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# A SELF-CENTRALIZING CHARACTERISTIC SUBGROUP

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**Abstract**

In this note we introduce a self-centralizing characteristic subgroup, associated with quasinilpotent injectors, of a finite group.

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All groups in this paper are assumed to be finite. Most of our notation is standard and may be found in [9] and [10]. Let $d(G)$, $d_\infty(G)$ denote, respectively, the maximum of the orders of the abelian and nilpotent subgroups of $G$. Further, $\mathbf{A}(G)$ $[\mathbf{A}_\infty(G)]$ denotes the set of all abelian [nilpotent] subgroups of order $d(G)$ $[d_\infty(G)]$ in $G$ and $J(G) = \langle \mathbf{A}(G) \rangle$ is the Thompson subgroup of $G$.

Let $p$ be a prime and let $P$ be a Sylow $p$-subgroup of a group $G$. In [8] Glauberman introduces the characteristic subgroup $ZJ^*(P)$, and proves that $ZJ^*(P)$ has some analogous properties to $ZJ(P) = Z(J(P))$, and moreover that it is self-centralizing, that is, $C_G(ZJ^*(P)) \leq ZJ^*(P)$. Some related results were obtained by Ezquerro in [6], where Glauberman's definition of $ZJ^*$ is set forth for an arbitrary finite group $G$ (instead of $P$).

DEFINITIONS. For any group $K$ define two sequences of characteristic subgroups of $K$ as follows. Let $ZJ^0(K) = 1$ and $K_0 = K$. Given $ZJ^i(K)$ and $K_i$, $i \geq 0$, let $ZJ^{i+1}(K)$ and $K_{i+1}$ be the subgroups of $K$ that contain $ZJ^i(K)$ and satisfy

$$ZJ^{i+1}(K)/ZJ^i(K) = ZJ(K_i/ZJ^i(K)),$$
$$K_{i+1}/ZJ^i(K) = C_{K_i/ZJ^i(K)}(ZJ^{i+1}(K)/ZJ^i(K)).$$

Let $n$ be the smallest integer such that $ZJ^n(K) = ZJ^{n+1}(K)$; then $ZJ^n(K) = ZJ^{n+r}(K)$ and $K_n = K_{n+r}$, for every $r \geq 0$. Set $ZJ^*(K) = ZJ^n(K)$ and $K_* = K_n$.

The main aim of this paper is to prove the following

THEOREM. *Let $G$ be a group such that for every odd $p$ in $\Pi(G/Z(G))$, the special affine group $SA(2,p)$ is not involved in $G$. Let $K$ be an $\mathbf{N}^*$-injector of $G$ and assume that $|F(G)| \neq 1$ is odd. Then*

(i) *$ZJ^i(K)\,\mathrm{char}\,G$, for every $i \geq 0$; in particular, $ZJ^*(K)\,\mathrm{char}\,G$;*

(ii) *$C_G(K_*) \leq K_* = E(G)ZJ^*(K)$; indeed, $C_G(K_*) = Z(ZJ^*(K))$.*

Here $\Pi$ denotes a set of primes and $\mathbf{N}$, $\mathbf{N}_\Pi$ and $\mathbf{N}^*$ denote the classes of nilpotent, nilpotent $\Pi$-groups and quasinilpotent groups, respectively. It is well known that all groups have a unique conjugacy class of $\mathbf{N}^*$-injectors, and these are the maximal $\mathbf{N}^*$-subgroups containing the quasinilpotent radical of the group [5].

The following simple observations are due to L. M. Ezquerro [6].

LEMMA 1. (i) *If $\psi: K \to H$ is an isomorphism, then for every $i \geq 0$, $ZJ^i(H) = \psi(ZJ^i(K))$.*

(ii) *$ZJ^{i+1}(K)/ZJ^1(K) = ZJ^i(K_1/ZJ^1(K))$ for every $i \geq 0$.*

(iii) *$C_K(ZJ^*(K)) \leq K_i$ for every $i \geq 0$.*

LEMMA 2. *Let $K$ be an $\mathbf{N}^*$-injector of the group $G$.*

(i) *For every $i \geq 0$, $ZJ^i(K)$ is nilpotent.*

(ii) *$K_i = E(G)F(K_i)$ and $F(K_i/ZJ^i(K)) = F(K_i)/ZJ^i(K)$, for every $i \geq 0$.*

(iii) *$ZJ^{i+1}(K) = ZJ^i(K)$ if and only if $F(K_i) = ZJ^i(K)$. Thus, $K_* = E(G)ZJ^*(K)$.*

PROOF. (i) By induction on $i$ we may assume that for some $i$, $ZJ^i(K)$ is nilpotent. Then $ZJ^{i+1}(K) \in \mathbf{N}^*$ is soluble and hence nilpotent.

(ii) Notice that for every $i \geq 0$, $E(G) \leq K_i \trianglelefteq K = E(G)F(K)$, so $K_i = E(G)F(K_i)$. For each $i \geq 0$, put $F/ZJ^i(K) = F(K_i/ZJ^i(K))$. Then, since $ZJ^i(K) \in \mathbf{N}$, $F \in \mathbf{N}^*$ is soluble. Consequently, $F$ is nilpotent and $F(K_i/ZJ^i(K)) = F(K_i)/ZJ^i(K)$.

(iii) For each $i \geq o$, we obtained from (ii) that

$$K_i/ZJ^i(K) = (E(G)ZJ^i(K)/ZJ^i(K))(F(K_i)/ZJ^i(K))$$
$$= (E(G)ZJ^i(K)/ZJ^i(K))(F(K_i/ZJ^i(K))).$$

Thus our claim is clear from $ZJ(K_i/ZJ^i(K)) \neq 1 \leftrightarrow F(K_i/ZJ^i(K)) \neq 1$.

DEFINITION [14, Definition 1]. Let $\mathbf{F}$ be a Fitting class. A group $G$ is said to be $\mathbf{F}$-stable if whenever $A$ is an $\mathbf{F}$-subgroup of $G$ and $B$ is an $\mathbf{F}$-subgroup of $N_G(A)$ such that $[A, B, B] = 1$, then $BC_G(A)/C_G(A) \leq (N_G(A)/C_G(A))_{\mathbf{F}}$.

LEMMA 3 [15]. *The class* $\mathbf{N}^*_\Pi = (G = F(G)E(G)$ *such that* $F(G)$ *is a* $\Pi$-*group) is a Fitting class.*

LEMMA 4 [15]. *If* $G$ *is an* $\mathbf{N}_\Pi$-*stable group, then* $G$ *is* $\mathbf{N}^*_\Pi$-*stable, and if* $A$ *and* $B$ *are* $\mathbf{N}^*_\Pi$-*subgroups such that* $B$ *normalizes* $A$ *and* $[A, B, B] = 1$, *then* $BC_G(A)/C_G(A) \leq O_\Pi(F(N_G(A)/C_G(A)))$.

LEMMA 5. *Let* $G$ *be an* $\mathbf{N}_{2'}$-*stable group. Let* $K$ *be a subgroup of* $G$ *which contains* $F^*(G)$. *Then* $O_{2'}(ZJ(K)) \lhd\lhd G$. *Moreover, if* $|F(G)|$ *is odd, then* $ZJ(K) \lhd\lhd G$.

PROOF. Using Lemma 4 on $F = E(G)O_{2'}(F(G))$ and $Z = O_{2'}(ZJ(K))$ we obtain $ZC_G(F)/C_G(F) \leq F(G/C_G(F))$. Hence $ZC_G(F^*(G)) = ZC_G(F) \cap C_G(O_2(F(G))) \lhd\lhd G$. Since $C_G(F^*(G)) \leq F^*(G)$, $Z \lhd\lhd G$. Moreover if $|F(G)|$ is odd, then $O_2(ZJ(K)) \leq C_G(F^*(G)) \leq F(G)$ and $O_2(ZJ(K)) = 1$.

Now, arguing like in the proof of [14, Theorem A], with small changes, we obtain

PROPOSITION 1. *Let* $G$ *be an* $\mathbf{N}_{2'}$-*stable group such that* $F(G) \neq 1$ *is not a 2-group. If* $K$ *is an* $\mathbf{N}^*$-*injector of* $G$, *then* $1 \neq O_{2'}(ZJ(K)) \lhd G$. *In particular if* $|F(G)|$ *is odd, then* $ZJ(K) \lhd G$.

PROOF OF THE THEOREM. (i) Assume that the result is false. Let $G$ be counterexample of least order and put $T = ZJ(K)$, $T^i = ZJ^i(K)$, $T^* = ZJ^*(K)$. By [7, Theorem A] $G$ is $\mathbf{N}_{2'}$-stable. Therefore, because of Proposition 1 we have $1 \neq T \lhd G$. Thus, $C = C_G(T) \neq 1$. Assume that $C < G$.

By our minimal choice of $G$, we have that for each $i \geq 0$, $ZJ^i(K \cap C)$ char $C$. Since $J(K) \leq K \cap C$, it follows that $J(K) = J(K \cap C)$ and $T = ZJ(K \cap C)$. Moreover, $K_1 = C_K(T) = C_{K \cap C}(ZJ(K \cap C))$.

By induction on $i$, it is clear that for each $i \geq 0$, $T^i = ZJ^i(K \cap C)$. Thus, for every $i \geq 0$, $T^i \trianglelefteq G$. But Lemma 1 (i) and the conjugacy of all $\mathbf{N}^*$-injectors of

$G$ imply that for every $i \geq 0$, $T^i \operatorname{char} G$, contrary to our choice of $G$. Therefore we can assume that $G = C$, (that is, $Z(G) = T$), and that $G/Z(G)$ satisfies the hypothesis of the theorem. By our minimal choice of $G$, we obtain that for every $i \geq 0$, $ZJ^i(K/Z(G)) \operatorname{char} G/Z(G)$. Since $K_1 = C_K(T) = K$, and by Lemma 1 (ii), it follows that for each $i \geq 0$, $T^{i+1}/Z(G) = ZJ^i(K/Z(G))$. Thus for every $i \geq 0$, $T^{i+1} \operatorname{char} G$, contrary to our choice of $G$.

(ii) By Lemma 2 (iii) we know that $K_* = E(G)T^*$. Now, in view of Lemma 1 (iii), it is clear that for every $i \geq 0$, $C_K(K_*) \leq C_K(T^*) \leq K_i$. In particular, $C_K(K_*) \leq K_*$. Thus, $C_K(K_*) = Z(K_*) = Z(T^*)$.

Part (i) and Lemma 2 (iii) imply that $C_G(K_*) \trianglelefteq G$. Hence $Z(K_*) = K \cap C_G(K_*)$ in an $\mathbf{N}^*$-injector of $C_G(K_*)$. Moreover, since $Z(K_*) \leq Z(C_G(K_*))$, we get that $Z(K_*) = Z(C_G(K_*)) = F(C_G(K_*))$. But from $C_G(K_*) \trianglelefteq C_G(E(G))$ we get that $C_G(K_*)$ is $\mathbf{N}$-constrained in the sense of [12]; see [11], [13]. Therefore, $C_G(K_*) = C_{C_G(K_*)}(Z(K_*)) = Z(K_*)$.

If $G$ is an $\mathbf{N}$-constrained group, then $E(G) = 1$ and the $\mathbf{N}$-injectors are the $\mathbf{N}^*$-injectors (see [12], [13] and [11]). So, we obtain immediately

COROLLARY 1. *Under the same assumptions as in the theorem and if moreover $G$ is $N$-constrained, then $K$ is an $\mathbf{N}$-injector of $G$, $ZJ^*(K) \operatorname{char} G$ and*

$$C_G(ZJ^*(K)) \leq ZJ^*(K).$$

Next we shall consider an analogue of our Theorem for arbitrary Fitting classes $\mathbf{F}$.

PROPOSITION 2 [6, THEOREM II.3.10]. *Assume that $2 \notin \Pi$ and $G$ is a $\Pi$-soluble group whose Sylow 2-subgroups are abelian and such that $O_{\Pi'}(G) = 1$. If $K$ is a Hall $\Pi$-subgroup of $G$, then $ZJ^*(K) \operatorname{char} G$ and $C_G(ZJ^*(K)) \leq ZJ^*(K)$.*

LEMMA 6. *Assume that $2 \notin \Pi$ and $G$ is a $\Pi$-soluble group and $H$ is a subgroup of $G$ contained in a Hall $\Pi$-subgroup $K$ of $G$ such that $d_\infty(H) = d_\infty(K)$. Let $A \in \mathbf{A}_\infty(H)$. Then $ZJ^i(A) = ZJ^i(H) = ZJ^i(K)$, for every $i \geq 0$. In particular, $ZJ(H) = ZJ(K)$ and $ZJ^*(H) = ZJ^*(K)$.*

PROOF. Notice first that if $G$ is a group of odd order, then $\mathbf{A}_\infty(G)$ is the set of $\mathbf{N}$-injectors of $G$. (See [4, Corollary 5] and [12, Theorem 1].)

We proceed by induction on $i$. Because of [3, Proposition (1.3)], it follows that $ZJ(H) = ZJ(A) = ZJ(K)$. Assume now that for some $i \geq 1$, $ZJ^i(H) = ZJ^i(A) = ZJ^i(K)$. Set $Z = ZJ(A) = ZJ(H) = ZJ(K)$. Note that $A_1 = C_A(Z) = A \cap H_1 = A \cap K_1$ is an $\mathbf{N}$-injector of $H_1 = C_H(Z)$ and $K_1 = C_K(Z)$, and obviously $Z \le A_1$. Hence, from [12, Theorem 1] we get that $A_1/Z$ is an $\mathbf{N}$-injector of $H_1/Z$ and of $K_1/Z$. Therefore, $A_1/Z \in \mathbf{A}_\infty(H_1/Z) \subseteq \mathbf{A}_\infty(K_1/Z)$.

The group $C_G(Z)/Z$ satisfies the hypothesis concerning $G$. Thus,

$$ZJ^i(A_1/Z) = ZJ^i(H_1/Z) = ZJ^i(K_1/Z).$$

Now, from Lemma 1 (ii) we have

$$ZJ^{i+1}(A) = ZJ^{i+1}(H) = ZJ^{i+1}(K).$$

In what follows let $\Pi(\mathbf{F})$ denote the set of all primes dividing the order of some $\mathbf{F}$-group. The next lemma and proposition are proved following a process analogous to the proofs of Lemma IV.2.3 and Theorem IV.2.4 in [13]. We denote by $\mathbf{F}$ a Fitting class.

LEMMA 7. *Let $G$ be a $\Pi(\mathbf{F})$-soluble group and $N \trianglelefteq G$ such that $G/N \in \mathbf{N}$. If $V_1$ and $V_2$ are $\mathbf{F}$-maximal subgroups of $G$ which contain an $\mathbf{F}$-maximal subgroup $W$ of $N$, then there exists $g \in G$ such that $V_1^g = V_2$.*

PROPOSITION 3. *If $G$ is a $\Pi(\mathbf{F})$-soluble group, then $G$ has a unique conjugacy class of $\mathbf{F}$-injectors.*

REMARK. If $2 \notin \Pi = \Pi(\mathbf{F})$ and $H$ is an $\mathbf{F}$-injector of the $\Pi$-soluble group $G$ contained in the Hall $\Pi$-subgroup $K$ of $G$ such that $d_\infty(H) = d_\infty(K)$, then by Lemma 6, for every $i \ge 0$ and $A \in \mathbf{A}_\infty(H)$,

$$ZJ^i(A) = ZJ^i(H) = ZJ^i(K)$$

and in particular, $ZJ(H) = ZJ(K)$ and $ZJ^*(H) = ZJ^*(K)$.

From Proposition 2 and the preceding remark we obtain the following

COROLLARY 2. *Assume that $2 \notin \Pi = \Pi(\mathbf{F})$ and $G$ is a $\Pi$-soluble group with abelian Sylow 2-subgroups and $O_{\Pi'}(G) = 1$. If $H$ is an $\mathbf{F}$-injector of $G$ contained in the Hall $\Pi$-subgroup $K$ of $G$ such that $d_\infty(H) = d_\infty(K)$, then $ZJ^*(H)$ char $G$ and $C_G(ZJ^*(H)) \le ZJ^*(H)$.*

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