Ergod. Th. & Dynam. Sys. (1988), **8**, 155–172
Printed in Great Britain

Minimal flows of finite almost periodic rank

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(*Received* 7 *March* 1986 *and revised* 13 *January* 1987)

Abstract. The totally minimal flow (X, T) is said to have finite almost periodic rank if there is a positive integer n such that whenever $(x_1, x_2, \ldots, x_{n+1})$ is an almost periodic point of the product flow $(X^{n+1}, T\times\cdots\times T)$ then, for some $i\neq j$, x_i and x_j are in the same orbit. The rank of (X, T) is the smallest such integer. If (Y, S) is a graphic flow, (Y, S^n) has rank $|n|$ and it is shown that every finite rank flow has, modulo a proximal extension, a graphic power factor. Various classes of finite rank flows are defined, and characterized in terms of their Ellis groups. There are four disjoint types which have basic structural differences.

This paper continues a program initiated by the authors in [**2**]. In that paper the graphic minimal flows were studied. These are the totally minimal flows (X, T) for which the only minimal subsets of the product flow are graphs of powers of the generating homeomorphism. Equivalently, if (x, x') is an almost periodic point of the product flow, then $x' = T^m(x)$ for some integer m (so x and x' are not 'independent'). It is natural to consider the somewhat more general situation in which there are only a finite number of 'independent' almost periodic points. That is, there is a positive integer n such that if $(x_1, \ldots, x_{n+1})$ is an almost periodic point in the product flow $(X^{n+1}, T\times\cdots\times T)$, then for some i, j with $1\leq i<j\leq n+1$, x_i and x_j are on the same orbit. The study of such 'finite almost periodic rank' minimal flows is the purpose of the present paper.

An almost periodic set mod T is defined to be a subset of X such that no two points are in the same orbit and which defines an almost periodic point in the appropriate product flow. The almost periodic rank is then defined to be the cardinality of a maximal almost periodic set mod T (all such sets have the same cardinality). Our interest here is in the case that the rank is finite, but we expect that the notion of almost periodic rank will be useful for arbitrary flows. For this reason we have developed the basic properties of the almost periodic rank in a general setting.

If (Y, S) is a graphic minimal flow and n is an integer, it is immediate that the flow (Y, S^n) has rank $|n|$. Our main result (theorem 3.5) is that every finite rank totally minimal flow (X, T) has a proximal extension which has a graphic power factor and that the structure of (X, T) can be described by means of this graphic power factor, which is an invariant of the proximal class of (X, T). This structure of (X, T) is based on finite group extensions, regular minimal flows, and the regularizer of a minimal flow. The latter turns out to be intimately connected with

https://doi.org/10.1017/S0143385700004399 Published online by Cambridge University Press

the almost periodic rank of a flow and a new *ab initio* treatment of regularizers is included.

As indicated above, this paper is a successor to [**2**], which was itself inspired by [**6**]. We assume familiarity with the first half of [**2**], as well as the standard notions of topological dynamics. (A brief discussion is in § 1 of [**2**], and a general reference is the monograph of Glasner, [**4**].)

A large portion of this paper relies only on the standard topological notions of dynamics. However, because our point of view is more or less to regard proximally equivalent flows as the same, there are important connections with the algebraic theory of minimal sets - specifically the Ellis group, which is a complete invariant for proximal equivalence. We have tried to organize the material so that the proof of the main theorem on the structure of finite rank minimal flows requires the least background and that the algebraic components are developed.

We would like to thank our colleagues Gertrude Ehrlich and James Schafer for valuable discussions concerning group theory. We also wish to thank Professor Andres del Junco for a number of penetrating observations on the subject of this paper. In particular he suggested proposition 1.11, which has several consequences including a proof of proposition 2.8 which is shorter and more elegant than our original proof.

1. *The rank of a flow*

Given a flow (X, T) and an index set Γ, we consider the flow (X^Γ, T_Γ) on the product space X^Γ in which T acts on each coordinate. Specifically, X^Γ is the set of all functions from Γ into X with the Tychonoff topology and $T_\Gamma : X^\Gamma \to X^\Gamma$ is defined by $[T_\Gamma(z)]_\gamma = T(z_\gamma)$ where w_γ denotes the value of $w \in X^\Gamma$ at $\gamma \in \Gamma$. Let z be an almost periodic point of (X^Γ, T_Γ). Because permutations of Γ define automorphisms of (X^Γ, T_Γ) and projections are homomorphisms, only the range of z denoted by $z_\Gamma = \{z_\gamma : \gamma \in \Gamma\}$ is relevant for almost periodicity. In particular it is easy to prove the following remarks.

Remark 1.1. Let z be a point of X^Γ, let Γ' be a subset of Γ, and $w \in X^{\Gamma'}$ be the restriction of z to Γ'. If z is an almost periodic point of (X^Γ, T_Γ), then w is an almost periodic point of $(X^{\Gamma'}, T_{\Gamma'})$. Conversely, if w is an almost periodic point of $(X^{\Gamma'}, T_{\Gamma'})$ and $w_{\Gamma'} = z_\Gamma$, then z is an almost periodic point of (X^Γ, T_Γ).

Remark 1.2. Let Γ and Γ' be any two index sets and let $z \in X^\Gamma$ and $w \in X^{\Gamma'}$ be such that $z_\Gamma = w_{\Gamma'}$. Then z is an almost periodic point of (X^Γ, T_Γ) if and only if w is an almost periodic point of $(X^{\Gamma'}, T_{\Gamma'})$.

A subset Λ of X is said to be an *almost periodic set* if there exists an index set Γ and an almost periodic point z of (X^Γ, T_Γ) such that $z_\Gamma = \Lambda$. It follows from the second remark that if Λ is an almost periodic set, then any point z in any X^Γ such that $z_\Gamma = \Lambda$ is an almost periodic point of (X^Γ, T_Γ). When it is convenient we can choose Γ and $z \in X^\Gamma$ such that $z_\Gamma = \Lambda$ and z is injective. Also note that by the first remark non-empty subsets of almost periodic sets are themselves almost periodic sets.

Using the basis for the Tychonoff topology, we see that Λ is an almost periodic set if and only if every finite subset of Λ is an almost periodic set. It follows that if Λ_i is a family of almost periodic sets which are linearly ordered by inclusion, then $\bigcup \Lambda_i$ is also an almost periodic set. Hence by Zorn's lemma maximal almost periodic sets exist. If Λ is an almost periodic set and $x \in \Lambda$, then $\Lambda \cup \{T^j(x): j \in Z\}$ is also an almost periodic set. Thus any maximal almost periodic set Λ is an invariant set.

A general approach to finding dynamical invariants is to count orbits in some way. We will show that the cardinality of the set of orbits in a maximal almost periodic set is an invariant of the flow. To this end we say a subset Λ of X is an *almost periodic set* mod T if it is an almost periodic set and distinct points of Λ lie on distinct orbits, i.e. $x, y \in \Lambda$ and $x = T^j(y)$ implies $j = 0$. Clearly maximal almost periodic sets mod T exist. The following remarks will be useful:

Remark 1.3. Let $z \in X^\Gamma$ and let $w \in \overline{O(z)}$. If $z_\gamma = T^j(z_\delta)$ for some $\gamma, \delta \in \Gamma$, then $w_\gamma = T^j(w_\delta)$.

Remark 1.4. Let Λ be an almost periodic set, let $z \in X^\Gamma$ be such that $z_\Gamma = \Lambda$, let $w \in \overline{O(z)}$, and let $\Lambda' = w_\Gamma$. Then the following hold:

(a) Λ' is an almost periodic set and $|\Lambda'| = |\Lambda|$ where $|\ |$ denotes the cardinality of a set.

(b) Λ' is maximal if and only if Λ is maximal.

(c) Λ' is an almost periodic set mod T if and only if Λ is an almost periodic set mod T.

(d) Λ' is a maximal almost periodic set mod T if and only if Λ is a maximal almost periodic set mod T.

THEOREM 1.5. *All maximal almost periodic sets* mod T *of the flow* (X, T) *have the same cardinality.*

Proof. Let Λ and Λ' be maximal almost periodic sets mod T. Choose index sets Γ and Γ' and points $z \in X^\Gamma$ and $z' \in X^{\Gamma'}$ such that $\Gamma \cap \Gamma' = \emptyset$, $z_\Gamma = \Lambda$ and $z'_{\Gamma'} = \Lambda'$. Consider $(z, z') \in X^\Gamma \times X^{\Gamma'}$ and let $T_\Gamma \times T_{\Gamma'}$ act on this space. There exists a $T_\Gamma \times T_{\Gamma'}$ minimal set Y in the orbit closure of (z, z') which projects onto $\overline{O(z)}$. Hence there exists a $T_\Gamma \times T_{\Gamma'}$ almost periodic point of the form (z, w). Set $\Lambda_0 = w_{\Gamma'}$. Since $w \in \overline{O(z')}$, Λ_0 is a maximal almost periodic set mod T and $|\Lambda'| = |\Lambda_0|$. Now it suffices to show that $|\Lambda_0| = |\Lambda|$. But (z, w) can be thought of as an almost periodic point of $(X^{\Gamma \cup \Gamma'}, T_\Gamma \times T_{\Gamma'})$. Hence $\Lambda \cup \Lambda_0$ and all its non-empty subsets are almost periodic sets. Because Λ is a maximal almost periodic set mod T, for each $x \in \Lambda_0$ there exists a unique $y \in \Lambda$ and j such that $x \in T^j(y)$. This defines a map $\sigma: \Lambda_0 \to \Lambda$. Because Λ_0 is a maximal almost periodic set mod T, σ is a bijection.

We define the *almost periodic rank* (or just *rank*) of a flow to be the cardinality of a maximal almost periodic set mod T.

Recall that a minimal flow (X, T) is *graphic* if it is totally minimal, and every almost periodic point of the product flow $(X \times X, T \times T)$ is of the form $(x, T^n(x))$

for $x \in X$ and some integer n. Thus a totally minimal flow is graphic if and only if it has rank one.

Moreover, it is easy to see that a totally minimal flow is graphic if and only if whenever x and y are not proximal there is a $k \neq 0$ such that $T^k(x)$ and y are proximal. If we replace 'x and y are not proximal' in the preceding sentence by '$x \neq y$' we obtain the definition of a POD flow [**2**], [**5**]. Thus graphic may be regarded as the 'proximal weakening' of POD.

We omit the simple proof of the following proposition.

PROPOSITION 1.6. *Let (X, T) be a flow and let Λ be a maximal almost periodic set* mod T. *If $h \neq 0$, then $\Lambda \cup T(\Lambda) \cup \cdots \cup T^{h-1}(\Lambda)$ is a maximal almost periodic set* mod T^h *and* rank $T^h = |h|$ rank T. *In particular, if (X, T) is a graphic minimal flow and $h \neq 0$, then (X, T^h) has rank $|h|$.*

If, however, we calculate the rank of a proper T^h invariant subset, we can get a different answer as the next proposition shows.

PROPOSITION 1.7. *Let (X, T) be a flow, let h be a positive integer and suppose X_0 is a closed T^h invariant subset of X such that X_0, $T(X_0)$, $T^{h-1}(X_0)$ are pairwise disjoint sets whose union is X. Then* rank $(X_0, T^h|X_0) =$ rank (X, T).

Proof. Let Λ be a maximal almost periodic subset mod T, and let $\Lambda_j = \Lambda \cap T^j(X_0)$, $j = 0, 1, \ldots, h-1$. Set

$$\Lambda^* = \Lambda_0 \cup T^{-1}(\Lambda_1) \cup \cdots \cup T^{-h+1}(\Lambda_{h-1}).$$

Then Λ^* is a maximal almost periodic set mod $T^h|X_0$ and $|\Lambda^*| = |\Lambda|$.

In a similar manner we can define the almost periodic rank of a homomorphism. Let (X, T) and (Y, S) be flows with (Y, S) minimal and $|Y|$ infinite and let $\pi: X \to Y$ be a homomorphism. The rank of π is defined to be the cardinality of a maximal almost periodic subset of a fiber (that is, rank $\pi = |\Lambda|$, where Λ is an almost periodic subset of $\pi^{-1}(y)$ for $y \in Y$, and Λ is maximal with respect to this property. Since Y is infinite, any such almost periodic set is necessarily almost periodic mod T). The following proof that the rank of π is well defined is very similar to the proof of theorem 1.5: Let Λ and Λ' be maximal almost periodic sets of the fibers $\pi^{-1}(y)$ and $\pi^{-1}(y')$. If $y = y'$, proceed as in the proof of theorem 1.5 to construct Λ_0 with $|\Lambda_0| = |\Lambda'|$ and check that $\Lambda_0 \subset \pi^{-1}(y)$. Because $\Lambda \cap \Lambda_0$ is again an almost periodic set, $\Lambda = \Lambda_0$ by maximality. If $y \neq y'$, first choose $z'' \in \overline{O(z')}$ such that $\Lambda'' = z''_{\Gamma'} \subset \pi^{-1}(y)$ and check that Λ'' is a maximal almost periodic subset of $\pi^{-1}(y)$. Then $|\Lambda'| = |\Lambda''|$ and by the above $|\Lambda''| = |\Lambda|$.

THEOREM 1.8. *Let (X, T) and (Y, S) be infinite minimal flows and let $\pi: X \to Y$ be a homomorphism. Then*

(a) rank $(X, T) =$ (rank π) rank (Y, S).
(b) rank $(Y, S) \leq$ rank (X, T).
(c) *If π is a group extension by the group H, then* rank $\pi = |H|$.
(d) *The homomorphism π is proximal if and only if* rank $\pi = 1$.

Proof. (a) Let Λ_0 be a maximal almost periodic set mod S. We need to construct Λ, a maximal almost periodic set mod T, such that $\pi(\Lambda)=\Lambda_0$ and $\pi^{-1}(y)\cap\Lambda$ is a maximal almost periodic subset of $\pi^{-1}(y)$ for all $y\in\Lambda_0$. This can be done directly with a suitable almost periodic point of '$X^{\pi^{-1}(\Lambda_0)}$', or with the algebraic machinery of $\beta\mathbb{Z}$. We choose the latter. Let M be a minimal left ideal in $\beta\mathbb{Z}$. There exists an idempotent u in M such that $uy=y$ for all $y\in\Lambda_0$. For each $y\in\Lambda_0$ let Λ_y be a maximal almost periodic subset of $\pi^{-1}(y)$. Since $u\Lambda_y=\{ux\colon x\in\Lambda_y\}$ is also a maximal almost periodic subset of $\pi^{-1}(y)$, we can assume $ux=x$ for all $x\in\Lambda_y$.

Let $\Lambda=U(\Lambda_y\colon y\in\Lambda_0\}$. Clearly Λ is an almost periodic set mod T. We will show that Λ is maximal. If not, there is an $x'\in X-\Lambda$ such that $\{x\}\cup\Lambda$ is an almost periodic set mod T.

If $y'=\pi(x')$ then $\{y'\}\in\Lambda_0$ is an almost periodic set, so by maximality of Λ_0, $S^r(y')=y\in\Lambda_0$ for some integer r. Then $\pi(T^r(x'))=y$, so $T^r(x')\in\pi^{-1}(y)$ and $T^r(x')\cup\Lambda_y$ is an almost periodic set. By maximality of Λ_y, $T^r(x')\in\Lambda_y\subset\Lambda$. This is a contradiction, so Λ is maximal mod T. This proves (a) and (b) follows immediately.

(c) If E is any set of automorphisms, then $[\phi(x)\,|\,\phi\in E]$ is an almost periodic set. It follows that $\pi^{-1}(y)$ is itself an almost periodic set of cardinality $|H|$.

(d) If π is proximal and $x, x'\in\pi^{-1}(y)$ with $x\neq x'$, then (x, x') is not an almost periodic point of $T\times T$ because $\Delta\subset\overline{O(x,x')}$. So rank π must be 1. Now assume rank $\pi=1$ and consider $x, x'\in\pi^{-1}(y)$. There exists an almost periodic point $(w, w')\in\overline{O(x,x')}$. If $w\neq w'$, then the rank $\pi\geq 2$. Hence $w=w'$ and x and x' are proximal.

Note that (d) implies that the rank of a minimal flow depends only on its proximal class (equivalently on its Ellis group). Moreover, the proof of (a) in theorem 2 yields the following corollary.

COROLLARY 1.9. *Let (X, T) and (Y, S) be flows with (Y, S) minimal and infinite and let $\pi: X\to Y$ be a homomorphism. Let Λ_0 be an almost periodic set* mod S. *Then there exists Λ an almost periodic set* mod T, *such that $\pi(\Lambda)=\Lambda_0$. If Λ_0 is maximal then Λ can be chosen to be maximal.*

The connection between the rank of a homomorphism and the Ellis groups of the flows involved is given by the next proposition. Recall that the Ellis group $\mathcal{G}(X, T, x)$ (where $ux=x$) is the set of $g\in G=uM$ such that $gx=x$. (A brief discussion is on page 491 of [**2**]; for more details consult [**4**].)

PROPOSITION 1.10. *Let (X, T) and (Y, S) be infinite minimal flows, let $\pi\colon X\to T$ be a homomorphism, and let $x\in X$. The rank of π is the index of the Ellis group $\mathcal{G}(X, T, x)$ in $\mathcal{G}(Y, S, \pi(x))$.*

Proof. Let Γ be a set of coset representatives for $\mathcal{G}(X, T, x)$ in $\mathcal{G}(Y, S, \pi(x))$ and check that $\Lambda=\{\gamma x\colon\gamma\in\Gamma\}$ is a maximal almost periodic subset of $\pi^{-1}(\pi(x))$.

Recall that the minimal flows (X, T) and (Y, S) are *disjoint* if the product flow $(X\times Y, T\times S)$ is minimal. Disjointness properties of finite rank minimal flows will be discussed at length in § 5, but now we obtain a simple consequence of non-disjointness.

PROPOSITION 1.11. *Let (X, T) be a totally minimal flow of rank k and let (Y, S) be minimal. Suppose X and Y are not disjoint. Let N be a minimal subset of $X \times Y$. Then the projection π from N to Y has rank at most k, and* rank $N \leq$ (rank X) (rank Y).

Proof. Suppose $\{(x_1, y), (x_2, y), \ldots, (x_{k+1}, y)\}$ is an almost periodic set in N (so $\pi(x_i, y) = y$, $i = 1, 2, \ldots, k+1$). Then $\{x_1, x_2, \ldots, x_{k+1}\}$ is an almost periodic set in X, and since X has rank k, $x_i = T^n(x_j)$ for some i, j with $i \neq j$ and some $n \neq 0$. Then $(x_j, y) \in N$ and $(T^n(x_j), y) = (x_i, y) \in N$ so $(T^n \times I)(N) \subset N$. Since (X, T) is totally minimal, it follows that $X \times (y) \subset N$ and therefore $N = X \times Y$ which contradicts the assumed non-disjointness of X and Y. By theorem 1.8(a), rank $N =$ (rank π) $\times$ (rank Y) $\leq$ (rank X)(rank Y).

Note that the above proof is valid even if k is not finite.

COROLLARY 1.12. *If (X, T) is a totally minimal flow of finite rank, and n is a positive integer, then every minimal subset of X^n has finite rank less than or equal to* (rank X)^n.

This corollary will be applied in the next section to obtain a bound on the rank of the regularizer of a finite rank minimal flow.

PROPOSITION 1.13. *Let (X, T) be a minimal flow whose only equicontinuous factors are finite. Then there is a minimal $q > 0$ such that the T^q-minimal subsets of X are totally minimal.*

Proof. Let q be the largest cardinality of the equicontinuous factors. (Such a q exists, for if Z_j denotes the flow on j points and if Z_j is a factor of X for j arbitrarily large, then some minimal subset of the product of these Z_j is a factor of X. Such a minimal set must be infinite.) Write $Z_q = \{0, 1, \ldots, q-1\}$ and let $\pi : X \to Z_q$. If $X_0 = \pi^{-1}(0)$ and $S = T^q | X_0$, then (X_0, S) is totally minimal, because otherwise X would have a finite factor of more than q points.

PROPOSITION 1.14. *Let (X, T) be a minimal flow of finite or countable rank. Then*
(a) *all equicontinuous factors of X are finite,*
(b) *X is weakly mixing if and only if it is totally minimal.*

Proof. Suppose Y is an infinite equicontinuous factor of (X, T) with $\pi : X \to Y$. Then Y is uncountable and so has uncountable many orbits. Let Λ_0 be a set consisting of a point from each orbit. Since Y is uniformly almost periodic [**3**], Λ_0 is an almost periodic set mod S. Let Λ be an almost periodic set mod T such that $\pi(\Lambda) = \Lambda_0$. Hence Λ is uncountable. This contradiction proves (a) and (b) follows immediately from (a).

The last two propositions imply that a finite rank minimal flow (X, T) has subsets which are totally minimal for some power of T. This is important for two reasons. First it reduces the study of finite rank minimal sets to the totally minimal case. Secondly, our method of studying finite rank totally minimal sets depends on the regularizer which need not be totally minimal. (The regularizer is defined and discussed in the next section.) These propositions will guarantee that some positive power of T has a totally minimal regularizer.

2. *The regularizer revisited*

Let (X, T) be a minimal flow and let ϕ be an automorphism of (X, T), that is, a homeomorphism of X onto itself such that $\phi T = T\phi$. Obviously $\{(x, \phi(x)): x \in X\}$, the graph of ϕ, is a minimal subset of $(X \times X, T \times T)$ and is isomorphic to (X, T). When all minimal subsets of $(X \times X, T \times T)$ are of this form, we say (X, T) is *regular.* In other words, (X, T) is regular if whenever (x, x') is a $T \times T$ almost periodic point, then there exists an automorphism ϕ of (X, T) such that $x' = \phi(x)$.

If (X, T) is a minimal flow, a *regularizer* of (X, T) is a regular minimal flow $(\tilde{X}, \tilde{T})$ such that (X, T) is a factor of $(\tilde{X}, \tilde{T})$ and $(\tilde{X}, \tilde{T})$ is a factor of every regular minimal set having (X, T) as a factor. (Thus $(\tilde{X}, \tilde{T})$ is a 'smallest' regular minimal flow which has (X, T) as a factor.) Regular minimal sets were introduced by the first author [1]. Although the existence of the regularizer was established in his paper (page 477) using the enveloping semi-group and the lattice of regular minimal sets, we prefer to reconstruct the regularizer in a more elementary way using almost periodic sets. We begin with a few immediate consequences of the definitions.

Remark 2.1. If (X, T) is a regular minimal set, then every minimal subset of every (X^Γ, T_Γ) is isomorphic to (X, T).

Remark 2.2. Let $\psi_1, \psi_2: (X, T) \to (Y, S)$ be homomorphisms of minimal flows. If (X, T) is regular, then there exists an automorphism ϕ of (X, T) such that $\psi_1 = \psi_2\phi$.

Remark 2.3. If ϕ is an endomorphism of a regular minimal flow, then ϕ is an automorphism.

Remark 2.4. Any two regularizers of a minimal flow are isomorphic.

We can now use the phrase 'the regularizer of (X, T)' to refer to the isomorphism class of all the regularizers of (X, T). However, when we are working with a particular representative of this isomorphism class we will continue to refer to it as 'a regularizer of (X, T)'.

Remark 2.5. Let $\pi: (X, T) \to (Y, S)$ be a homomorphism of minimal flows, and let $(\tilde{X}, \tilde{T})$ and $(\tilde{Y}, \tilde{S})$ be regularizers of (X, T) and (Y, S) with homomorphisms $\alpha: (\tilde{X}, \tilde{T}) \to (X, T)$ and $\beta: (\tilde{Y}, \tilde{S}) \to (Y, S)$. Then there exists a homomorphism $\tilde{\pi}: (\tilde{X}, \tilde{T}) \to (\tilde{Y}, \tilde{S})$ such that $\beta\tilde{\pi} = \pi\alpha$.

THEOREM 2.6. *Let (X, T) be a minimal flow and let Λ_0 be a maximal almost periodic set* mod *T. If $z \in X^\Gamma$ is an almost periodic point satisfying $z_\Gamma \supset \Lambda_0$, then $\overline{O(z)}$ is a regularizer of (X, T). (It is understood that the homeomorphism on $\overline{O(z)}$ is just the restriction of T_Γ.)*

Proof. We first establish the case when $z_\Gamma = \Lambda_0$ and $z: \Gamma \to \Lambda_0$ is one-to-one. Let $w, w' \in \overline{O(z)}$ such that (w, w') is an almost periodic point of $T_\Gamma \times T_\Gamma$. We will construct an automorphism ϕ of (X^Γ, T_Γ) such that $w' = \phi(w)$. It then follows that $\phi(\overline{O(z)}) = \overline{O(z)}$ and $\overline{O(z)}$ is regular. Set $\Lambda = w_\Gamma$ and $\Lambda' = w'_\Gamma$. By remarks 1.3 and 1.4 both w and w' are one-to-one and both Λ and Λ' are maximal almost periodic sets mod *T*. Since (w, w') is almost periodic point of $T_\Gamma \times T_{\Gamma'}$, $\Lambda \cup \Lambda'$ is an almost periodic set. Now as in the proof of Theorem 1.5, for each $\gamma \in \Gamma$ there exists a unique point $x \in \Lambda$ (which gives us a unique $\sigma(\gamma) \in \Gamma$ such that $x = w_{\sigma(\gamma)}$), and $k(\gamma) \in \mathbb{Z}$ such that

$w'_\gamma = T^{k(\gamma)}(x) = T^{k(\gamma)}(w_{\sigma(\gamma)})$. It is easy to check that $\sigma:\Gamma\to\Gamma$ is a bijection and hence the formula

$$[\phi(z')]_\gamma = T^{k(\gamma)}(z'_{\sigma(\gamma)})$$

for $z'\in X^\Gamma$ defines an automorphism of (X^Γ, T_Γ). Clearly $\phi(w)=w'$.

Now let $\psi:(Y,S)\to(X,T)$ with (Y,S) regular. Then ψ induces $\tilde{\psi}:(Y^\Gamma,S_\Gamma)\to(X^\Gamma,T_\Gamma)$ by $[\tilde{\psi}(z)]_\gamma=\psi(z_\gamma)$. Let W be a minimal subset of (Y^Γ,S_Γ) such that $\tilde{\psi}(W)=\overline{O(z)}$. Now by Remark 2.1, $\overline{O(z)}$ is a factor of (Y,S) and $\overline{O(z)}$ is a regularizer of (X,T).

For the general case there exists a subset of Γ' of Γ such that the restriction z' of z to Γ' is a bijection of Γ' onto Λ_0. For each $\gamma\in\Gamma\backslash\Gamma'$, there exists a unique $\sigma(\gamma)\in\Gamma'$ and $k(\gamma)\in\mathbb{Z}$ such that $z_\gamma=T^{k(\gamma)}(z_{\sigma(\gamma)})$. Set $\sigma(\gamma)=\gamma$ and $k(\gamma)=0$ for $\gamma\in\Gamma'$. Then

$$[\psi(w)]_\gamma = T^{k(\gamma)}(w_{\sigma(\gamma)})$$

defines an isomorphism ψ of $\overline{O(z')}$ onto $\overline{O(z)}$. (The inverse is just the restriction of the projection of X^Γ onto $X^{\Gamma'}$.) Since $\overline{O(z')}$ is a regularizer by the first part of the proof, we are done.

PROPOSITION 2.7. *Let (X,T) be a minimal flow, let k be a positive integer and let $(\tilde{X},\tilde{T})$ be a regularizer of (X,T). If X^* is a minimal subset of $(\tilde{X},\tilde{T}^k)$, then $(X^*,\tilde{T}^k)$ is a regularizer for the minimal subsets of (X,T^k).*

Proof. First the result is a property of the isomorphism class of $(\tilde{X},\tilde{T})$, so we can construct a convenient $(\tilde{X},\tilde{T})$. Let Y be a minimal subset of (X,T^k) and let Λ be a maximal almost periodic set mod T^k of Y. Now choose $z\in Y^\Gamma\subset X^\Gamma$ such that $z_\Gamma=\Lambda$. Note that Y^Γ is a closed T^k_Γ invariant set of X^Γ. Since a T^k_Γ-almost periodic point is automatically a T_Γ-almost periodic point, Λ is an almost periodic set of (X,T). Then by theorem 2.6 the closure of $O(z,T^k_\Gamma)$ is a regularizer of (Y,T^k) and a T^k_Γ minimal subset of $\overline{O(z,T_\Gamma)}$. Now by theorem 2.6 it suffices to show that Λ contains a maximal almost periodic set mod T.

Clearly Λ contains Λ' an almost periodic set mod T such that every point in Λ lies in the T orbit of exactly one point in Λ'. Suppose Λ' is not maximal, that there exists $x\in X$ such that $x\notin\Lambda'$ and $\Lambda'\cup\{x\}$ is an almost periodic set mod T. Without loss of generality $x\in Y$. Because every point in Λ is in the T orbit of some point in Λ', $\Lambda\cup\{x\}$ is an almost periodic set, and then by the maximality of Λ, $x=T^{nk}(y)$ for some $y\in\Lambda$ and $n\in\mathbb{Z}$. It follows that $\Lambda'\cup\{y\}$ is an almost periodic set mod T contained in Λ. Thus $y\in\Lambda'$ which contradicts the choice of x.

PROPOSITION 2.8. *If a minimal flow has finite rank, then its regularizer has finite rank.*

Proof. By theorem 2.6 and corollary 1.12 the rank of the regularizer of a minimal flow of rank k is at most k^k. (Actually, the rank is at most $k!$. This can be established by using theorem 2.6 to represent automorphisms of $(\tilde{X},\tilde{T})$ as permutations composed with powers of T and then applying the next proposition.)

PROPOSITION 2.9. *Let (X,T) be a regular minimal set with automorphism group $\mathscr{A}(T)$, and let $[T]$ be the cyclic subgroup generated by T. Then $[T]$ is a normal subgroup of*

$\mathscr{A}(T)$ and rank $(X, T) = |\mathscr{A}(T)/[T]|$. *Moreover, $\mathscr{A}(T)$ is isomorphic to $\mathbb{Z}$ if and only if there exists $S \in \mathscr{A}(T)$ such that $T = S^k$ for some $k > 0$ and* rank (X, S) *is* 1.

Proof. $[T]$ is normal because T is in the center of $\mathscr{A}(T)$. Let $\{h_i: i \in \Gamma\}$ be representatives of the cosets of $[T]$ in $\mathscr{A}(T)$, and let $x \in X$. It is easy to see that $\{h_i(x): i \in \Gamma\}$ is a maximal almost periodic set mod T.

To prove the second statement we first assume $\mathscr{A}(T) \cong \mathbb{Z}$. Then there exists a generator S of $\mathscr{A}(T)$ such that $S^k = T$ for some $k > 0$. If (x, x') is an $S \times S$ almost periodic point, then it is a $T \times T$ almost periodic point and $x' = S^i(x)$ for some i. For the converse let $\varphi \in \mathscr{A}(T)$ and $x \in X$. Then $(x, \varphi(x))$ is a $T \times T$ and hence $S \times S$ almost periodic point. Because rank (X, S) is 1, $\varphi(x) = S^i(x)$ for some i, $\varphi = S^i$, and $\mathscr{A}(T) = \{S^j : j \in \mathbb{Z}\}$.

The *proximal class* of a minimal flow is the equivalence class of minimal flows determined by proximal equivalence. (Two minimal flows are proximally equivalent if they have a common proximal extension.) The Ellis group is a complete invariant for proximal equivalence (see [**4**] for further details).

A proximal class is called regular if it contains a regular minimal flow. The regular proximal classes will play an important role in the determination of the structure of totally minimal finite rank flows described in the next two sections.

The following proposition gives several characterizations of regular proximal classes.

PROPOSITION 2.10. *Let $\mathscr{P}$ be a proximal class of minimal flows. Then the following are equivalent:*

(i) *$\mathscr{P}$ is regular.*

(ii) *If (X, T) is in $\mathscr{P}$ then its regularizer is a proximal extension of (X, T).*

(iii) *If (X, T) is in $\mathscr{P}$, then for every index set Γ and every minimal subset W of (X^Γ, T_Γ), every projection of W onto X is proximal.*

(iv) *If (X, T) is in $\mathscr{P}$ and W is a minimal subset of $(X \times X, T \times T)$, then both projections of W onto X are proximal.*

(v) *If (X, T) is in $\mathscr{P}$ and u is an idempotent in the minimal left ideal M of $\beta\mathbb{Z}$, then the Ellis group is a normal subgroup of the group $G = uM$.*

This proposition can easily be established using the algebraic theory. Alternatively the first four equivalences can be proved topologically. The details are left to the reader.

The fundamental relationship between the Ellis groups of (X, T) and its regularizer $(\tilde{X}, \tilde{T})$ is also easy to establish from our construction of $(\tilde{X}, \tilde{T})$.

PROPOSITION 2.11. *Let $(\tilde{X}, \tilde{T})$ be the regularizer of the minimal flow (X, T). Then the Ellis group of $(\tilde{X}, \tilde{T})$ is the largest normal subgroup of G contained in the Ellis group of (X, T). In particular, the Ellis group of a regular minimal flow is normal.*

Proof. Choose $x \in X$ such that $ux = x$ and set $\Lambda = \{gx: g \in G\}$. Since $ugx = gx$, Λ is an almost periodic set. Choose $z \in X^\Gamma$ such that $z_\Gamma = \Lambda$. Then

$$\mathscr{G}(\overline{O(z)}, T_\Gamma, z) = \bigcap_{\gamma \in \Gamma} \mathscr{G}(X, T, z_\gamma) = \bigcap_{g \in G} \mathscr{G}(X, T, gx) = \bigcap_{g \in G} g\mathscr{G}(X, T, x)g^{-1}.$$

It now suffices by theorem 2.6 to show that Λ contains a maximal almost periodic set mod T.

Let Λ_0 be a maximal almost periodic set mod T, and let $w \in X^{\Gamma'}$ with $w_{\Gamma'} = \Lambda_0$. Then $uw \in \overline{O(w)}$ and $(uw)_{\Gamma'}$ is a maximal almost periodic set mod T contained in $\{y: uy = y\}$. Since $uy = y$ implies $y = gx$ for some $g \in G$, $(uw)_{\Gamma'} \subset \Lambda$ and the proof is complete.

3. *The graphic invariant*

In this section we show the proximal classes of the graphic minimal sets can be used as a set of invariants for finite rank totally minimal flows. Actually we are primarily interested in the proximal classes of the latter. This is not surprising since rank cannot be used to distinguish non-isomorphic flows which are in the same proximal class. Our point of view is that proximal equivalence is somewhat analogous to measure theoretic isomorphism in ergodic theory.

We first restrict our attention to regular finite rank totally minimal flows. The key result of this section occurs in this setting and depends on the following group theoretic fact:

LEMMA 3.1. *Let H be a group whose center has finite index. Then the commutator subgroup of H is finite.*

The proof of this result can be found in 'Group Theory' by Scott [7, page 443].

THEOREM 3.2. *Let (X, T) be a regular totally minimal flow of finite rank. Then (X, T) is a finite group extension of a power of a graphic flow. Moreover, the finite subgroup is a normal subgroup of $\mathscr{A}(T)$.*

Proof. We first suppose that the automorphism group $\mathscr{A}(T)$ of (X, T) is abelian. Since (X, T) has finite rank, $[T]$ (the cyclic subgroup of $\mathscr{A}(T)$ generated by T) has finite index in $\mathscr{A}(T)$ and $\mathscr{A}(T) = \mathbb{Z} \oplus F$, where F is a finite group. (Note $[T]$ can be a proper subgroup of the direct summand $\mathbb{Z}$.) Since $\mathscr{A}(T)$ is abelian, F is normal and the quotient flow $(X/F, T/F)$ is regular. Moreover, the automorphism group of $(X/F, T/F)$ is $\mathscr{A}(T)/F$ which is isomorphic to $\mathbb{Z}$. Therefore, T/F is a power of a graphic (proposition 2.9 and the remarks following theorem 1.5).

Now consider the general case. As observed above $[T]$ has finite index in $\mathscr{A}(T)$ and $[T]$ is a subgroup of the center of $\mathscr{A}(T)$. Therefore, the center has finite index in $\mathscr{A}(T)$. By the group theoretic lemma, the commutator subgroup K of $\mathscr{A}(T)$ is finite. Since K is normal, the quotient flow $(X/K, T/K)$ is regular, with abelian automorphism group $\mathscr{A}(T)/K$. Thus we may apply the previous case to conclude that $(X/K, T/K)$ is a finite group extension of a power of a graphic by some finite normal subgroup F_0 of $\mathscr{A}(T)/K$. There exists a finite normal subgroup F of $\mathscr{A}(T)$ such that $F_0 = F/K$. Clearly $\mathscr{A}(T)/F \cong (\mathscr{A}(T)/K)/(F/K) \cong \mathbb{Z}$ and as in the abelian case $(X/F, T/F)$ is the desired power of a graphic.

LEMMA 3.3. *Let (X, T) and (Z, U) be infinite minimal flows with (Z, U) regular, and let $\pi: (Z, U) \to (X, T)$ be a homomorphism. If (Z, U) has finite rank, then (Z, U) is a finite group extension of a proximal extension of (X, T).*

Proof. Let $x \in X$ and let $\Lambda = \{z_1, \ldots, z_m\}$ be a maximal almost periodic subset of $\pi^{-1}(x)$. Set $H = \{\varphi \in \mathcal{A}(U): \varphi(z_1) = z_i$ for some $i\}$. By minimality $\pi(\varphi(z)) = \pi(z)$ for all $z \in Z$ and $\varphi \in H$. It is easy to see that $\Lambda \cup \varphi(\Lambda)$ is also an almost periodic subset of $\pi^{-1}(x)$. Therefore, $\varphi(\Lambda) = \Lambda$ from which it follows that H is a subgroup of $\mathcal{A}(U)$. Because (Z, U) is regular, for each i there exists a unique $\varphi \in \mathcal{A}(U)$ such that $z_i = \varphi(z_1)$. Thus $|H| = m$ and $\Lambda = \{\varphi(z_1): \varphi \in H\}$.

Now form the flow $(Z/H, U/H)$. There exists a homomorphism $\pi': (Z/H, U/H) \to (X, T)$ since $\pi\varphi(z) = \pi(z)$ for all $z \in Z$ and $\varphi \in H$. Since rank $\pi = |H|$, we have by theorem 1.8

$$\begin{aligned}\text{rank}\,(Z/H, U/H) &= \text{rank}\,(Z, U)/|H| \\ &= \text{rank}\,(Z, U)/\text{rank}\,\pi = \text{rank}\,(X, T).\end{aligned}$$

Applying this theorem again gives rank $\pi' = 1$ and π' is proximal.

LEMMA 3.4. *Let (Z, U) be a totally minimal finite rank flow which has a factor (Y, S^k) such that (Y, S) is graphic. If (X, T) is any non-trivial factor of (Z, U), then (Y, S^k) is a factor of a proximal extension of (X, T).*

Proof. Let $\psi: (Z, U) \to (Y, S^k)$ and $\varphi: (Z, U) \to (X, T)$ denote the homomorphisms, and consider the minimal subset $M = \{(\psi(z), \varphi(z)): z \in Z\}$ of $(Y \times X, S^k \times T)$. It suffices to show that the projection $\pi: M \to X$ has rank 1. If not there exists an almost periodic set $\Lambda = \{(y, x), (y', x)\} \subset M$, $y \neq y'$. It follows that (y, y') is an almost periodic point for $S^k \times S^k$ and $y' = S^p(y)$, $p \neq 0$. Therefore, $S^p \times T^0(M) \cap M \neq \phi$, $S^p \times T^0(M) = M$, and $M = Y \times X$ by the minimality of S^p. It follows that $\{(S^k(y), x) \mid k = 0, \pm 1, \ldots\}$ is an almost periodic set in $\pi^{-1}(x)$ so the rank of π, and hence the rank of M, is infinite. But M is a factor of the finite rank flow Z, so the rank of M is finite (theorem 1.8(b)). This contradiction completes the proof.

THEOREM 3.5. *Let (X, T) be a totally minimal flow of finite rank. Then there is a graphic minimal flow (Y, S) and a positive integer r such that (Y, S^r) is a factor of a proximal extension of (X, T). The integer r and the proximal class of (Y, S) are unique and depend only on the proximal class of (X, T).*

Proof. Let $(\tilde{X}, \tilde{T})$ be the regularizer of (X, T) and let p be the smallest positive integer such that the minimal sets of $(\tilde{X}, \tilde{T}^p)$ are totally minimal (propositions 1.13, 1.14, and 2.8). Let Z be a $\tilde{T}^p$ minimal set and let V denote the restriction of $\tilde{T}^p$ to Z. Then (Z, V) is a regular totally minimal flow of finite rank (proposition 1.7), and the restriction of the known homomorphism of $(\tilde{X}, \tilde{T})$ onto (X, T) defines a homomorphism $\theta: (Z, V) - (X, T^p)$.

Since (Z, V) is regular, it has an automorphism U satisfying $\theta U = T\theta$ (so θ may also be regarded as a homomorphism from (Z, U) to (X, T)). Consider the automorphism $\varphi = U^p V^{-1}$ of (Z, V) and let $z \in Z$. Then $\{z, \varphi(z), \varphi^2(z), \ldots\}$ is an almost periodic set for (Z, V) and since the rank of V is finite, we must have $\varphi^m(z) = V^j(z)$ for some integers m and j with $m > 0$. Hence $\varphi^m = V^j$. Note that $\theta\varphi = \theta U^p V^{-1} = T^p T^{-p}\theta = \theta$, so $\theta\varphi^m = \theta$. Since also $\theta\varphi^m = \theta V^j = T^{pj}\theta$, this forces $j = 0$, so $\varphi^m =$ identity and $U^{pm} = V^m$ and (Z, U) is totally minimal. If we put $q = pm$

then $U^q = V^m$ is regular and by lemma 3.3, (Z, U^q) is a finite group extension of a proximal extension of (X, T^q).

By theorem 3.2 there is a graphic minimal flow (Y, S) and a positive integer k such that the regular finite rank flow (Z, V^m) is a finite group extension of (Y, S^k). Since U is an automorphism of V^m it induces an automorphism R of S^k, and since all automorphisms of a graphic power are themselves powers of the generating homeomorphism, $R = S^r$ (see the discussion preceding theorem 4 in [**2**]). Thus if π is the homomorphism of (Z, V^m) to (Y, S^k), π may also be regarded as a homomorphism of (Z, U) to (Y, S^r). Furthermore $U^q = V^m$ implies $S^{rq} = S^k$, so $rq = k$ and $r > 0$. Thus, both (X, T) and (Y, S^r) are factors of (Z, U). By lemma 3.4, (Y, S^r) is a factor of a proximal extension of (X, T).

Now we use lemma 3.4 to prove the uniqueness assertions. If (Y, S) and (Y', S') are graphic flows such that (Y, S^r) and (Y', S'^j), (r and j positive integers), are factors of proximal extensions of (X, T) and (X', T') where (X, T) and (X', T') are proximally equivalent totally minimal flows of finite rank, then both (Y, S^r) and (Y', S'^j) are factors of some (Z, U), a common proximal extension of both (X, T) and (X', T'). By lemma 3.4 each of (Y, S^r) and (Y', S'^j) is a factor of a proximal extension of the other. Thus they have the same Ellis group and are proximally equivalent. Since r and j are positive, (Y, S) and (Y', S') are proximally equivalent [**2**, theorem 1.8] and $r = j$ [**2**, theorem 1.9] completing the proof.

In the course of proving Theorem 3.5, we have proved the following theorem.

THEOREM 3.6. *Let (X, T) be a totally minimal flow of finite rank. Then*

(i) *there is a totally minimal finite rank extension (Z, U) of (X, T) and a positive integer q such that (Z, U^q) is regular. (Z, U^q) is a finite group extension of a proximal extension of (X, T^q).*

(*ii*) *If (Y, S) is the graphic minimal flow whose existence was established in theorem* 3.5, *then there is a positive integer r such that (Z, U) is a finite to one extension of (Y, S^r). (Z, U^q) is a finite group extension of (X, S^{rq}) and the fibers of the extension are the orbits of a finite normal subgroup of $\mathscr{A}(U^q)$.*

Proof. The first part summarizes what was accomplished in the initial steps of the proof of theorem 3.5 and the second part summarizes the relationship between $U^q = V^m$ and $S^{rq} = S^k$. The normality of the finite subgroup follows from theorem 3.2.

COROLLARY 3.7. *Let (X, T) be a finite rank totally minimal flow. Then the Ellis group of (X, T) is a maximal τ-closed subgroup if and only if (X, T) is proximally equivalent to a power of a graphic flow.*

Proof. If (X, T) is proximally equivalent to a graphic power, then its Ellis group is a maximal τ-closed subgroup by Corollary 16 of [**2**]. If (X, T) is not proximally equivalent to a graphic power, then it is proximally equivalent to a flow (X', T') which has a graphic power factor $\pi : (X', T') \to (Y, S^m)$, with π not proximal. Hence the Ellis group of (Y, S^m) properly contains the Ellis group of (X', T') (which equals the Ellis group of (X, T)).

Motivated by theorem 3.5 we define the *graphic invariants* of a finite rank totally minimal flow to be the positive integer k and the proximal class of any graphic flow (Y, S) such that (Y, S^k) is a factor of a proximal extension of (X, T). Equivalently we may define the graphic invariants as a pair $\{k, B\}$ where B is the Ellis group of (Y, S) and k is as above. By theorem 3.5 the graphic invariant is uniquely defined and depends only on the proximal class of (X, T).

If (X, T) is proximally equivalent to a power of a graphic flow, and (Y, S) is minimal, then (X, T) is either disjoint from (Y, S) or is a factor of a proximal extension of (Y, S) ([**2**, corollary 17]; a proof can also be given using corollary 3.7 above). Thus there is a dichotomy for graphic power flows. An interesting general question is to determine the class of minimal flows which satisfy such a 'dichotomy' condition. (It is known that this class is larger than the the graphic power flows.) Related ideas are discussed in [**5**].

4. *Structure of minimal finite rank flows*

In this section (X, T) will always be a minimal flow of finite rank. If (X, T) is not totally minimal, by propositions 1.13 and 1.14 there exists $p>0$ and a T^p-minimal set X_0 such that $X = X_0 \cup T(X_0) \cup \cdots \cup T^{p-1}(X_0)$ (disjoint) and (X_0, T^p) is totally minimal. If we set $Y = X_0 \times \{0, \ldots, p-1\}$ and define $S: Y \to Y$ by

$$S(x, i) = \begin{cases} (x, i+1) & 0 \le i < p-1 \\ (T^p(x), 0) & i = p-1, \end{cases}$$

then (Y, S) is isomorphic to (X, T). Consequently, all the finite rank flows which are not totally minimal can be constructed from those which are totally minimal and distinguished by the invariant p. From now on we will assume (X, T) is also totally minimal.

Now we partition the totally minimal finite rank flows into four classes. These classes are suggested by the results of the previous section and, as we shall see, can be characterized in terms of Ellis groups.

Definition. Let (X, T) be a totally minimal flow of finite rank. We say that (X, T) is of

(i) type PG if (X, T) is proximally equivalent to a power of a graphic flow,

(ii) type RE if (X, T) is proximally equivalent to a regular non-trivial finite group extension of a power of a graphic flow,

(iii) type RR if (X, T) is proximally equivalent to a minimal flow (Z, V) where for some integer $q>1$, (Z, V^q) is regular and of type RE, and V is not in the center of the automorphism group of (Z, V^q),

(iv) type QR if it is not of type PG, RE, or RR.

The abbreviations PG, RE, RR, and QR stand for, respectively, 'power of a graphic,' 'regular extension,' 'root of a regular,' and 'quotient of a root' (a dynamical characterization of type QR will be given in theorem 4.2).

The four types are exhaustive and mutually exclusive. This is not difficult to see directly, but it will follow immediately from the next theorem, in which these types are characterized in terms of their Ellis groups. To this end, let $A = \mathcal{G}(X, T, x)$ be the Ellis group of (X, T) and let $A_q = \mathcal{G}(X, T^q, x_q)$, as $A_1 A$. (As usual, $ux = x$ and $ux_q = x_q$.)

THEOREM 4.1. *Let (X, T) be a totally minimal flow of finite rank. Then (X, T) is of*

(i) *type* PG *if and only if A_1 is a maximal τ-closed normal subgroup of G,*

(ii) *type* RE *if and only if A_1 is a normal subgroup of G, but not maximal, τ-closed,*

(iii) *type* RR *if and only if A_1 is not a normal subgroup of G, but for some $q > 1$, A_q is normal,*

(iv) *type* QR *if and only if no A_q $(q \geq 1)$ is a normal subgroup of G.*

Proof. Assertion (i) is just a restatement of corollary 3.7. If (X, T) is of type RE, then the proximal class of (X, T) is regular, so A_1 is normal and, since a proximal extension has a non-trivial group factor, A_1 is not maximal. On the other hand, if A_1 is normal but not maximal, then it follows from proposition 2.10 and theorem 3.2 that (X, T) is of type RE.

Suppose A_1 is not normal, but for some $q > 1$, A_q is normal. We show (X, T) is of type RR. Since A_q is normal, the proximal class of (X, T^q) is regular, and there exists a regular minimal set (Z, U) and a proximal homomorphism $\pi:(Z, U) \to (X, T^q)$. By regularity, there is an automorphism V of (Z, U) such that $\pi V = T\pi$, and hence $\pi V^q = T^q \pi$. Because π is proximal it follows that $V^q = U$. Clearly $\pi:(Z, V) \to (X, T)$ is a proximal homomorphism. If V is in the center of the automorphism group of (Z, V^q), then (Z, V^q) and (Z, V) have the same automorphism group. It follows that (Z, V) is regular because $V \times V$ and $V^q \times V^q$ have the same almost periodic points; hence A_1 is normal. Thus V cannot be in the center of the automorphism group of (Z, V^q). Clearly (Z, V^q) is of type PG or RE. If (Z, V^q) is of type PG, then its automorphism group is $\mathbb{Z}$ which is abelian. This proves sufficiency for (iii) and necessity follows similarly. Having established the first three parts, (iv) becomes trivial.

THEOREM 4.2. *The flow (X, T) is type* QR *if and only if it is proximally equivalent to a flow of the form $(Z/H, U/H)$ where*

(a) *(Z, U^q) is regular and of type* RE *for some $q > 0$; i.e. (Z, U) is of type* RR.

(b) *H is a finite non-normal subgroup of the automorphism group of (Z, U^q).*

(c) *U is in the normalizer of H. (This condition is necessary and sufficient for U to induce a homeomorphism U/H on Z/H by $U/H(Hx) = HU(x)$.)*

Proof. Suppose (X, T) is of type QR. By Theorem 3.6 there exists $\pi:(Z, U) \to (X, T)$ where (Z, U) is a finite rank totally minimal flow and a positive integer q such that (Z, U^q) is regular. Furthermore, there exists a subgroup H of $\mathcal{A}(U^q)$ and a proximal homomorphism $\pi':(Z/H, U^q/H) \to (X, T^q)$ such that $\pi = \pi'\sigma$ where $\sigma:(Z, U^q) \to (Z/H, U^q/H)$ is canonical. Hence A_q is also the Ellis group of $(Z/H, U^q/H)$. Since A_q is not normal, $(Z/H, U^q/H)$ is not regular and H is not a normal subgroup of $\mathcal{A}(U^q)$.

The next step is to show that U is in the normalizer of H. If $\varphi \in H$, then

$$\pi U\varphi U^{-1} = T\pi\varphi U^{-1} = T\pi'\sigma\varphi U^{-1} = T\pi'\sigma U^{-1} = T\pi U^{-1} = TT^{-1}\pi = \pi.$$

Thus $\pi'\sigma U\varphi U^{-1} = \pi'\sigma$. Because π' is proximal $\sigma U\varphi U^{-1}(z) = \sigma(z)$ for some z and hence for all z. Clearly $\sigma U\varphi U^{-1} = \sigma$ implies $U\varphi U^{-1} \in H$.

It follows that U induces a minimal homeomorphism U/H on Z/H and π' is a homomorphism of $(Z/H, U/H)$ onto (X, T). Finally, because π' is a proximal homomorphism from $U^q/H = (U/H)^q$ to T^q it is a proximal homomorphism from U/H to T.

Once again the converse is straightforward.

Note that in the definition of types PG, RE, and RR and in the dynamical characterization of type QR in theorem 4.2, 'proximally equivalent' can be strengthened to 'a proximal factor of'. For types RR and QR this follows from the proofs of theorems 4.1 and 4.2, respectively. The other two types, PG and RE, have regular proximal classes, so we may apply theorem 3.2 to the regularizer of the flow, (which is a proximal extension of it).

It is also evident that the two types which are not regular, RR and QR, are intimately connected with the structure of the group of automorphisms of a regular finite rank flow. We conclude this section by determining the structure of such a group of automorphisms.

PROPOSITION 4.3. *Let (X, T) be a regular flow of type RE with automorphism group $\mathscr{A}(T)$ and with graphic invariants k and (Y, S). Then the following hold:*

(a) *The group $\mathscr{A}(T)$ is a semi-direct product of $\mathbb{Z}$ and a finite group.*

(b) *The group $\mathscr{A}(T)$ is a direct product of $\mathbb{Z}$ and a finite group if and only if there exists U in the center of $\mathscr{A}(T)$ satisfying $U^p = T^q$ for some positive integers p and q such that $p/q = k$.*

(c) *If $k = 1$, then $\mathscr{A}(T)$ is a direct product of $\mathbb{Z}$ and a finite group.*

Proof. (a) There exists a homomorphism $\pi: (X, T) \to (Y, S^k)$. Let H be the kernel of the induced onto homomorphism $\tilde{\pi}: \mathscr{A}(T) \to [S]$ where $[S]$, the infinite cyclic subgroup generated by S, is the automorphism group of (Y, S^k). (For the definition and properties of the induced homomorphism see [**2**, pages 487 and 488].) Because (X, T) has finite rank, H is a finite group. (By theorem 1.8 rank $(X, T) = k|H|$.) Let U be an element of $\mathscr{A}(T)$ such that $\tilde{\pi}(U) = S$. Then $[U] \cap H = \{\mathrm{Id}\}$, $[U]H = \mathscr{A}(T)$, and $\mathscr{A}(T)$ is a semi-direct product of $[U]$ and H. (For a concrete representation of $\mathscr{A}(T)$ as a semi-direct product of $\mathbb{Z}$ and H let $\psi: H \to H$ by $\psi(h) = U^{-1}hU$, define $(n, h)(m, h') = (n+m, \psi^m(h)h')$ on $Z \times H$, and note that $(n, h) \to U^n h$ is an isomorphism.)

(b) Suppose $\mathscr{A}(T)$ is the direct product of an infinite cyclic group $[U]$ and a finite group H. It follows that H must be the kernel of $\tilde{\pi}$ and $\tilde{\pi}(U) = S$. Clearly U is in the center of $\mathscr{A}(T)$ and $T = U^k h$ for some $h \in H$. Since H is finite and U is in the center, $T^q = U^{kq}$ where q is the order of h.

For the converse, suppose $\tilde{\pi}(U) = S^n$. Then $S^{np} = S^{qk}$, $np = qk$, and $n = 1$. Hence $\tilde{\pi}(U) = S$ and because U is in the center the semi-direct product constructed above is a direct product.

(c) Let $U = T$, let $p = q = 1$, and apply part (b).

Let (Y, S) be graphic and let H be a finite group. Suppose we can construct (X, T) a regular totally minimal group extension of (Y, S) by H, (e.g. with a cocycle $f: Y \to H$). Then by the previous proposition the automorphism group $\mathscr{A}(T)$ of (X, T) is the direct sum of $[T]$ and H. Let $h \in H$ and set $U = Th$. Then (X, U) is also totally minimal with finite rank. Moreover, (X, U) is regular if and only if h is in the center of H. Hence, if h is not in the center of H, (X, U) is of type RR. If K is a non-normal subgroup of H and $hKh^{-1} = K$, then $(X/K, U/K)$ would be of type QR. Consequently to construct examples of type RR and QR it is only necessary to construct one of type RE over a graphic with a finite group H containing a non-normal subgroup. This can be done with a flow proximally equivalent to (Y, S) and will appear elsewhere.

5. *Disjointness*

Two minimal flows (X, T) and (Y, S) are said to be disjoint, written $(X, T) \perp (Y, S)$, if $(X \times Y, T \times S)$ is minimal. Let $A = \mathscr{G}(X, T, x)$ and $B = \mathscr{G}(Y, S, y)$ where as usual $ux = x$ and $uy = y$ for a given idempotent u. Then (X, T) and (Y, S) are disjoint if and only if $AB = G$. (For an elementary proof see [**2**, lemma 13].) Since A and B depend only on the proximal classes of (X, T) and (Y, S), (X, T) is disjoint from (Y, S) if and only if every flow proximally equivalent to (X, T) is disjoint from every flow proximally equivalent to (Y, S). We will use also use the obvious fact that factors of disjoint flows are disjoint.

In this section we will show that the class of flows disjoint from a totally minimal finite rank flow is determined by its graphic invariant flow. We start with a flow of type RE.

THEOREM 5.1. *Let (X, T), a finite rank totally minimal flow, be of type* RE *with graphic invariants k and (Y, S) and let (Z, U) be a minimal flow. Then (Z, U) is disjoint from (X, T) if and only if (Z, U) is disjoint from (Y, S^k).*

Proof. The 'only if' is trivial. Assume $(Z, U) \perp (Y, S^k)$ and without loss of generality that $\pi: (X, T) \to (Y, S^k)$ is a finite group extension. Let M be a minimal subset of $(X \times Z, T \times U)$. Hence $\pi \times \mathrm{id}(M) = Y \times Z$ and there exist (x, z) and $(x', z) \in M$ such that $\pi(x') = S\pi(x) = \pi T(x)$. Since π is a finite group extension, there exists an automorphism φ of finite order such that $x' = \varphi Tx$. Thus $\varphi T \times \mathrm{id}(M) = M$ and for some k, $(\varphi T)^k = T^k$. Therefore, $T^k \times \mathrm{id}(M) = M$ and $M = X \times Z$ because (X, T) is totally minimal.

PROPOSITION 5.2. *Let (X, T) and (Y, S) be disjoint minimal flows and p a positive integer. If (X, T^p) and (Y, S^p) are minimal, then (X, T^p) and (Y, S^p) are disjoint.*

Proof. We suppose that p is prime. Let $V = T \times S$ and let M_0 be a minimal set for V^p. Suppose $M_0 \neq X \times Y$. Then because $(X \times Y, V)$ is minimal

$$X \times Y = \bigcup_{j=0}^{p-1} V^j(M_0)$$

where the union is disjoint and each $V^j(M_0)$ is a V^p-minimal set. Since $T \times I$,

I = identity map, is also an automorphism of V^p,

$$T\times I(M_0)=V^j(M_0)$$

for some j and hence $T^p\times I(M_0)=V^{jp}(M_0)=M_0$. Pick $(x_0, y_0)\in M_0$. By the above $(T^{kp}(x_0), y_0)\in M_0$ for all $k\in\mathbb{Z}$ and hence $X\times\{y_0\}\subset M_0$ because (X, T^p) is minimal. Finally by the minimality of (Y, S^p), $X\times Y\subset M_0$ completing the proof for p prime. The general case follows easily.

PROPOSITION 5.3. *Let (X, T) and (Y, S) be minimal flows, let $p>0$ and let X_0 and Y_0 be T^p and S^p minimal subsets of X and Y respectively. If (X, T) and (Y, S) are disjoint, then (X_0, T^p) and (Y_0, S^p) are disjoint.*

Proof. We will show there is a $\rho>0$ such that $\rho|p$, $T^\rho(X_0)=X_0$, $S^\rho(Y_0)=Y_0$ and $(X_0, T^\rho)\perp(Y_0, S^\rho)$. The proof then follows immediately from proposition 5.2.

The set of integers k such that $T^k(X_0)=X_0$ is a non-trivial subgroup of $\mathbb{Z}$, so there is a $q>0$ such that $T^k(X_0)=X_0$ if and only if $q|k$. Similarly, there is an $r>0$ such that $S^k(Y_0)=Y_0$ if and only if $r|k$. In particular, both q and r divide p. Note that the finite flows $(Z_q, 1_q)$ and $(Z_r, 1_r)$ are factors of (X, T) and (Y, S) respectively, so that $(Z_q\times Z_r, 1_q\times 1_r)$ is a factor of the minimal flow $(X\times Y, T\times S)$. Therefore $(Z_q\times Z_r, 1_q\times 1_r)$ is minimal, and so q and r must be relatively prime. If $\rho=qr$, then $\rho|p$ and (X_0, T^ρ), (Y_0, S^ρ) are both minimal.

Now $X\times Y=\bigcup_{i=0}^{q-1}\bigcup_{j=0}^{r-1} T^i\times S^j(X_0\times Y_0)$, and this union is disjoint. Moreover, each $T^i\times S^j(X_0\times Y_0)$ is a closed $T^\rho\times S^\rho$ invariant set. Hence there are at least $\rho=qr$ minimal subsets of $(X\times Y, T^\rho\times S^\rho)$. But since $(X\times Y, T\times S)$ is minimal, $(X\times Y, T^\rho\times S^\rho)$ is pointwise almost periodic and contains at most ρ minimal sets. Therefore, there are exactly ρ minimal sets for $T^\rho\times S^\rho$, and they must coincide with the sets $T^i\times S^j(X_0\times Y_0)$, $0\le i<q$, $0\le j<r$. In particular, $(X_0\times Y_0, T^\rho\times S^\rho)$ is minimal. Therefore, (X_0, T^ρ) and (Y_0, S^ρ) are disjoint, and as mentioned above, we may apply proposition 5.2 to complete the proof.

Although the converse of proposition 5.3 does not hold, there is a partial converse which is the content of the next proposition.

PROPOSITION 5.4. *Let (X, T) and (Y, S) be minimal, let $p>0$ and let X_0 and Y_0 be T^p and S^p minimal subsets of X and Y respectively. Let $q=\min\{k>0\colon T^k(X_0)=X_0\}$ and $r=\min\{k>0\colon S^k(Y_0)=Y_0\}$. If (X_0, T^p) and (Y_0, S^p) are disjoint and q and r are relatively prime, then (X, T) and (Y, S) are disjoint.*

Proof. As in the proof of proposition 5.3, $X\times Y$ is the disjoint union of $T^i\times S^j(X_0\times Y_0)$ $(0\le i\le q-1,\ 0\le j\le p-1)$. Now each $T^i\times S^j(X_0\times Y_0)$ is a $T^p\times S^p$ minimal set so $X\times Y$ is $T^p\times S^p$ pointwise almost periodic, and hence $T\times S$ is pointwise almost periodic.

Now there is a homomorphism π of $(X\times Y, T\times S)$ onto $(Z_q\times Z_r, 1_q\times 1_r)$ which maps $X_0\times Y_0$ to the point $(0,0)$. Since $(q, r)=1$, $(Z_q\times Z_r, 1_q\times 1_r)$ is minimal. Moreover, since (X_0, T^p) and (Y_0, S^p) are disjoint, $X_0\times Y_0$ is contained in the $T\times S$ orbit closure of (x_0, y_0), for any $(x_0, y_0)\in X_0\times Y_0$. Hence $\pi^{-1}(0,0)=X_0\times Y_0$ is contained in a $T\times S$ minimal set. The proof is completed by applying the following lemma.

LEMMA 5.5. *Let (Z, U) and (Z', U') be flows with (Z, U) pointwise almost periodic, and (Z', U') minimal, and let $\pi: Z \to Z'$ be a homomorphism. Let Z^* be a minimal subset of Z and suppose there is a $z_0 \in Z'$ such that $\pi^{-1}(z_0) \subset Z^*$. Then (Z, U) is minimal.*

Proof. Each point of Z is in some minimal subset M_0 of Z. Then $\pi(M_0) = Z'$, $M_0 \cap \pi^{-1}(z_0)$, $M_0 = Z^* \neq \emptyset$, so $M_0 \cap Z^* \neq \emptyset$, and $Z \subset Z^*$.

COROLLARY 5.6. *Let (X, T) be totally minimal and let (Y, S) be minimal. Let $p > 0$ and let Y_0 be an S^p minimal subset. Then (X, T) and (Y, S) are disjoint if and only if (X, T^p) and (Y_0, S^p) are disjoint.*

Proof. The 'only if' part follows from proposition 5.3 with $X_0 = X$. The 'if' part follows from proposition 5.4 because $q = 1$.

THEOREM 5.7. *Let (X, T) be a finite rank totally minimal flow with graphic invariants k and (Y, S) and let (Z, U) be a minimal flow. Then (Z, U) is disjoint from (X, T) if and only if (Z, U) is disjoint from (Y, S^k).*

Proof. Assume $(Z, U) \perp (Y, S^k)$. By theorem 5.1 we need only consider the cases when (X, T) is of type RR or QR. Since a flow of type QR is a factor of one of type RR, it suffices to prove the result for (X, T) of type RR. Because disjointness depends only on the proximal class we can assume (Y, S^k) is a factor of (X, T) and (X, T^p) is regular for some $p > 0$. The graphic invariants of (X, T^p) are kp and (Y, S). Let Z_0 be a U^p minimal set. By corollary 5.6, $(Y, S^{kp}) \perp (Z_0, U^p)$; by theorem 5.1, $(X, T^p) \perp (Z_0, U^p)$; and finally by corollary 5.6, $(X, T) \perp (Z, U)$.

COROLLARY 5.8. *Let (X, T) and (X', T') be finite rank totally minimal flows with graphic invariants k, (Y, S) and l, (Y', S'), respectively. Then*

(i) *$(X, T) \perp (X', T')$ if and only if $(Y, S^k) \perp (Y', S'^l)$.*

(ii) *If $m, n \in \mathbb{Z}$ with $m, n \neq 0$ and $mk \neq \pm nl$, then $(X, T^m) \perp (X', T'^n)$.*

Proof. (i) Apply theorem 5.7 twice, first to obtain $(X, T) \perp (Y', S'^l)$ and then $(X', T') \perp (X, T)$. (ii) Use theorem 18 of [**2**] and (i) of this corollary.

In [**2**] it is shown that if (Y, S) is a graphic minimal flow, and i and j are non-zero integers with $i \neq \pm j$ then (Y, S^i) and (Y, S^j) are disjoint (theorem 19). Combining this with (i) of corollary 5.8, we immediately obtain

COROLLARY 5.9. *Let (X, T) be a finite rank totally minimal flow and let m and n be non-zero integers with $m \neq \pm n$. Then $(X, T^m) \perp (X, T^n)$.*

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