\pi(U_p) \subset \mathbb{H}^2$.

Proposition 2.3. *There is a Killing field $\mathcal{U}$ of $\mathbb{H}^2$ such that $Y_v^H|_{\pi(U_p)} = \mathcal{U}|_{\pi(U_p)}$.*

Proof. We are to show that Y_v^H satisfies the Killing equation. To do so, since Y_u^H, Y_v^H are linearly independent, it is enough to show that

$$\langle Y_{vv}^H, Y_v^H\rangle = \langle Y_{vu}^H, Y_u^H\rangle = 0, \quad \langle Y_{vu}^H, Y_v^H\rangle + \langle Y_{vv}^H, Y_u^H\rangle = 0.$$

Since $Y_{uu}^H = Y_{uu}$, $Y_{uv}^H = Y_{uv}$, $Y_{vv}^H = Y_{vv}$,

$$\begin{aligned}\langle Y_{vv}^H, Y_v^H\rangle &= \langle Y_{vv}^H, Y_v\rangle = \langle Y_{vv}, Y_v\rangle = \frac{1}{2}\frac{\partial}{\partial v}\langle Y_v, Y_v\rangle = \frac{1}{2}\frac{\partial}{\partial v}\lambda(u) = 0,\\ \langle Y_{vu}^H, Y_u^H\rangle &= \langle Y_{vu}, Y_u\rangle = \langle Y_{uv}, Y_u\rangle = \frac{1}{2}\frac{\partial}{\partial v}\langle Y_u, Y_u\rangle = \frac{1}{2}\frac{\partial}{\partial v}\lambda(u) = 0,\end{aligned}$$

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and

$$\langle Y^H_{vu}, Y^H_v\rangle + \langle Y^H_{vv}, Y^H_u\rangle = \langle Y_{vu}, Y_v\rangle + \langle Y_{vv}, Y_u\rangle = \langle Y_{uv}, Y_v\rangle + \langle Y_u, Y_{vv}\rangle$$
$$= \frac{\partial}{\partial v}\langle Y_u, Y_u\rangle = \frac{\partial}{\partial v}\lambda(u) = 0.$$

This completes the proof. □

Let U, V be the two neighbourhoods satisfying (a) and (b). Then there are Killing fields $\mathcal{U}, \mathcal{V}$ of $\mathbb{H}^2$ such that

$$\mathcal{U}|_{\pi(U)} = Y^H_v|_{\pi(U)}, \quad \mathcal{V}|_{\pi(V)} = Y^H_v|_{\pi(V)}.$$

Since $\mathcal{U}|_{\pi(U\cap V)} = Y^H_v|_{\pi(U\cap V)} = \mathcal{V}|_{\pi(U\cap V)}$, it follows that the two Killing fields $\mathcal{U}$ and $\mathcal{V}$ are in fact the same field. Hence one can see that the vector field $Y^H_v|_{\Sigma^+}$ is a well-defined vector field on $\pi(Y(\Sigma^+)\subset \mathbb{H}^2$ and is a restriction of a Killing field on $\mathbb{H}^2$. Moreover, since the vector field Y^H_v is C^∞ on the line (u_0, v), one can see that the vector field $Y^H_v|_{\Sigma^+\cup\Sigma^0}$ is (a restriction of) a Killing field on $\mathbb{H}^2$. In exactly the same way, one can see that the vector field $Y^H_v|_{\Sigma^-\cup\Sigma^0}$ is (a restriction of) a Killing field on $\mathbb{H}^2$. Since these two Killing vector fields are the same on Σ^0, they are in fact the same Killing field. Hence we have the following result.

Proposition 2.4. *An associate minimal surface to a helicoidal minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ is helicoidal.*

Moreover, the following result is evident.

Proposition 2.5. *For every helicoidal minimal immersion $X : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$, there exists a helicoid associated to X.*

Proof. Let $Y : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ be the immersion associated to X corresponding to the angle θ. Choose θ in (2.1) so that

$$\langle Y_u, \xi\rangle = a\cos\theta - b\sin\theta = 0.$$

Then

$$\xi = \frac{\sqrt{a^2+b^2}}{\lambda(u)}Y_v + \frac{\sqrt{\lambda(u)-(a^2+b^2)}}{\lambda(u)}N.$$

Now, from the minimality, we have

$$\langle Y_{uu}, Y_v\rangle = -\langle Y_{vv}, Y_v\rangle = -\frac{\partial}{\partial v}\langle Y_v, Y_v\rangle = -\frac{\partial}{\partial v}\lambda(u) = 0.$$

Moreover, since $\langle Y_u, \xi\rangle = 0$,

$$\langle Y_{uu}, \xi\rangle = 0.$$

Hence for every v, the u-curve $Y(\cdot, v)$ is a horizontal geodesic and $Y(\Sigma)$ is a horizontally ruled minimal surface, which must be a helicoid by [5]. □

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Summarising this subsection, we have the following theorem.

THEOREM 2.6. *A minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ is helicoidal if and only if it lies in the associate family of helicoids (or a catenoid).*

3. Conjugate surfaces of helicoids in $\mathbb{H}^2 \times \mathbb{R}$

In this section we compute the conjugate surfaces of the parabolic helicoid and hyperbolic helicoids.

By taking the hyperboloid model of $\mathbb{H}^2$, let us consider $\mathbb{H}^2 \times \mathbb{R}$ as the hypersurface in the Lorentz–Minkowski space $\mathbb{R}^{3,1}$ given by the equation $-x^2 + y^2 + z^2 = -1$, $x > 0$. We first give the conformal parametrisations of the parabolic helicoids PH, the hyperbolic helicoids HH_β, and the hyperbolic catenoids HC_α. Conformal parametrisations of any other types of helicoids and catenoids are given in [2].

3.1. Parabolic helicoids. It was shown in [5] that every parabolic helicoid has the parametrisation

$$X_b(t, s) = \begin{bmatrix} \cosh t + \frac{1}{2}s^2 e^{-t} \\ \sinh t + \frac{1}{2}s^2 e^{-t} \\ s e^{-t} \\ bs \end{bmatrix}$$

for a constant $b \neq 0$. We first show that all parabolic helicoids are congruent.

PROPOSITION 3.1. *Any parabolic helicoid X_b, $b \neq 0$, is congruent to the parabolic helicoid X_1.*

PROOF. One may assume that $b = e^a > 0$ for some a. Note first that

$$\begin{bmatrix} \cosh a & \sinh a & 0 & 0 \\ \sinh a & \cosh a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

gives a congruence motion in $\mathbb{H}^2 \times \mathbb{R}$. Reparametrising X_b by $\tilde{s} = e^a s$, $\tilde{t} = t + a$ and moving by the congruence in $\mathbb{H}^2 \times \mathbb{R}$,

$$\begin{bmatrix} \cosh a & \sinh a & 0 & 0 \\ \sinh a & \cosh a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh t + \frac{1}{2}s^2 e^{-t} \\ \sinh t + \frac{1}{2}s^2 e^{-t} \\ s e^{-t} \\ bs \end{bmatrix} = \begin{bmatrix} \cosh \tilde{t} + \frac{1}{2}\tilde{s}^2 e^{-\tilde{t}} \\ \sinh \tilde{t} + \frac{1}{2}\tilde{s}^2 e^{-\tilde{t}} \\ \tilde{s} e^{-\tilde{t}} \\ \tilde{s} \end{bmatrix} = X_1(\tilde{t}, \tilde{s}).$$

This completes the proof. □

This proposition can be observed more easily by using the upper half-plane model for $\mathbb{H}^2$ as follows: considering $\mathbb{H}^2 \times \mathbb{R}$ as $\{(x, y, z) \mid y > 0\}$ with the metric $ds^2 = (dx^2 + dy^2)/y^2 + dz^2$, the parabolic helicoids can be given by the equations $z = bx$, $b \neq 0$

since the map $(x, y) \mapsto (x + t, y)$ is the parabolic rotation in $\mathbb{H}^2$. Noting that the map $(x, y) \mapsto (\lambda x, \lambda y)$, $\lambda > 0$, is the hyperbolic rotation in $\mathbb{H}^2$, one can see that the surface $z = bx$ is congruent to the surface $z = x$ through the isometry $(x, y, z) \mapsto (bx, by, z)$ of $\mathbb{H}^2 \times \mathbb{R}$.

Now consider the parametrisation of the parabolic helicoid

$$\begin{bmatrix} \cosh f(u) + \frac{1}{2}v^2 e^{-f(u)} \\ \sinh f(u) + \frac{1}{2}v^2 e^{-f(u)} \\ ve^{-f(u)} \\ v \end{bmatrix}$$

which is conformal if

$$f'(u)^2 = 1 + e^{-2f(u)}.$$

A solution of this equation is $f(u) = \log \sinh u$, $u > 0$. Then we have the conformal parametrisation of the parabolic helicoid

$$\mathrm{PH}(u, v) = \frac{1}{2 \sinh u} \begin{bmatrix} \sinh^2 u + 1 + v^2 \\ \sinh^2 u - 1 + v^2 \\ 2v \\ 2v \sinh u \end{bmatrix}, \quad u > 0.$$

The induced metric is written as

$$ds^2 = \frac{\cosh^2 u}{\sinh^2 u}(du^2 + dv^2) \tag{3.1}$$

and the normal to $\mathbb{H}^2 \times \mathbb{R}$ in $\mathbb{R}^{3,1}$ is

$$\bar{N}(u, v) = \frac{1}{2 \sinh u} \begin{bmatrix} \sinh^2 u + 1 + v^2 \\ \sinh^2 u - 1 + v^2 \\ 2v \\ 0 \end{bmatrix}.$$

The unit normal vector field $N(u, v)$ of PH in $\mathbb{H}^2 \times \mathbb{R}$ is computed as

$$N(u, v) = \frac{1}{\cosh u} \begin{bmatrix} v \sinh u \\ v \sinh u \\ \sinh u \\ -1 \end{bmatrix}$$

and the matrix of the shape operator S_{PH} with respect to $\partial/\partial u, \partial/\partial v$ is computed as

$$S_{\mathrm{PH}} = -\frac{\sinh u}{\cosh^2 u} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

from which we can see that the immersion PH is minimal. Furthermore, the tangential component T_{PH} and the normal component ν_{PH} of $\partial/\partial t$ are computed as

$$T_{\mathrm{PH}} = \frac{\sinh^2 u}{\cosh^2 u} \frac{\partial}{\partial v}, \qquad \nu_{\mathrm{PH}} = -\frac{1}{\cosh u}.$$

3.2. Hyperbolic helicoids. Since the hyperbolic rotation in $\mathbb{H}^2$ in the hyperboloid model is represented as

$$\begin{bmatrix} \cosh t & \sinh t & 0 \\ 0 & 0 & 1 \\ \sinh t & \cosh t & 0 \end{bmatrix},$$

we have the following parametrisation of hyperbolic helicoids:

$$\mathrm{HH}_\beta(u, v) = \begin{bmatrix} \cosh\beta v & \sinh\beta v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\beta v & \cosh\beta v & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh f(u) \\ 0 \\ \sinh f(u) \\ v \end{bmatrix} = \begin{bmatrix} \cosh f(u)\cosh\beta v \\ \sinh f(u) \\ \cosh f(u)\sinh\beta v \\ v \end{bmatrix}$$

which is conformal if

$$f'(u)^2 = \beta^2 \cosh^2 f(u) + 1. \tag{3.2}$$

The normal to $\mathbb{H}^2 \times \mathbb{R}$ in $\mathbb{R}^{3,1}$ is

$$\bar{N}(u, v) = \begin{bmatrix} \cosh f(u)\cosh\beta v \\ \sinh f(u) \\ \cosh f(u)\sinh\beta v \\ 0 \end{bmatrix}.$$

The unit normal vector field $N(u, v)$ of HH_β in $\mathbb{H}^2 \times \mathbb{R}$ is computed as

$$N(u, v) = -\frac{1}{f'(u)} \begin{bmatrix} \sinh\beta v \\ 0 \\ \cosh\beta v \\ -\beta\cosh f(u) \end{bmatrix}.$$

Now assume that $\mathrm{HH}_\beta(u, v)$ is conformal. Then the matrix of the shape operator S_{HH_β} with respect to $\partial/\partial u$, $\partial/\partial v$ is computed as

$$S_{\mathrm{HH}_\beta} = -\frac{\beta\sinh f(u)}{f'(u)^2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

from which we see that the immersion HH_β is minimal. Furthermore, the tangential component T_{HH_β} and the normal component ν_{HH_β} of $\partial/\partial t$ are computed as

$$T_{\mathrm{HH}_\beta} = \frac{1}{f'^2(u)}\frac{\partial}{\partial v}, \qquad \nu_{\mathrm{HH}_\beta} = -\frac{\beta\cosh f(u)}{f'(u)}.$$

3.3. Hyperbolic catenoids. The conformal parametrisations of the hyperbolic helicoids HC_α and the geometric properties when $0 < \alpha < 1$ are given in [2]. However, we repeat the computation here for completeness and explain some of the geometric properties when $\alpha \geq 1$.

Let Π be a vertical geodesic plane containing the origin of $\mathbb{H}^2$ and let γ be a smooth curve in Π. Assume that γ is a graph over the t axis. Let u denote the height along the t axis. Then the curve γ can be written as

$$\begin{bmatrix} \cosh r(u) \\ 0 \\ \sinh r(u) \\ u \end{bmatrix}$$

for a smooth function r. Then we have the following parametrisation of the hyperbolic catenoids HC_α:

$$\begin{aligned} \mathrm{HC}_\alpha(u, v) &= \begin{bmatrix} \cosh \alpha v & \sinh \alpha v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \alpha v & \cosh \alpha v & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh r(u) \\ 0 \\ \sinh r(u) \\ u \end{bmatrix} \\ &= \begin{bmatrix} \cosh r(u) \cosh \alpha v \\ \sinh r(u) \\ \cosh r(u) \sinh \alpha v \\ u \end{bmatrix} \end{aligned}$$

which is conformal if

$$1 + r'(u)^2 = \alpha^2 \cosh^2 r(u). \tag{3.3}$$

The normal to $\mathbb{H}^2 \times \mathbb{R}$ in $\mathbb{R}^{3,1}$ is

$$\bar{N}(u, v) = \begin{bmatrix} \cosh r(u) \cosh \alpha v \\ \sinh r(u) \\ \cosh r(u) \sinh \alpha v \\ 0 \end{bmatrix}.$$

One can see that the unit normal vector field $N(u, v)$ of HC_α in $\mathbb{H}^2 \times \mathbb{R}$ is computed as

$$N(u, v) = -\frac{1}{\alpha \cosh r(u)} \begin{bmatrix} \sinh r(u) \cosh \alpha v \\ \cosh r(u) \\ \sinh r(u) \sinh \alpha v \\ -r'(u) \end{bmatrix}.$$

Now assume that $\mathrm{HC}_\alpha(u, v)$ is conformal. Then the matrix of the shape operator S_{HC_α} with respect to $\partial/\partial u$, $\partial/\partial v$ is computed as

$$S_{\mathrm{HC}_\alpha} = \frac{\sinh r(u)}{\alpha \cosh^2 r(u)} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence we see that the immersion HC_α in $\mathbb{H}^2 \times \mathbb{R}$ is minimal if $\alpha \neq 0$. Furthermore,

$$T_{\mathrm{HC}_\alpha} = \frac{1}{\alpha^2 \cosh^2 r(u)} \frac{\partial}{\partial u}, \quad \nu_{\mathrm{HC}_\alpha} = \frac{r'(u)}{\alpha \cosh r(u)}.$$

Let us consider two cases separately: (i) when $\alpha = 1$, and (ii) when $\alpha > 1$.

(i) When $\alpha = 1$, since the function

$$r(u) = \log \frac{e^u + 1}{e^u - 1}, \quad u > 0,$$

is a solution of (3.3), we have a conformal parametrisation

$$\mathrm{HC}_1(u, v) = \frac{1}{\sinh u} \begin{bmatrix} \cosh u \cosh v \\ 1 \\ \cosh u \sinh v \\ u \sinh u \end{bmatrix}$$

(see Figure 1) whose induced metric is written as

$$ds^2 = \frac{\cosh^2 u}{\sinh^2 u}(du^2 + dv^2). \tag{3.4}$$

We also have

$$S_{\mathrm{HC}_1} = \frac{\sinh u}{\cosh^2 u} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T_{\mathrm{HC}_1} = \frac{\sinh^2 u}{\cosh^2 u} \frac{\partial}{\partial u}, \quad \nu_{\mathrm{HC}_1} = -\frac{1}{\cosh u}.$$

(ii) Consider the case $\alpha > 1$. One may assume in (3.3) that $r'(u) > 0$. Now (3.3) can be written

$$\frac{dr}{\sqrt{\alpha^2 \sinh^2 r(u) + \alpha^2 - 1}} = du.$$

Let u^+ be the supremum of the domain of $r(u)$ and let u^- be the infimum of the domain of $r(u)$. Then, by putting $x = \sinh r(u)$,

$$\begin{aligned} u^+ - u^- &= \int_{r(u^-)}^{r(u^+)} \frac{dr}{\sqrt{\alpha^2 \sinh^2 r + \alpha^2 - 1}} \\ &= \frac{1}{\alpha} \int_{\sinh r(u^-)}^{\sinh r(u^+)} \frac{1}{\sqrt{x^2 + \frac{\alpha^2 - 1}{\alpha^2}} \sqrt{x^2 + 1}} \, dx \\ &< \frac{1}{\alpha} \int_{\sinh r(u^-)}^{\sinh r(u^+)} \frac{1}{x^2 + \frac{\alpha^2 - 1}{\alpha^2}} \, dx < \infty, \end{aligned}$$

which shows that, when $\alpha > 1$, the height of the surface of the hyperbolic catenoid HC_α is finite (see Figure 2).

3.4. Conjugate correspondences. We prove the following theorem.

THEOREM 3.2. (i) *The conjugate minimal surface of the parabolic helicoid PH is the hyperbolic catenoid HC_1. Moreover, PH and HC_1 are isometric.*

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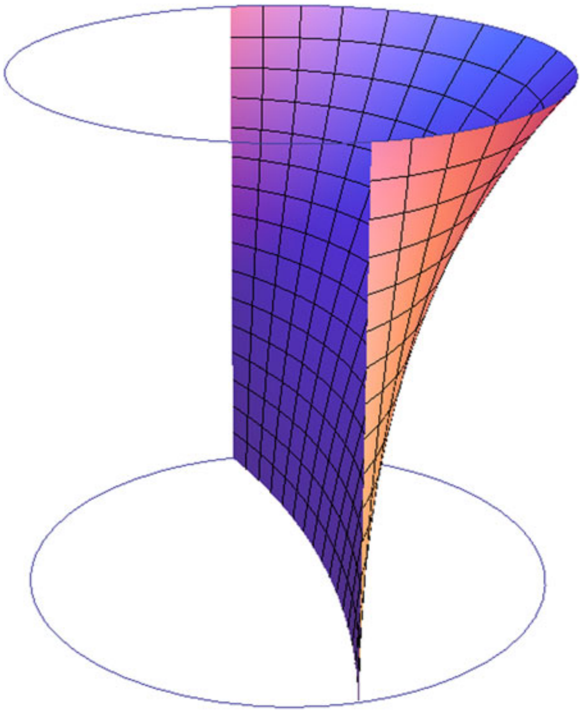

FIGURE 1. Hyperbolic catenoid HC_1

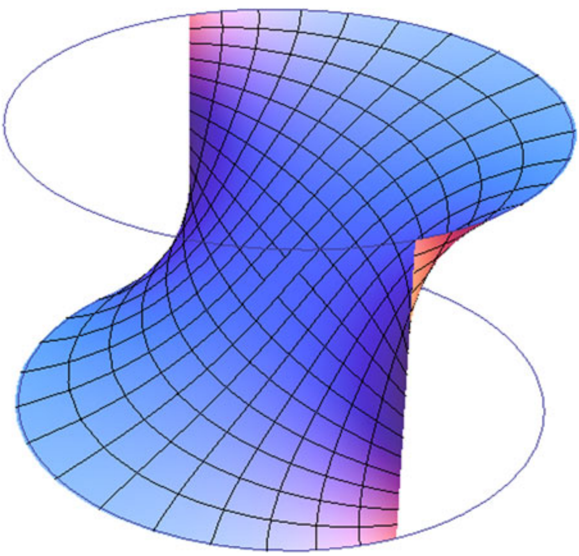

FIGURE 2. Hyperbolic catenoid HC_2.

(ii) *The conjugate minimal surface of the hyperbolic helicoid HH_β is the hyperbolic catenoid HC_α if $\alpha^2 = 1 + \beta^2$. Moreover, these two surfaces are isometric.*

Proof. Take $\partial/\partial u$, $\partial/\partial v$ to be positively oriented so that $J\partial/\partial u = \partial/\partial v$.

(i) Equations (3.1) and (3.4) show that PH is locally isometric to HC_1. Since

$$S_{\mathrm{PH}} = JS_{\mathrm{HC}_1}, \quad T_{\mathrm{PH}} = JT_{\mathrm{HC}_1}, \quad \nu_{\mathrm{PH}} = \nu_{\mathrm{HC}_1},$$

HC_1 is conjugate to PH. Moreover, as the parametrisations of PH and HC_1 are injective in the common domains, the conjugate surfaces are globally isometric to each other.

(ii) We may assume that $\beta > 1$, $f'(u) > 0$, $r'(u) > 0$ and that f and r satisfy the initial condition

$$\begin{cases} \beta \sinh f(0) = \alpha \sinh r(0), \\ \beta f'(0) \cosh f(0) = \alpha r'(0) \cosh r(0). \end{cases} \tag{3.5}$$

We set $y_1(u) = \beta \sinh f(u)$, $y_2(u) = \alpha \sinh r(u)$. Then, since $\alpha^2 = \beta^2 + 1$, a computation shows that both y_1 and y_2 satisfy the equation

$$y'^2 = (y^2 + \alpha^2)(y^2 + \beta^2),$$

and hence they satisfy the equation

$$y'' = y(2y^2 + \alpha^2 + \beta^2).$$

From (3.5), one can see that $y_1(u) = y_2(u)$, which shows from (3.2) and (3.3) that HH_β and HC_α are locally isometric. Moreover, since it follows that

$$S_{\mathrm{HH}_\beta} = JS_{\mathrm{HC}_\alpha}, \quad T_{\mathrm{HH}_\beta} = JT_{\mathrm{HC}_\alpha}, \quad \nu_{\mathrm{HH}_\beta} = \nu_{\mathrm{HC}_\alpha},$$

HH_β is conjugate to HC_α when $\alpha^2 = 1 + \beta^2$. Moreover, as the parametrisations of HH_β and HC_α are injective in the common domains, the conjugate surfaces are globally isometric to each other. □

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