Can. J. Math. Vol. 44 (1), 1992 pp. 135–153

LOCAL PROPERTIES OF LIGHT HARMONIC MAPPINGS

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ABSTRACT. The object of this paper is to study the local properties of light harmonic mappings.

1. **Introduction.** Let W be a simply connected domain of the complex plane $\mathbb{C}$, and let u and v be real-valued harmonic functions of W. We call a *harmonic mapping* of W every function of the form $f = u + iv$. For such functions it is immediate that there exist analytic functions g and h of W such that $f = \bar{g} + h$, where $\bar{g}$ denotes the complex conjugate of g. The Jacobian of f is given by

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2. \tag{1.1}$$

Unlike analytic functions, a nonconstant harmonic mapping f may neither be open nor light (f is light means that $f^{-1}(y)$ is either empty or totally disconnected for all y). For instance, the mappings $z \longrightarrow (z^2 + \bar{z}^2)/2$ and $z \longrightarrow (2z + z^2 + \overline{2z - z^2})/4$ collapse $\mathbb{C}$ to the real axis and the imaginary axis to the origin, respectively.

This paper deals mostly with the class, denoted by $D(W)$ or simply D, of all light harmonic mappings of W. Obviously, this class contains all analytic and anti-analytic functions of W, and all functions $L \circ g$ where g is analytic in W and L is an affine transformation $z \longrightarrow az + b\bar{z} + c$ satisfying $|a| \neq |b|$.

The purpose of this paper is to investigate the local mapping properties of light harmonic mappings. We classify the critical points of these mappings and study their behaviour rigorously at the different kinds of critical points. Generally speaking, we conclude that a light harmonic mapping satisfies, in addition to the properties of open continuous functions, the property that it may fold a disc several times along special folds which originate from some point in the disc. The local study then yields a surface structure, called folded coverings, that wraps the mapping properties of light harmonic mappings into a global setting. These coverings allow both folds and branch points, and generalize the classical notion of a covering surface.

The paper is organized as follows. In Section 2 we study some basic properties of harmonic mappings, and introduce the so called critical points of light harmonic mappings. We classify these points into two classes: the folding and nonfolding critical points, and we further divide the former class into critical points of the first, second, and third kinds.

This research was supported by grants from Kuwait University and the University of Tenessee. The author gratefully acknowledges that support.

Received by the editors November 26, 1989; revised May 2, 1991.

AMS subject classification: 30C60.

Before we embark on the study of the local properties of a light harmonic mapping at each kind of critical point, we introduce in Section 3 standard mappings which we later find suitable to describe these properties. Also in this section we use these mappings to describe the general local behaviour of light harmonic mappings. Sections 4, 5 and 6 form the paper's core and are devoted to a detailed investigation of the mapping properties of light harmonic mappings at the nonfolding critical points, folding critical point of the first and second kind, and folding critical points of the third kind, respectively. Finally in Section 7 we introduce the notion of a folded covering, and use this notion to show every light harmonic function, now defined on a Riemann surface W, defines W as a folded covering of $\mathbb{C}$.

Throughout this paper we use $\bar{A}$, int(A), and ∂A to denote the closure, interior, and boundary of a subset A of a toplogical space, and we also use I as the unit interval $[0, 1]$. For a function $f\colon \Delta \longrightarrow \mathbb{C}$ where Δ is a subset of $\mathbb{C}$, we say f is *N-valent* in Δ if f admits every value in $\mathbb{C}$ at most N times in Δ and some value exactly N times. We also say that the *valency of f at* $z_0 \in \Delta$ is N, written as

$$V_f(z_0, \Delta) = N,$$

if there exists a positive ε such that for any neighbourhood U of z_0 of diameter less than ε, f is *N-valent* in U. If Δ is open, we use the notation $V_f(z_0)$ instead of $V_f(z_0, \Delta)$.

2. **Critical points of light harmonic mappings.** In this section we introduce the critical points of light harmonic mappings and investigate some of their properties. In doing so, we study some of the general features of harmonic mappings.

We begin with the preliminary result:

LEMMA 2.1. *Let $f = \bar{g} + h$ be a harmonic mapping of W. Then J_f is identical to zero in some local neighbourhood if, and only if, there exist complex constants λ, $u \in \mathbb{C}$, with $|\lambda| = 1$, such that*

$$f(z) = \mu + 2\lambda^{-1}\operatorname{Re}\lambda h(z) \tag{2.1}$$

for all $z \in W$. In this case f is not light.

Note that because of the analyticity of h and g, J_f is identical to zero in some local neighbourhood if, and only if, J_f is identical to zero in W.

PROOF. Equation (2.1) means that f collapses W to a straight line, segment or a point, hence f is not light. Then by direct computation of J_f or the implicit function theorem we infer that J_f is identical to zero.

Now suppose that J_f is identical to zero in some disc U of W. Then $|h'|$ and $|g'|$ are identical in U, and by the maximum principle $g(z) = \mu + e^{i\theta} h(z)$ where $z \in U$, for some real θ and $\mu \in \mathbb{C}$. This yields (2.1) in U, which extends to W because of the connectedness of W. ∎

It follows at once that if f is light, then J_f is not identical to zero, and that the converse statement is not true as the mapping $z \longrightarrow (2z + z^2 + \overline{2z - z^2})/4$ shows.

Suppose now that $f = \bar{g} + h$ is a harmonic mapping of W whose Jacobian admits zero in W but is not identical to zero. The former assumption is meant to exclude the possibility that f is locally 1-1 which we discard completely in our investigation. Then the quotient

$$\psi(z) = (h'/g')(z) \quad (z \in W) \tag{2.2}$$

determines a nonconstant meromorphic function or is identically constant with modulus different from unity.

Denote by J the set of all z such that $J_f(z) = 0$, and by N the set of all $z \in J$ such that $|\psi(z)| \neq 1$. Note that N can possibly be empty or equals J, but for the sake of generality we avoid either case unless otherwise specified, which would then discard the possibility that ψ is constant..

DEFINITION 2.1. For a light harmonic mapping f of W every $z \in N$ is called a *non-folding critical point of f*.

LEMMA 2.2. *Every $z_0 \in N$ belongs to a neighbourhood that contains no other value of J.*

The proof follows immediately by noting that z_0 is a zero of h' (and g') which isolates z_0 from the rest of N, and that z_0 is also isolated from $J \backslash N$ since $|\psi|$ is continuous and $|\psi(z_0)| \neq 0$.

It is easy to see that $J \backslash N$ is a set, Γ_f, on which $|\psi|$ has modulus 1 and which consists of curves which are analytic except possibly for algebraic singularities. Throughout, we assume tht Γ_f has the direction induced via ψ by the positive direction of the unit circle, and assume that this direction is inherited by the Jordan subarcs of Γ_f.

Now let γ be a directed Jordan subarc of Γ_f given by an analytic path $z(t)$, $t \in I$ then the identity

$$\psi \circ z(t) = \exp\big(i\phi(t)\big) \quad (t \in I) \tag{2.3}$$

defines $\phi : I \longrightarrow \mathbb{R}$ as a continuously increasing function.

LEMMA 2.3. *With the above notation we have*

$$\big(f \circ z(t)\big)' = \big(2\operatorname{Re}\omega(t)\big)\exp\big(i\phi(t)/2\big) \quad (t \in I), \tag{2.4}$$

where

$$\omega(t) = h'\big(z(t)\big)z'(t)\exp\big(-i\phi(t)/2\big) \tag{2.5}$$

is an analytic path which satisfies the property that $\operatorname{Re}\omega$ *either admits zero finitely many times or is identical to zero. In the latter case, f is constant on γ and f is not light.*

PROOF. It is immediate that

$$\big(f \circ z(t)\big)' = \overline{g'\big(z(t)\big)z'(t)} + h'\big(z(t)\big)z'(t) \quad (t \in I).$$

https://doi.org/10.4153/CJM-1992-008-0 Published online by Cambridge University Press

Using (2.3) we obtain (2.4) where ω is as given by (2.5). Also (2.3) yields the analyticity of ω since ψ is analytic on γ. Hence the curve determined by ω either meets the imaginary axis in finitely many points or is a subset of the imaginary axis. This is equivalent to the desired property of $\operatorname{Re}\omega$. The rest of the proof follows obviously. ∎

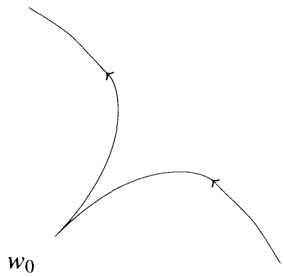

Figure 2.1

Note that the mapping $z \longrightarrow (2z + z^2 + \overline{2z - z^2})/4$ collapses the imaginary axis (which is its corresponding Γ_f) to the origin.

As an immediate consequence of Lemmas 2.1 and 2.3 we have:

THEOREM 2.1. *A harmonic mapping f of W is either (a) light, (b) has a zero Jacobian, or (c) is constant on some analytic subarcs of Γ_f.*

Lemma 2.3 also motivates the remaining critical points of f as follows.

DEFINITION 2.2. Let $f = \bar{g} + h \in D$ and let $z_0 \in \Gamma_f$.

(A) If $\psi'(z_0) \neq 0$, then z_0 is interior to a Jordan subarc γ of Γ_f which we assume given as above, with $z_0 = z(t_0)$, $0 < t_0 < 1$, together with the function ω. We call z_0 a *critical point of f of the*

(a) *first kind* if $\operatorname{Re}\omega$ changes signs at t_0.

(b) *second kind* if z_0 is a zero of h', or equivalently g', which yields $\operatorname{Re}\omega(t_0) = 0$, and $\operatorname{Re}\omega$ changes no signs at t_0.

(B) If $\psi'(z_0) = 0$, then we call z_0 a *critical point of f of the third kind.*

It is easy to see that Definitions 2.2A(a) and 2.2A(b) are independent of γ and its parametrization.

Let F_j where $j = 1, 2$ or 3, denote the set of all folding points of f of the jth kind, and let $F = \cup_{j=1}^{3} F_j$. We call F the set of *folding critical points* of f.

By virtue of Lemma 2.3 we also conclude:

THEOREM 2.2. *Let $f = \bar{g} + h \in D$. Then the critical points of f are isolated.*

PROOF. Let z_0 be a critical point of f. If z_0 is nonfolding, then by Lemma 2.2 z_0 is isolated from the rest of critical points. If z_0 is folding, then z_0 is isolated from the nonfolding critical points since the latter do not accumulate in W. It is easy to see that z_0 has a neighbourhood Δ which meets Γ_f in a finite number of Jordan arcs. By Lemma 2.3, a sufficiently small Δ can contain no folding critical points but z_0. ∎

Further consequences of Lemma 2.3 need

DEFINITION 2.3. A directed arc α is called (a) *convex* if α is simple and the slope of its tangent is continuously increasing (b) *locally convex at* z_0 if z_0 belongs to some (relatively) open convex subarc of β, (c) *locally convex* if β is locally convex at every point, (d) a *harmonic cusp* if there exist a parameterization $w = w(t)$, $t \in I$, and a real t_0, $0 < t_0 < 1$, such that $\arg w'(t)$ determines a continuously increasing function in $I \setminus \{t_0\}$ with a simple discontinuity of jump π at t_0. We call $w(t_0)$ the *vertex* of the cusp (see Figure 2.1.)

Suppose now that f is light, and let $z_0 = z(t_0) \in \operatorname{int}(\gamma)$ where γ is given as above. In view of Lemma 2.3, we can assume, without loss of generality, that $\operatorname{Re}\omega$ either (i) changes no signs in I, or (ii) changes signs in I only once and at t_0. In either case there exists an open interval (t_1, t_2) containing t_0 such that the argument of the tangent vector, $\arg \frac{d}{dt} f \circ z(t)$ (see (2.4)), to $f(\gamma)$ at $f \circ z(t_0)$ is strictly increasing in (t_1, t_2) if (i) holds, and in (t_1, t_0) and (t_0, t_2) with a discontinuity jump of size π at t_0 if (ii) holds. Geometrically, this means that if $\alpha\colon z = z(t)$, $t_1 < t < t_2$, then the arc $\beta = f(\alpha)$ is locally convex in case (i), and is a harmonic cusp with vertex $f(z_0)$ in case (ii). It follows that f is 1-1 on every sufficiently small Jordan subarc of Γ.

We summarize this discussion as follows:

THEOREM 2.3. *Let $f \in D$ and $z_0 \in \Gamma_F$. Then $f|_{\Gamma_f}$ is locally convex at every $z_0 \in \Gamma_f \setminus (F_1 \cup F_3)$, maps a neighbourhood of every $z_0 \in F_1$ to a harmonic cusp, and is locally homeomorphic at every $z_0 \notin F_3$.*

3. **Standard mappings and general local behaviour.** As with analytic functions where the local behaviour is described by the elementary mappings $z \longrightarrow z^n$ where n is a positive integer, local properties of harmonic mappings can also be described by some suitable mappings. The purpose of this section is to introduce these mappings, and use them to describe the general local behaviour of $f \in D$ on Γ_f, where f and Γ_f are as in Section 2.

In this section we let Δ be a Jordan domain in $\mathbb{C}$ and let n and m be positive integers.

DEFINITION 3.1. Let f be a function of Δ containing z_0. The notation

$$f_{z_0} \sim z^n \tag{3.1}$$

means that there exist an open subset U of Δ containing z_0 and sense preserving homeomorphisms $h_1\colon U \longrightarrow (|\zeta| < 1)$ and $h_2\colon \mathbb{C} \longrightarrow \mathbb{P}$ such that $h_1(z_0) = h_2 \circ f(z_0) = 0$ and

$$h_2 \circ f \circ h_1^{-1}(\zeta) = \zeta^n \quad (|\zeta| < 1). \tag{3.2}$$

If instead of (3.2) we have

$$h_2 \circ f \circ h_1^{-1}(\zeta) = \bar{\zeta}^n \quad (|\zeta| < 1), \tag{3.3}$$

https://doi.org/10.4153/CJM-1992-008-0 Published online by Cambridge University Press

then we use the notation

$$f_{z_0} \sim \bar{z}^n. \tag{3.4}$$

Clearly, (3.1) ((3.4)) says that f has the same local properties near z_0 as the mapping $z \longrightarrow z^n$ ($z \longrightarrow \bar{z}^n$) near the origin; hence n is unique, and (3.1) and (3.4) cannot occur simultantiously at z_0.

It also follows from (3.1) ((3.4)) that f is open in some neighbourhood of z_0. On the other hand, a result of Stöilow [9] states, in terms of (3.1) and (3.4), that an open sense preserving (reversing) function f in a neighbourhood of z_0 satisfies $f_{z_0} \sim z^n$ ($f_{z_0} \sim \bar{z}^n$) for some n. Therefore f satisfies (3.1) ((3.4)) if, and only if, f is open and sense preserving (reversing) in a neighbourhood of z_0.

We shall use the following result whose proof can be found in [2, p. 27] and which can be stated in terms of (3.1) and (3.4) as follows.

LEMMA 3.1. *Let $f: \Delta \longrightarrow \mathbb{C}$ where $z_0 \in \Delta$, be a continuous function which is locally 1-1 in $\Delta \setminus \{z_0\}$. Then there exists n such that either (3.1) or (3.4) holds.*

DEFINITION 3.2. Let f be a function of $\bar{\Delta}$ and let $z_0 \in \partial\Delta$. The notation

$$f_{z_0} \sim z^n \quad (z \in \bar{\Delta}) \tag{3.5}$$

means that there exist an open subset U of $\mathbb{C}$ containing z_0 and sense preserving homeomorphisms $h_1: U \cap \bar{\Delta} \longrightarrow (|\zeta| < 1, \operatorname{Im}\zeta \geq 0)$ and $h_2: \mathbb{C} \longrightarrow \mathbb{C}$ such that $h_1(z_0) = h_2 \circ f(z_0) = 0$ and

$$h_2 \circ f \circ h_1^{-1}(\zeta) = \zeta^n \quad (|\zeta| < 1, \operatorname{Im}\zeta \geq 0). \tag{3.6}$$

If instead of (3.6) we have

$$h_2 \circ f \circ h_1^{-1}(\zeta) = \bar{\zeta}^n \quad (|\zeta| < 1, \operatorname{Im}\zeta \geq 0), \tag{3.7}$$

then we use the notation

$$f_{z_0} \sim \bar{z}^n \quad (z \in \bar{\Delta}). \tag{3.8}$$

Suppose that (3.5) ((3.8)) holds. Then according to the definition, f becomes $z \longrightarrow z^n$ ($z \longrightarrow \bar{z}^n$) for some n, upon finding a suitable U containing z_0 and sense preserving homeomorphisms which deform $U \cap \bar{\Delta}$ and $\mathbb{C}$ to the semi-disc: $|z| < 1$, $\operatorname{Im} z \geq 0$ and $\mathbb{C}$, respectively, and which map z_0 and $f(z_0)$ to the origin. This implies at once that n is unique, and that (3.5) and (3.8) cannot hold simultaneously. Furthermore, in either case ((3.5) or (3.8)) f is 1-1 on the Jordan arc $U \cap \partial\Delta$ if n is odd, and 1-1 on each of the arc components of $(U \cap \partial\Delta) \setminus \{z_0\}$ if n is even. In the latter case f maps these arcs to the same Jordan arc.

It is also immediate from (3.5) and (3.8) that f is continuous on $U \cap \bar{\Delta}$ and open on $U \cap \Delta$. We shall need a partial converse of this statement whose proof can be found in [1] and which can be stated in terms of (3.5) and (3.8) as follows.

LEMMA 3.2. *Let γ be an open boundary arc of Δ and let $z_0 \in \gamma$. Suppose that $f: \Delta \cup \gamma \longrightarrow \mathbb{C}$ is a continuous function which is open on Δ and 1-1 on γ. Then there exists n such that either (3.5) or (3.8) holds.*

We call an *endcut* of Δ every simple arc that has one endpoint on $\partial\Delta$ and lies otherwise in Δ. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be endcuts of Δ which have a common endpoint $z_0 \in \Delta$ and are otherwise disjoint in $\bar{\Delta}$. Then $\bar{\Delta}$ partitions into m (closed) topological sectors which have a common vertex z_0 and which can be read as we go positively about z_0, starting from some sector, in the order $\bar{\Delta}_1, \bar{\Delta}_2, \ldots, \bar{\Delta}_m$. We call this sequence a *sector subdivision of* $\bar{\Delta}$ *from* z_0.

DEFINITION 3.3. Let f be a function of $\bar{\Delta}$ and let $z_0 \in \Delta$. The notation

$$f_{z_0} \sim (z^*)^{n_1}, (z^*)^{n_2}, \ldots, (z^*)^{n_m} \tag{3.9}$$

where $n_1, n_2, \ldots, n_m$ are positive integers and $(z^*)^{n_k}$ equals z^{n_k} or $\bar{z}^{n_k}$ depending only on k, means that there exists a sector subdivision $\bar{\Delta}_1, \bar{\Delta}_2, \ldots, \bar{\Delta}_m$ of $\bar{\Delta}$ from z_0 such that

$$f_{z_0} \sim (z^*)^{n_k} \quad (z \in \bar{\Delta}_k) \tag{3.10}$$

for all k.

Note that this definition is of local nature since $\bar{\Delta}$ can be replaced by any Jordan subdomain of itself containing z_0. Suppose that (3.9) holds and let the endcuts associated with the sector subdivision $\bar{\Delta}_1, \bar{\Delta}_2, \ldots, \bar{\Delta}_m$ be $\alpha_1, \alpha_2, \ldots, \alpha_m$ which are labeled so that α_k is the common edge of $\bar{\Delta}_k$ and $\bar{\Delta}_{k+1}, (\Delta_{m+1} = \Delta_1)$, for $1 \leq k \leq m$. In view of (3.10) we can choose Δ sufficiently small so that f is 1-1 on each $\alpha_k \cup \alpha_{k+1}$ ($\alpha_{m+1} = \alpha_1$) where n_k is odd, and 1-1 on α_k and α_{k+1} and $f(\alpha_k) = f(\alpha_{k+1})$ where n_k is even. In either case f maps Δ_k to a topological sector, S_k, which can be viewed spread over $\mathbb{C}$ with vertex over $f(z_0)$ and arms ℓ_k and ℓ_{k+1} over $f(\alpha_k)$ and $f(\alpha_{k+1})$, respectively, and which covers $f(\alpha_k)\setminus\{f(z_0)\}$ exactly $\frac{n_k}{2} - 1$ times if n_k is even and $\frac{n_k-1}{2}$ otherwise. In addition, if ℓ_k and ℓ_{k+1} are directed away from $f(z_0)$, then S_k lies to the left (right) of ℓ_k and to the right (left) of ℓ_{k+1} in case $(z^*)^{n_k} = z^{n_k}$ ($(z^*)^{n_k} = \bar{z}^{n_k}$). It follows that sectors $\bar{S}_k$ and $\bar{S}_{k+1}$ meet along a fold over $f(\alpha_k)$ if f changes orientation as it moves from Δ_k to Δ_{k+1}.

We conclude this section with a result which describes the general local behaviour of a light harmonic mapping near its folding critical points.

THEOREM 3.1. *Let $f = \bar{g} + h \in D$ and let $z_0 \in \Gamma_f$. Then there exist positive integers $n_1, n_2, \ldots, n_{2m}$ such that*

$$f_{z_0} \sim (z^*)^{2n_1-1}, (z^*)^{2n_2-1}, \ldots, (z^*)^{2n_{2m}-1} \tag{3.11}$$

where

(i) *$(z^*)^{2n_k-1}$ equals z^{2n_k-1} if k is odd and $\bar{z}^{2n_k-1}$ otherwise.*
(ii) *$m = 1$ if $z_0 \notin F_3$.*
(iii) *$m - 1$ is the order of z_0 as a zero of ψ' if $z_0 \in F_3$ (see (2.2)).*

PROOF. We consider two cases: (a) Let $z_0 \in F_3$. Since Γ_f consists of analytic arcs, there exists a Jordan domain Δ containing z_0 such that Γ_f separates Δ into two Jordan domains, Δ^+ and Δ^-. By using Theorems 2.2 and 2.3(c) and labeling Δ^+ and Δ^- suitably, we conclude that Δ can be chosen sufficiently small so that f is 1-1 on $\Delta \cap \Gamma_f, J_f > 0$ on Δ^+, and $J_f < 0$ on Δ^-. Then by the invariance of domains theorem, Lemma 3.2 applies and we conclude that $f_{z_0} \sim z^{2n_1-1}(z \in \bar{\Delta}^+)$ and $f_{z_0} \sim \bar{z}^{2n_2-1}(z \in \bar{\Delta}^-)$, or, $f_{z_0} \sim z^{2n_1-1}, \bar{z}^{2n_2-1}$.

(b) Suppose that $z_0 \in F_3$ and that $m-1$ is the order of z_0 as a zero of ψ'. Let $\lambda = \psi(z_0)$ and let E be an open disc whose center is λ and which meets the unit circle in a proper subarc, μ. Then for a sufficiently small E there exists a Jordan domain Δ containing z_0 such that $(\bar{\Delta}, f)$ is an m-fold covering of $\bar{E}$ which has only one branch point located at z_0 and is of order $m-1$. Let μ be endowed with the positive direction of the unit circle, and denote by μ_1 and μ_2 the subarcs of μ terminating and starting at λ, respectively. It follows that there exist analytic endcuts $\alpha_1, \alpha_2, \ldots, \alpha_{2m}$ of Δ such that (i) these arcs have a common endpoint z_0 and are mutually disjoint otherwise, (ii) ψ maps each α_1 to μ_1 if j is odd and to μ_2 if j is even, (iii) each α_j is directed according to the direction induced by μ via ψ, and (iv) the arcs α_j are located so that as we go positively about z_0 they can be read in the order $\alpha_1, \alpha_2, \ldots, \alpha_{2m}$. Hence the arcs α_j determine a sector subdivision $\bar{\Delta}_1, \bar{\Delta}_2, \ldots, \bar{\Delta}_{2m}$ of $\bar{\Delta}$ from z_0, where each $\bar{\Delta}_j$ is the sector bounded by α_j, α_{j+1} $(\alpha_{2m+1} = \alpha_1)$ and a subarc of $\partial\Delta$. Note that (iii) implies that α_j is directed towards (away from) z_0 if j is odd (even). This permits us to define the arc products $\gamma_j = \alpha_j\alpha_{j+1}$ and $\gamma_j = \alpha_{j+1}\alpha_j$ if j is odd and even, respectively. It follows at once that each γ_j is a directed Jordan arc which is analytic except at z_0 (where it makes an angle of size π/m) and which maps under ψ to μ in a direction preserving homeomorphism. Hence γ_j can be defined by a path $z = z_j(t)$, $t \in I$, such that $z_0 = z(t_0)$ for some t_0, $z'(t) \neq 0$ for all $t \neq t_0$, and $\psi \circ z_j(t) = e^{i\phi_j(t)}$ where $\phi_j \colon I \longrightarrow \mathbb{R}$ is a continuously increasing function. Now Lemma 2.3 essentially applies and yields

$$(f \circ z_j(t))' = \big(2\operatorname{Re}\omega_j(t)\big)\exp(i\phi_j(t)/2) \quad (t \in I \setminus \{t_0\}) \tag{3.12}$$

where

$$\omega_j(t) = h'\big(z_j(t)\big)z_j'(t)\exp\big(-i\phi_j(t)/2\big). \tag{3.13}$$

Suppose now that Δ is chosen so that z_0 is the only critical of f there and $\operatorname{Re}\omega_j(t) \neq 0$ for all $t \neq t_0$. Then $\operatorname{Re}\omega_j$ does not change signs in $[0, t_0)$ or $(t_0, 1]$. Hence $\operatorname{Re}\omega_j$ either admits the same sign in both intervals or opposite signs. In either case $f|_{\gamma_j}$ is locally homeomorphic at z_0; in fact, $f|_{\gamma_j}$ maps a local neighbourhood of z_0 to a convex arc in the first case and to a harmonic cusp of vertex $w_0 = f(z_0)$ in the latter.

Finally, note that if f is sense preserving (reversing) in every sector $\bar{\Delta}_j$ where j is odd (even), then the result follows by the same argument of (a). ∎

REMARK 3.1. The set of sectors in a sector subdivision from z_0 of the above $\bar{\Delta}$ which is associated with (3.11) is unique. This is because Γ_f locally separates W at z_0 into $2m$ topological sectors such that the sectors where f preserves and reverses orientation are located alternately about z_0.

4. **Behaviour at the nonfolding critical points.** The local behaviour of a light harmonic mapping at a nonfolding critical point resembles the mapping $z \longrightarrow z^m$ or $z \longrightarrow \bar{z}^m$. This justifies the term 'nonfolding' in Definition 2.1. This behaviour extends to pseudo-analytic and anti-pseudo-analytic functions of the second kind (see [3]). So we content ourselves with the following result without proof.

THEOREM 4.1. *Let $f = \bar{g} + h \in D$, $z_0 \in N$, and k and ℓ be the order of z_0 as a zero of h' and g' respectively.*

(a) If $k < \ell$, then $f_{z_0} \sim z^{k+1}$.

(b) If $k > \ell$, then $f_{z_0} \sim \bar{z}^{\ell+1}$.

(c) If $k = \ell$, then $f_{z_0} \sim (z^*)^{\ell+1}$ where z^* equals z or $\bar{z}$ depending respectively on whether $J_f > 0$ or $J_f < 0$ in a deleted neighbourhood of z_0.

5. **Behaviour at the points in $\Gamma_f \setminus F_3$.** We are now going to study the precise mapping properties of a light harmonic mapping f at the points of $\Gamma_f \setminus F_3$. Our result in this direction is as follows.

THEOREM 5.1. *Let $f = \bar{g} + h \in D$, $z_0 \in \Gamma_f$, and $\ell \geq 0$ the order of z_0 as a zero of h', or equivalently g'.*

(a) Suppose that $z_0 \in \Gamma_f \setminus (F_1 \cup F_3)$. Then f satisfies $f_{z_0} \sim z^{\ell+1}, \bar{z}^{\ell+1}$ if ℓ is even (including zero), and $f_{z_0} \sim z^{\ell+2}, \bar{z}^{\ell}$ or $f_{z_0} \sim z^{\ell}, \bar{z}^{\ell+2}$ if ℓ is odd. In either case $V_f(z_0) = \ell + 2$.

(b) Suppose that $z_0 \in F_1$. Then f satisfies $f_{z_0} \sim z^{\ell+1}, \bar{z}^{\ell+3}$ or $f_{z_0} \sim z^{\ell+3}, \bar{z}^{\ell+1}$ if ℓ is even, and $f_{z_0} \sim z^{\ell+2}, \bar{z}^{\ell+2}$ if ℓ is odd. In either case $V_f(z_0) = \ell + 3$.

Regarding (a), note that if $z_0 \in \Gamma_f \setminus F$ then $\ell = 0$ (See Definition 2.2.)

PROOF. We treat (a) and (b) simultanteously unless otherwise specified. We give the proof in four steps as follows.

STEP 1. Here we consider some implications of the general behaviour of f at z_0 as given by Theorem 3.1.

Since $z_0 \notin F_3$, there exists a Jordan domain Δ containing z_0 such that Γ_f separates Δ into two Jordan domains Δ^+ and Δ^- which satisfy the property that f is sense preserving on Δ^+ and reversing on Δ^-.

Let $\gamma = \Delta \cap \Gamma_f$. By Theorem 2.3 Δ can be chosen so that f maps γ homeomorphically to a Jordan arc, β, which is convex in (a) and a harmonic cusp with vertex $w_0 = f(z_0)$ in (b). In either case β separates into two components an arbitrarily small open disc B centred at w_0. Now by Theorem 3.1 there exists a positive integer n such

$$f_{z_0} \sim z^{2n-1} \quad (z \in \overline{\Delta^+}), \tag{5.1}$$

that is, there exist a neighbourhood N of z_0 and sense preserving homeomorphisms $h_1 \colon \Delta^+ \cap N \longrightarrow (|\zeta| < 1, \operatorname{Im} \zeta \geq 0)$ and $h_2 \colon \mathbb{C} \longrightarrow \mathbb{C}$ which satisfy $h_1(z_0) = h_2 \circ f(z_0) = 0$ and

$$\eta = h_2 \circ f \circ h_1^{-1}(\zeta) = \zeta^{2n-1} \quad (|\zeta| < 1, \operatorname{Im} \zeta \geq 0). \tag{5.2}$$

It follows that if B is chosen such that $|h_2(w)| < 1$ for all $w \in \bar{B}$, then the set $\overline{\Delta^+} \cap f^{-1}(\bar{B})$ is itself the set of all $z \in \overline{\Delta^+}$ satisfying $[h_1(z)]^{2n-1} \in h_2(\bar{B})$. Observe that this set is the closure of a Jordan domain, S^+, which has as boundary arc a subarc, α, of γ that contains z_0 in its interior.

Recall the direction of Γ, and let α and β be endowed with the directions induced by Γ_f via the identity map and f respectively. Write $\psi(z_0) = \lambda$ (see (2.2)) where $|\lambda| = 1$, and define the line L: $w = w_0 + t\tau$ $(t \in \mathbb{R})$, where τ is a chosen value of $\lambda^{1/2}$. Then by Lemma 2.3 we conclude that L is tangent to β at w_0. Denote by $[w_1, w_2]$ the diameter $B \cap L$ of B.

Suppose that (a) holds. Then by Theorem 2.3 L lies to the right of β, and consequently the Jordan arc $h_2([w_1, w_2])$ lies in the semi-disc $\operatorname{Im} \eta > 0$ with the exception of the origin. Now the inverse image under f of $[w_1, w_2]$ in $\overline{S^+}$ can be easily traced by taking the inverse image of $h_2([w_1, w_2])$ under $\eta(\zeta)$ then followed by h_1^{-1}. This yields two sets of Jordan arcs $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1n}$ and $\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2n}$ that satisfy the properties: (i) Each α_{ij} is a cross-cut of S^+ with one end point z_0, (ii) the arcs α_{ij} are mutually disjoint except for z_0, and (iii) f maps every α_{ij} homeormorphically to $[w_0, w_i]$.

Suppose now that (b) holds. Then L crosses β at w_0. We distinguish the radii $[w_0, w_1]$ and $[w_0, w_2]$ of B by letting $[w_0, w_1]$ be on the right of β. It follows that $h_2([w_0, w_1])$ and $h_2([w_0, w_2])$ are Jordan arcs in the disc $|\eta| < 1$ which end at the origin and lie otherwise in the semi-discs $\operatorname{Im} \eta > 0$ and $\operatorname{Im} \eta < 0$ respectively. As above, we also conclude that the inverse image under f of $[w_0, w_1]$ and $[w_0, w_2]$ in $\overline{S^+}$ yields two sets of Jordan arcs which for convenience are also denoted by $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1n}$ and $\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2(n-1)}$, and which satisfy the same properties of the previous sets with exception that the set $\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2(n-1)}$ is void if $n = 1$.

STEP 2. We study now the mapping properties at z_0 of the function F_λ defined by

$$f = F_\lambda + 2\tau \operatorname{Re}(\tau g) \tag{5.3}$$

where $\tau^2 = \lambda$ and $F_\lambda = h - \lambda g$.

First observe that z_0 is a zero of F'_λ of order $\ell + 1$. This follows since in a neighbourhood of z_0 we can write

$$g'(z) = \sum_{k=\ell}^{\infty} b_k (z - z_0)^k$$

where $b_\ell \neq 0$, and

$$\psi(z) = \lambda + \sum_{k=1}^{\infty} a_k (z - z_0)^k,$$

where $a_1 \neq 0$, which give

$$F'_\lambda(z) = \sum_{k=\ell+1}^{\infty} c_k (z - z_0)^k$$

where $c_{\ell+1} = a_1 b_\ell \neq 0$.

Now we claim that there exists an arbitrarily small Jordan domain G that satisfies the properties of Δ, in addition to the property that $(\bar{G}, F_\lambda)$ is an $(\ell + 2)$-fold covering of a closed disc $\bar{D}$ centred at $\xi_0 = F_\lambda(z_0)$ such that $\bar{G}$ has z_0 as the only branch point. To see this, consider the single-valued analytic branch of the function ψ^{-1} satisfying $z_0 = \psi^{-1}(\lambda)$, and then define the function $H = F \circ \psi^{-1} \circ T$ where T is a Möbius transformation which maps $\mathbb{R}$ to the unit circle $|w| = 1$ and satisfies $T(0) = \lambda$. It follows that in some neighbourhood of the origin we can write $H = \xi_0 + \kappa^{\ell+2}$ where κ is a univalent function satisfying $\kappa(0) = 0$. Now observe that there exists an arbitrarily small convex region, K, containing the origin such that $\kappa(K)$ is an open disc [6, p. 44]. Then by letting $G = \psi^{-1} \circ T(K)$, we can easily verify that G satisfies the desired properties.

Let us now choose G sufficiently small so that Γ separates G into two components of which one, say G^+, has its closure in S^+. Then from (5.3) we conclude that $f(z)$ is a translation of $F_\lambda(z)$ in the direction of τ for all z. Since $f^{-1}|_{S^+}[w_1, w_2]$ is the union of all α_{ij}, the value of $F_\lambda(z)$, where $z \in G^+$, has the form $\xi_0 + t\tau$ $(t \in \mathbb{R})$ if, and only if, z belongs to some α_{ij}. It follows that F_λ maps α_{ij} homeomorphically into one of the semi-lines $\xi = \xi_0 + t\tau$ and $\xi = \xi_0 - t\tau$ $(t \geq 0)$. This implies that $\partial G^+ \setminus \gamma$ meets every α_{ij} at exactly one point, a_{ij}, for if $a, b \in (\partial G^+ \setminus \gamma) \cap a_{ij}$, then $F_\lambda(a) \neq F_\lambda(b)$ since F_λ is 1-1 in α_{ij}, and $F_\lambda(a) = F_\lambda(b)$ since each of the semi-lines $\xi = \xi_0 + t\tau$ and $\xi = \xi_0 - t\tau$ $(t \geq 0)$ meets ∂D at exactly one point.

It follows that the points a_{ij} are the only points of ∂G^+ which map under F_λ to the points $\xi_1 = \xi_0 + r\tau$ and $\xi_0 - r\tau$, where r is the radius of D and that the number of these points is $2n$ in (a) and $2n - 1$ in (b).

STEP 3. We now relate n and ℓ. By the choice of G there are exactly $2(\ell + 2)$ points $z_1, z_2, \ldots, z_{2(\ell+2)}$ of ∂G which map under F_λ to ξ_1 and ξ_2 and which contain the points a_{ij} as the only points z_k in ∂G^+. Denote by γ_1 and γ_2 the subarcs of γ terminating and starting at z_0 respectively, and let $\delta_j = F_\lambda(\gamma_j)$. Clearly, the arcs δ_j are analytic and Jordan for sufficiently small γ; hence the arcs $\delta_1 \setminus \{\xi_0\}$ and $\delta_2 \setminus \{\xi_0\}$ are either disjoint or coinciding. But by (5.3) we conclude that δ_1 and δ_2 lie on one side of $[\xi_1, \xi_2]$ if (a) holds, and on different sides otherwise. This implies that $\delta_1 \setminus \{\xi_0\}$ and $\delta_2 \setminus \{\xi_0\}$ can either be disjoint or coinciding in the former case, and that $\delta = \delta_1 \cup \delta_2$ forms a Jordan arc in the latter.

Now let $C = \partial G^+ \setminus \gamma$. We need to find the number of points z_j in C. Note that $D \setminus \delta$ consists of two components, U_1 and U_2, except possibly when (a) holds and ℓ is even in which case one of these components could be empty. In any case we can view U_1 and U_2 as topological sectors whose vertex is ξ_0 and arms δ_1 and δ_2. It follows that $F_\lambda^{-1}|_G(U_i)$ is a disjoint union of $\ell + 2$ components V_i which can also be viewed as sectors of vertex z_0. Note that the V_1 and V_2 sectors are located alternately around z_0 so that the sequence of sectors in G which is obtained by going positively about z_0 starts and ends either with V_1 sectors or V_2 sectors. Now we consider two cases: (1) If (a) or (b) holds and ℓ is even, then one of the sectors, say U_1, has vertex angle of size 2π whereas the other (which is possibly degenerate if (a) holds) has vertex angle of size zero. It follows that the sectors V_1 and V_2 have vertex angle of size $2\pi/(\ell + 2)$ and zero respectively. This implies that G^+ fits exactly $(\ell + 2)/2$ sectors V_1 and either $\ell/2$ or $\ell/2 + 2$ sectors V_2. Here we consider two cases:

(i) If (a) holds, then $\xi_1, \xi_2 \in \partial U_1$, and consequently each $\partial G \cap \partial V_1$, contains two points z_j and each $\partial G \cap \partial V_2$ contains none. Therefore C contains exactly $\ell + 2$ points z_j.

(ii) If (b) holds, then either $\zeta_1 \in U_1$ and $\xi_2 \in U_2$, or $\xi_1 \in U_2$ and $\zeta_2 \in U_1$, and consequently each $\partial G \cap \partial V_i$ contains exactly one point z_j. Therefore C contains either $\ell + 1$ or $\ell + 3$ points.

(2) If (a) or (b) holds and ℓ is odd, then both sectors U_1 and U_2 have the same vertex angle size π. It follows that each sector V_i has vertex angle of size $\pi / (\ell + 2)$. This implies that G^+ fits exactly $\ell + 2$ sectors V_1 and V_2 combined, which are divided into $(\ell + 1)/2$ sectors V_1 and $(\ell + 3)/2$ sectors V_2, or $(\ell + 3)/2$ sectors V_1 and $(\ell + 1)/2$ sectors V_2. Here again, we consider two cases:

(i) If (a) holds, then either $\xi_1, \xi_2 \in \partial U_1$ or $\xi_1, \xi_2 \in \partial U_2$. Suppose without loss of generality that the former holds, then each $\partial G \cap \partial V_1$ contains two points z_j and each $\partial G \cap \partial V_2$ contains none. Therefore C contains either $\ell + 1$ or $\ell + 3$ points.

(ii) If (b) holds, then either $\xi_1 \in \partial U_1$ and $\xi_2 \in \partial U_2$, or $\xi_1 \in \partial U_2$ and $\xi_1 \in \partial U_1$, and consequently each $\partial G \cap \partial V_i$ contains exactly one point z_j. Therefore C contains exactly $\ell + 2$ points.

STEP 4. Here we conclude the proof. The conclusion of Step 2 states that C contains $2n$ points z_j if (a) holds and $2n - 1$ points z_j if (b) holds. On the other hand, Step 3 states that C contains $\ell + 2$ points z_j if (a) holds and ℓ is even or (b) holds and ℓ is odd, and either $\ell + 1$ or $\ell + 3$ points z_j if (a) holds and ℓ is odd, or (b) holds and ℓ is even. By comparison, we obtain $2n - 1$ in terms of ℓ which in view of (5.1) yields the theorem except for $V_f(z_0)$ which then follows directly. ∎

REMARK 5.1. It easily follows from Theorem 5.1 that the degree of f at $z_0 \in \Gamma_f$ is zero if $z_0 \in \Gamma_f \setminus \cup_{j=1}^3 F_j$, or $z_0 \in F_1$ and ℓ is odd, or $z_0 \in F_2$ and ℓ is even, and ± 1 if $z_0 \in F_1$ and ℓ is even, or $z_0 \in F_2$ and ℓ is odd.

We close this section by illustrating the image surface of f in a neighbourhood of $z_0 \in \Gamma_f \setminus F_3$. For this purpose we recall from the proof of Theorem 5.1 $w_0 = f(z_0)$, $\beta = f(\gamma)$ and B.

Let n be a positive integer, $S(n)$ the n-fold simply connected covering of B which have only one branch point located over w_0 and of order $n - 1$, and $\Pi(n)$ the natural projection map of $S(n)$. Then there exists a lift $\beta(n)$ of β that divides $S(n)$ into two open topological sectors over B of which one, denoted by $R(n)$, lies on the right of $\beta(n)$, and has its vertex over w_0 and vertex angle of size $(2n-1)\pi$ if β is convex and $(2n-2)\pi$ if β is a harmonic cusp.

Now for two sectors $R(n)$ and $R(m)$ we identify the points of the boundary arcs $\beta(n)$ and $\beta(m)$ which have the same projection. This defines a surface, $T(n, m)$ or $T(m, n)$, which together with the projection map, $\Pi(n, m)$ or $\Pi(m, n)$ respectively, defined so that it coincides with $\Pi(n)$ and $\Pi(m)$ in the respective domains, yields a covering surface structure with "fold" over β.

It is easy to verify now that the image surface of f in a neighbourhood of z_0 is as follows.

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(a) $T(1,1)$ if $z_0 \in \Gamma_f \setminus F$.
(b) $T\big((\ell+2)/2, (\ell+4)/2\big)$ if $z_0 \in F_1$ and ℓ is even, and $T\big((\ell+3)/2, (\ell+3)/2\big)$ if $z_0 \in F_1$ and ℓ is odd.
(c) $T\big((\ell+2)/2, (\ell+2)/2\big)$ if $z_0 \in F_2$ and ℓ is even, and $T\big((\ell+1)/2, (\ell+3)/2\big)$ if $z_0 \in F_2$ and ℓ is odd.

6. **Behaviour at the critical points in F_3.** We devote this section for the study of the local behaviour of a light harmonic mapping f at a point $z_0 \in F_3$.

We begin our investigation by noting that we shall be using throughout this section the notation associated with f which appears in part (b) of the proof of Theorem 5.1.

Note that the paths $z_j(t)$, $t \in I$, can be chosen so that $z_0 = z_j(t_0)$ for all j. Then for a given j, $\operatorname{Re}\omega_j$ either (i) changes no signs in I, or (ii) changes signs only at t_0. We say that an arc γ_j is of *type A* if (i) holds and *type B* if (ii) holds. Obviously, $f(\gamma_j)$ is a convex arc if γ_j is of type A, otherwise $f(\gamma_j)$ is a harmonic cusp whose vertex is $w_0 = f(z_0)$. Furthermore, since $\lambda = \psi(z_0) = \psi \circ z_j(t_0) = e^{i\phi_j(t_0)}$ for all j, we conclude from (3.12) that every $f(\gamma_j)$ is tangent at $w_0 = f(z_0)$ to the straight line L: $w = w_0 + t\tau$, where $t \in \mathbb{R}$ and τ is a value of $\lambda^{1/2}$. So we have:

PROPOSITION 6.1. *Suppose that $f = \bar{g} + h \in D$, $z_0 \in F_3$ and $\gamma_1, \gamma_2, \ldots, \gamma_{2m}$ are as previously defined. Then every $f(\gamma_j)$ is a convex arc or a harmonic cusp whose vertex is w_0. Furthermore, all $f(\gamma_j)$ are tangent at w_0 to the same line.*

To have a better idea of $f(\Delta \cap \Gamma)$, we need to find the number of arcs γ_j of types A and B. For this purpose we define

$$\theta_j = \lim_{t \to t_0^-} \arg \omega_j(t) \tag{6.1}$$

when j is odd, and

$$\theta_j = \lim_{t \to t_0^+} \arg \omega_j(t)$$

when j is even. Observe that the values θ_j exist since the arcs α_j are analytic, and that $(-1)^{j+1}(\theta_{j+1} - \theta_j)$ is the discontinuity jump of $\arg \omega_j$ at t_0 for all j.

PROPOSITION 6.2. *Under the assumptions of Proposition 6.1, suppose that z_0 is a zero of h', or equivalently g', of order $\ell \geq 0$, c the number of harmonic cusps $f(\gamma_j)$, and s the integer satisfying $1 - m \leq s \leq m$ and $\ell - m + 1 = s \pmod{2m}$. Then $c = 2|s|$, unless s equals 0 or m and $\theta_j = \pi/2 \pmod{\pi}$ for some j in which case c could be any even integer satisfying $0 \leq c \leq 2m$.*

PROOF. By analyzing $\omega_j(t)$ it is readily seen that

$$\theta_{j+1} - \theta_j = (\ell - m + 1)\pi/m \pmod{2\pi}, \tag{6.3}$$

where $\theta_{2m+1} = \theta_1 \pmod{2\pi}$. It follows that for suitably chosen values θ_j the sequence $\theta_1, \theta_2, \ldots, \theta_{2m}$ can be rewritten in terms of s as

$$\theta_1, \theta_1 + s\pi/m, \ldots, \theta_1 + (2m-1)s\pi/m. \tag{6.4}$$

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Now we consider three cases:

CASE 1. If $s = 0$, then one of two cases hold:

(a) If $\theta_1 \neq \pi/2 \pmod{\pi}$, or equivalently $\theta_j \neq \pi/2 \pmod{\pi}$ for all j. Then we conclude at once that $\operatorname{Re}\omega_j$ changes no signs at t_0, and $c = 0$. It follows that $\operatorname{Re}\omega_j(t)$ $(t \neq t_0)$ is either positive for all j or negative for all j; in the first case $f(\Delta \cap \Gamma)$ is as in Figure 6.1 and in the second as in Figure 6.2.

(b) If $\theta_1 = \pi/2 \pmod{\pi}$, or equivalently $\theta_j = \pi/2 \pmod{\pi}$ for all j. In this case we conclude that each arc $\omega_j(\gamma_j)$ consists of two subarcs $\omega_j(\alpha_j)$ and $\omega_j(\alpha_{j+1})$ such that each has the origin as an endpoint at which it is tangent to the imaginary axis, and which lies otherwise either in the right or left half plane. It follows immediately that c could be an even integer between 0 and $2m$, without any restriction on which of the arcs γ_j are of type A or type B.

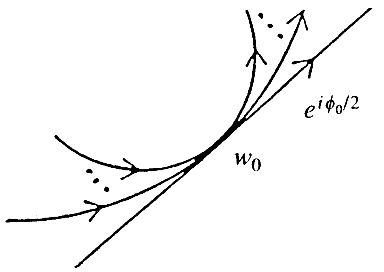

Figure 6.1

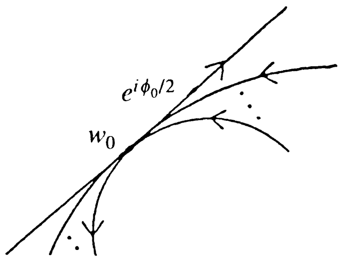

Figure 6.2

CASE 2. If $s = m$, then again one of the following two cases hold:

(a) If $\theta_1 \neq \pi/2 \pmod{\pi}$, then every $\operatorname{Re}\omega_j$ changes signs at t_0 and consequently $c = 2m$. It is easy to see that $\operatorname{Re}\omega_j(t < t_0)$ is either positive for all j or negative for all j; in the former case $f(\Delta \cap \Gamma)$ is as in Figure 6.3 and in the latter as in Figure 6.4.

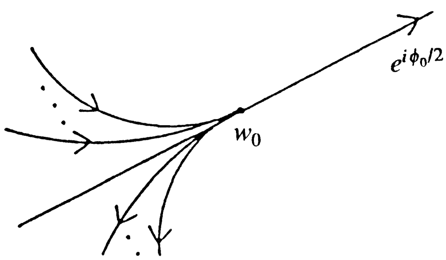

Figure 6.3

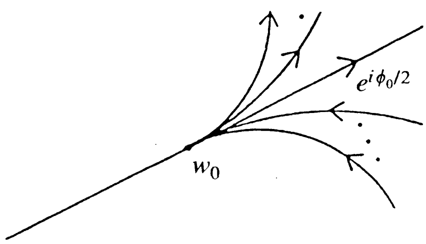

Figure 6.4

(b) If $\theta_1 = \pi/2 \pmod{\pi}$, then this case is identical to Case 1(b).

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CASE 3. If $s \neq 0, m$, then we contend that $c = 2|s|$. We assume that s is positive since otherwise the proof is essentially the same. Let C_j denote the minor subarc of the unit circle whose endpoints are $\exp i(\theta_1 + (j-1)s\pi/m)$ and $\exp i(\theta_1 + js\pi/m)$, where $1 \leq j \leq 2m$. Note that if $\operatorname{Re}\omega_j$ changes signs at t_0, then C_j meets the imaginary axis, but the converse is not necessarily true. For if C_j and C_{j+1} share their common point with the imaginary axis, then obviously only one of the functions $\operatorname{Re}\omega_j$ and $\operatorname{Re}\omega_{j+1}$ changes signs at t_0 since $s\pi/m < \pi$. Let us view for a moment that every such pair of arcs C_j is a single arc having i or $-i$ as an interior point. Since by tracing the arcs C_j positively we cover the unit circle s times, there are exactly $2s$ arcs C_j meeting the imaginary axis. This yields the claim. ∎

As a consequence of the argument of the proof of Case 3 of Proposition 6.2 we conclude:

PROPOSITION 6.3. *Under the assumptions of Proposition 6.2, suppose that γ_i and γ_j, where $1 \leq i < j \leq 2m$, are two arcs of type B satisfying the property that every γ_k, where $i < k < j$, is of type A. Then*

$$\Big|i - j - m/|s|\Big| \leq 1. \tag{6.5}$$

REMARK 6.1. As we go around w_0 setting the arcs $f(\gamma_j)$ in the order $f(\gamma_1), f(\gamma_2), \ldots, f(\gamma_{2m})$, the arc $f(\gamma_{2m})$, whether convex or a harmonic cusp, must be completely determined by $f(\gamma_1)$ and $f(\gamma_{2m-1})$. For assume with no loss of generality that $w_0 = 0$ and $\lambda = 1$, then each $f(\alpha_j)\setminus\{w_0\}$ lies in one of the four quadrants of $\mathbb{C}$. It is not hard to see that the quadrant of $f(\alpha_j)$ depends only on that of $f(\alpha_1)$ and the number of arcs $f(\gamma_k)$, where $1 \leq k \leq j$, of types A and B. Hence, to determine the quadrant of $f(\alpha_{2m})$ it suffices to consider one of the two cases where the union of the sectors $\bar{\Delta}_j$ bounded by the arcs γ_j of each type minus z_0 is connected. In either case we conclude that $f(\alpha_{2m})$ can be only a convex arc or a harmonic cusp.

The main result of this section is a strong version of Theorem 3.1 which we state as follows.

THEOREM 6.1. *Suppose that $f \in D$ and $z_0 \in F_3$, and let m, ℓ and c be as previously defined. Then there exist positive integers $n_1, n_2, \ldots, n_{2m}$ satisfying*

$$\sum_{j=1}^{2m} n_j = \ell + m + c/2 + 1, \tag{6.6}$$

such that f satisfies (3.11); furthermore

$$V_f(z_0) \leq \ell + m + c/2 + 1. \tag{6.7}$$

PROOF. By virtue of Theorem 3.1 we need only establish (6.6) and (6.7). Note that we shall continue to use the notation of part (b) of the proof of Theorem 3.1.

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In proving (6.6) we shall use without mention some of the details which have already appeared in the proof of Theorem 5.1.

Let $\beta_j = f(\gamma_j)$, $\beta = \cup_{j=1}^{2m}\beta_j$, and B an open disc centered at $w_0 = f(z_0)$ and whose boundary meets every β_j exactly twice. We choose B so that the set $S = \Delta \cap f^{-1}(B)$ is a Jordan subdomain of Δ which satisfies the same properties of Δ, in addition to the property that for every $S_j = S \cap \bar{\Delta}_j$ there exist sense preserving homeomorphisms $h_j\colon S_j \longrightarrow (|\zeta| < 1, \operatorname{Im}\zeta \geq 0)$ and $k_j\colon \mathbb{C} \longrightarrow \mathbb{C}$ satisfying $h_j(z_0) = k_j(w_0) = 0$ such that $k_j \circ f \circ h_j^{-1}(\zeta) = (\zeta^*)^{2n_j-1}$, where $\zeta^* = \zeta$ if j is odd and $\zeta^* = \bar{\zeta}$ otherwise. As before, denote by L the line $w = w_0 + t\tau$ $(t \in \mathbb{R})$, where τ is a value of $\lambda^{1/2}$, and by $[w_1, w_2]$, where $w_1, w_2, \in \partial B$, the diameter of ∂B in L, then $L \cap \beta = \{w_0\}$. Now the inverse images of the radii $[w_0, w_1]$ and $[w_0, w_2]$ under f in each S_j is a set of endcuts of S, denoted by $\alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jq}$, where $q = 2n_j$ if γ_j is a type A arc and $q = 2n_j - 1$ otherwise, which have z_0 as a common endpoint and are otherwise disjoint in $\bar{S}$, and which satisfy the property that each maps under f homeomorphically either to $[w_0, w_1]$ or $[w_0, w_2]$. Since there are c arcs γ_j of type B, the total number of arcs α_{jk} in S is

$$\sum_{j=1}^{2m}(2n_j - 1) + 2m - c = 2\sum_{j=1}^{2m} n_j - c. \tag{6.8}$$

Now recall the function F_λ for the assumed value λ. Then by arguing as in Step 2 of the proof of Theorem 5.1 we conclude that z_0 is a zero of F'_λ of order $\ell + m$. We claim that there exist a Jordan domain G containing z_0 and satisfies the properties of Δ, and an open disc D centred at $\xi_0 = F\lambda(z_0)$ and of radius, r, such that $(\bar{G}, F_\lambda)$ is an $(\ell + m + 1)$-fold covering of $\bar{D}$ which has only one branch point over $\bar{D}$ located at z_0 and is of order $\ell + m$. To see this, we argue again as in the same Step 2 above but by replacing the Möbius transformation $T(\nu)$ by $T(\nu^m)$.

Suppose now that $\bar{G}$ is chosen a subset of S. Then once more as in the above Step 2, each α_{jk} meets ∂G exactly once, say at a_{jk}, such that the points a_{jk} are the only points of ∂G which map under F_λ to $\xi_0 \pm r\tau$. It follows that the number of points a_{jk} is $2\sum_{j=1}^{2m} n_j - c$, and also $2(\ell + m + 1)$ since $(\bar{G}, F_\lambda)$ is an $(\ell + m + 1)$-fold covering of $\bar{D}$. Therefore

$$2\sum_{j=1}^{2m} n_j - c = 2(\ell + m + 1)$$

and (6.6) holds.

To prove (6.7), observe that the local valency of f in each $\bar{\Delta}_j$ is exactly n_j and is independent of the arc type of γ_j. Also, observe that these valences do not necessarily add up, for f may map distinct sectors $\bar{\Delta}_j$ to disjoint sets except for w_0. Therefore

$$V_f(z_0) \leq \sum_{j=1}^{2m} n_j$$

which by (6.6) gives (6.7). ■

7. **Light harmonic mappings and folded coverings.** The purpose of this section is twofold: first to introduce the notion of a folded covering, and second to show that a light harmonic mapping affects a folded covering of $\mathbb{C}$.

The mapping f of the open unit disc defined by $f(z) = f(\bar{z}) = z$ for $\operatorname{Im} z \geq 0$, maps the upper and lower semi-discs homeomorphically to the upper semi-disc so that every image point w with $\operatorname{Im} w > 0$ has only two inverse images while every real w corresponds only to itself. It seems appropriate to say that f defines a covering surface of the semi-disc: $|w| < 1$ and $\operatorname{Im} w \geq 0$, with a fold along $-1 < z < 1$.

DEFINITION 7.1. Let $\tilde{F}$ and F be Riemann surfaces and let $f: \tilde{F} \longrightarrow F$. The pair $(\tilde{F}, f)$ defines a *folded covering* of F if there exists a triangulation of $\tilde{F}$ with complex $\tilde{K}$ such that f maps every 2-simplex of $\tilde{K}$ homomorphically into F; we call f the *projection map of* f.

Suppose that $\tilde{s}_1$ and $\tilde{s}_2$ are 2-simplices of $\tilde{K}$ which are adjacent along a 1-simplex $\tilde{\sigma}$. If f is sense preserving in one and reversing in the other, then we call $\tilde{\sigma}$ a *part of the fold of the covering*, and we call the union of all such $\tilde{\sigma}$ the *fold of the covering*.

REMARK 7.1. This definition is satisfied if $(\tilde{F}, f)$ is a ramified covering surface. Suppose that $\tilde{K}$ is the complex of a triangulation of $\tilde{F}$. If $\tilde{K}$ is finite, then this follows at once since for every open covering U of $\tilde{F}$ there exists a triangulation of $\tilde{F}$ which is finer than U. This triangulation can be constructed by taking successive barycentric subdivisions of $\tilde{K}$. The definition also holds if $\tilde{K}$ is also infinite, but the proof is more complicated and uses the fact that every open polynhedron has a subdivision which allows a canonical exhaustion (see [2; pp. 61–64]).

REMARK 7.2. Restricted types of folded coverings seem to have been first introduced and studied by A. W. Tucker [10], and apparently never used until recently by A. Bouchet ([4], [5]) for the purpose of graph embedding. Tucker assumes that f is simplicial between $\tilde{F}$ and F which makes these coverings special so that they fail to contain the classical coverings, and hence not suitable to describe the geometry of general classes of functions.

Since every triangulation of a compact Riemann surface consists of finitely many triangles, we conclude at once:

PROPOSITION 7.1. *If $(\tilde{F}, f)$ is a compact folded covering of F, then f is N-valent for some N.*

We have considered so far the mapping properties of light harmonic mappings of simply connected domains in $\mathbb{C}$, and concluded that these properties are of purely local nature. This fact enables us to extend directly the notions and results of the previous sections to light harmonic mappings of Riemann surfaces.

Suppose now that (W, ϕ) is Riemann surface of conformal structure ϕ, and $f: W \longrightarrow \mathbb{C}$ is a light harmonic mapping of W. Then we can automatically assume the associated sets Γ_f, N and $F = \cup_{j=1}^{3} F_j$ and the local behaviour of f at points of these sets.

The main result of this section can now be stated as follows.

THEOREM 7.1. *Let f be a light harmonic mapping of (W, ϕ). Then (W,f) is a folded covering of $\mathbb{C}$ whose fold is Γ_f.*

PROOF. It follows by Theorem 2.2 that there exists a collection of mutually disjoint domains Δ such that every critical point p of f belongs to some Δ which contains no other critical point of f.

We choose each Δ sufficiently small so that (i) if $p \in N$ then $\bar{\Delta}$ admits a sector subdivision from p by some endcuts, β, from p such that f is homeomorphic on each of the sectors, and (ii) if $p \in F$ then $\bar{\Delta} \cap \Gamma$ is a finite set of endcuts α of Δ from p that yields a sector subdivision of $\bar{\Delta}$ from p, which can be refined by adding a finite number of analytic endcuts β of Δ from p to a new sector subdivision such that f becomes homeomorphic on each of the new sectors.

Now let Γ^* be the point-set union of Γ and all the additional arcs β, and let $p \in \Gamma^*$ be a noncritical point of f. If $p \in \operatorname{int}(\Gamma^*)$, then by Theorem 5.1 (a) there exists a Jordan domain Δ containing p such that Γ^* divides Δ into two Jordan domains in whose closure f is homeomorphic. If $p \notin \operatorname{int}(\Gamma^*)$, i.e. p is the endpoint not in Γ of the added arcs β, then f is locally 1-1 at p and there exists a Jordan domain Δ containing p such that $\Delta \cap \beta$ is an endcut of Δ and f is homeomorphic on $\bar{\Delta}$.

On the other hand, if $p \notin \Gamma^*$ then there exists a Jordan domain containing p such that $\bar{\Delta} \cap \Gamma^* = \emptyset$ and f is homeomorphic on $\bar{\Delta}$.

We conclude that every domain Δ thus defined has its closure either (A) admits a sector subdivision from the corresponding p via analytic arcs α and β in Γ^*, or (B) meets Γ^* only in a subarc of some β, or (C) satisfies $\bar{\Delta} \cap \Gamma^* = \emptyset$.

Now we choose the domains Δ small enough so that each $\bar{\Delta}$ lies in the domain of some $\varphi \in \phi$. Obviously, the collection O of all Δ forms an open covering of W. Consider now a triangulation of W whose complex, K_1, is finer than O, i.e. every 2-simplex s of K_1 is contained in some Δ (see [8; pp. 125–126].) In fact, by using ϕ every 1-simplex of K_1 can be chosen analytic so that it meets Γ^* in at most finitely many points.

We shall now construct a triangulation of W whose complex, K_2, is a refinement of K_1, contains Γ^* as a subset of the point-set union of its 1-simplices, and admits every critical point of f as a 0-simplex. Let $s = (a, b, c)$ be a 2-simplex of K_1 with 0-simplices a, b and c, then s is a subset of some Δ. Suppose that (A) above holds. Since s meets the arcs α and β in finitely many points, s meets α and β along a finite number of cross-cuts. For convenience, let us relabel the arcs α and β by $\delta_1, \delta_2, \ldots, \delta_r$. We consider first the cross-cuts on δ_1, if any. The first cross-cut divides s into two Jordan domains. One of these domains is further divided by the second cross-cut, and so on. It follows that δ_1 divides s into a finite number of Jordan domains. Each of these domains either does not meet δ_2 or in turn is divided by δ_2 into a finite number of Jordan domains. Since this process is finite, we finally reach a finite number of Jordan subdomains, G, of s whose boundary consists of finitely many subarcs, σ, of the arcs δ_j and the 1-simplices of s that satisfy the property that $\operatorname{int}(\sigma) \cap \delta_j = \emptyset$ for all j. Now we replace s and its 1-simplices in K_1 by a new set of 0, 1 and 2-simplices constructed as follows. For each G choose an interior point, which we join through G to the endpoints of the arcs σ bounding G and to p if $p \in s \cap \partial G$. This divides G into triangular closed domains. The totality of these domains over all G, together with their vertices and edges form the natural replacement of s and its 1-simplices in K_1. Suppose now that (B) holds. Then either $s \cap \beta = \emptyset$ and s remains unchanged, or $s \cap \beta \neq \emptyset$ and $p \notin \mathrm{int}(s)$ which is a special case of (A), or $p \in \mathrm{int}(s)$ and we replace β by the subarc of β extending from Γ to the last point of intersection of β with ∂s. This is again a special case of (A). Finally, if (C) holds, then s remains unchanged. By proceeding like this with every 2-simplex s of K_1, we obtain the desired complex K_2.

It is easy to verify that K_2 satisfies the desired properties that makes (W,f) a folded covering of $\mathbb{C}$ with Γ as the fold. ∎

REMARK 7.3. There is a large literature on harmonic mappings defined on varieties. The reason lies in the connection between harmonic mappings and minimal surfaces. Here we explain it for the simple case of $\mathbb{R}^3$. A surface S is a minimal surface if and only if there are isothermic parameters x, y on a domain D such that S can be expressed by $S = \big(u(z), v(z), w(z)\big)$ where $z = x + iy$, such that u, v and w are harmonic functions on D satisfying $u_z^2 + v_z^2 + w_z^2 = 0$. Put $f = u + iv = \bar{g} + h$ and $\psi = h'/g'$. Then f can be regarded as the projection of a minimal surface via the isothermic parameters and the normal vector at a given point P of S is then given by the Gauss map $n(z) = -(2\,\mathrm{Im}\,\psi, 2\,\mathrm{Re}\,\psi, |\psi| - 1)/(|\psi| + 1)$ which is locally well-defined for each z for which ψ has not a zero of odd order. Moreover, $|\psi| = 1$ implies that the normal direction of the minimal surface is horizontal. This also explains the folding property on Γ_f and also Theorem 2.3. (see[7]).

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