FINITE GROUPS WHOSE POWERS HAVE NO COUNTABLY INFINITE FACTOR GROUPS

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1. Introduction. Let P be the class of all finite groups G whose powers G^I have no countably infinite factor groups. Neumann and Yamamuro (**1**) proved that if G is a finite non-Abelian simple group, then $G \in P$. We generalize this result by proving the following theorem.

THEOREM. *A finite group $G \in P$ if and only if G is perfect.*

2. Inheritance properties of P.

P_1. If $G \in P$ and N is normal in G, then $G/N \in P$.

Proof. Since $(G/N)^I$ is isomorphic to G^I/N^I, it is clear that factor groups of $(G/N)^I$ are isomorphic to factor groups of G^I, and hence finite or uncountable.

P_2. If $G = HK$, where $H \in P$ and $K \in P$, then $G \in P$.

Proof. We show that homomorphic images of G^I are either finite or uncountable. Let ϕ be a homomorphism of G^I. Then $G^I\phi = (HK)^I\phi = (H^IK^I)\phi = (H^I\phi)(K^I\phi)$. Since $H^I\phi$ and $K^I\phi$ must be finite or uncountable, the conclusion follows.

3. Preliminaries. Our aim in this section is to establish some lemmas which are used to prove the main result.

LEMMA 1. *If G is a finite group and G^I/K is infinite, then there exists a g in G and a countable number of distinct cosets of K in G^I with representatives whose components are either e or g.*

Proof. Suppose the contrary. Then for all $g \in G$, there exists a finite set $S(g)$ of cosets of K in G^I which contain all elements whose components are either e or g. Let

$$S = \bigcup\{S(g) \mid g \in G\}$$

and

$$T = \{S_1, \ldots, S_m \mid S_j \in S, m < \text{order } (G)\}.$$

Since S is finite, so is T. Let $(x_\alpha)K \in G^I/K$. Let $U = \{g_i \in G \mid g_i \neq e$ and there

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exists $\alpha \in I$ such that $x_\alpha = g_i\}$. Then $O(U) = k < O(G)$. If $A_i = \{\alpha | x_\alpha = g_i\}$, $i = 1, 2, \ldots, k$, define $(z_\alpha{}^i)$ by

$$z_\alpha{}^i = \begin{cases} g_i, & \alpha \in A_i, \\ e, & \alpha \notin A_i, \end{cases} \qquad i = 1, 2, \ldots, k.$$

Then $(z_\alpha{}^i)K \in S$ for $i = 1, 2, \ldots, n$, and

$$\prod_{i=1}^{k} (z_\alpha{}^i)K = (x_\alpha)K.$$

Hence, $(x_\alpha)K \in T$, implying that $G^I/K = T$. However, G^I/K is infinite, contradicting the finiteness of T.

If $(x_\alpha) \in G^I$, then $\sigma(x_\alpha)$, the *support* of (x_α), is defined by

$$\sigma(x_\alpha) = \{\alpha | x_\alpha \neq e\}.$$

LEMMA 2. *If G is a finite, perfect group with unique maximal normal subgroup M, $N \lhd G^I$, $(x_\alpha) \in N$, and $P = \{\alpha | x_\alpha \notin M\}$, then $G^P \subseteq N$.*

Proof. Our first step is to reduce the proof to the case where $P = \sigma(x_\alpha)$. Since G/M is simple and non-Abelian, for all $x_\alpha \in G$, $x_\alpha \notin M$, there exists $x_\alpha{}^*$ such that $[x_\alpha, x_\alpha{}^*] \notin M$. Define (y_α) by

$$y_\alpha = \begin{cases} x_\alpha{}^*, & \alpha \in P, \\ e, & \alpha \notin P. \end{cases}$$

Then $(z_\alpha) = [(x_\alpha), (y_\alpha)] \in N$, $\sigma(z_\alpha) = P$ and $z_\alpha \notin M$, for all $\alpha \in \sigma(z_\alpha)$. Hence, without loss of generality, $P = \sigma(x_\alpha)$.

Let $(b_\alpha) \in G^P$. Let g_i, $i = 1, \ldots, k$, and h_j, $j = 1, \ldots, t$, be those non-identity elements which occur as components in (x_α) and (b_α), respectively. Let $A_i = \{\alpha | x_\alpha = g_i\}$, $i = 1, \ldots, k$, and $B_j = \{\alpha | b_\alpha = h_j\}$, $j = 1, \ldots, t$. Define $(c_\alpha{}^{(i,j)})$ by

$$c_\alpha{}^{(i,j)} = \begin{cases} h_j, & \alpha \in A_i \cap B_j, \\ e, & \alpha \notin A_i \cap B_j. \end{cases}$$

Then

$$(b_\alpha) = \prod_{i,j} (c_\alpha{}^{(i,j)}).$$

Thus, if we show that for each i and j, $(c_\alpha{}^{(i,j)}) \in N$, then $(b_\alpha) \in N$ and the proof is complete.

We may assume that $(c_\alpha{}^{(i,j)}) \neq (e)$. Since $g_i \notin M$, there exists $g_i{}^*$ such that $[g_i, g_i{}^*] \notin M$. Define (w_α) by

$$w_\alpha = \begin{cases} g_i{}^*, & \alpha \in A_i \cap B_j, \\ e, & \alpha \notin A_i \cap B_j. \end{cases}$$

Then $(v_\alpha) = [(x_\alpha), (w_\alpha)] \in N$, where

$$v_\alpha = \begin{cases} [g_i, g_i{}^*], & \alpha \in A_i \cap B_j, \\ e, & \alpha \notin A_i \cap B_j. \end{cases}$$

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Let $d = [g_i, g_i^*]$. Since $d \notin M$ and M is the unique maximal normal subgroup of G, the normal closure of d is G. Therefore, there exists n_i, $i = 1, \ldots, s$, such that

$$\prod_{i=1}^{s} n_i^{-1} d^{\pm 1} n_i = h_j.$$

If we let

$$n_\alpha{}^i = \begin{cases} n_i, & \alpha \in A_i \cap B_j, \\ e, & \alpha \notin A_i \cap B_j, \end{cases}$$

then

$$\prod_{i=1}^{s} (n_\alpha{}^i)^{-1} (v_\alpha)^{\pm 1} (n_\alpha{}^i) = (c_\alpha{}^{(i,j)}).$$

Since $(v_\alpha)^{\pm 1} \in N$ and $N \lhd G$, we have $(c_\alpha{}^{(i,j)}) \in N$.

LEMMA 3. *If G is a finite perfect group with unique maximal normal subgroup M, then $G \in P$.*

Proof. Suppose the contrary. Then there exists I and there exists $N \lhd G^I$ such that G^I/N is countably infinite. Since G/M is simple and non-Abelian, $G/M \in P$ (**1**, Corollary 9). Using this fact and applying an isomorphism theorem, we find that G^I/NM^I is finite and $M^I/(N \cap M^I)$ is countably infinite.

By Lemma 1, there exists $g \in M$ and a countable number of distinct cosets of $N \cap M^I$ in M^I of the form $(x_\alpha{}^i)(N \cap M^I)$, $i = 1, 2, \ldots$, where

$$x_\alpha{}^i = \begin{cases} g, & \alpha \in A_i, \\ e, & \alpha \notin A_i, \end{cases}$$

for some $A_i \subseteq I$. Since $(x_\alpha{}^i)(N \cap M^I) \neq (x_\alpha{}^j)(N \cap M^I)$ for $i \neq j$,

$$(x_\alpha{}^i)(x_\alpha{}^j)^{-1} = (x_\alpha{}^i x_\alpha{}^{j^{-1}}) \notin N \cap M^I,$$

where

$$x_\alpha{}^i x_\alpha{}^{j^{-1}} = \begin{cases} g \text{ or } g^{-1}, & \alpha \in A_i + A_j, \\ e, & \alpha \notin A_i + A_j. \end{cases}$$

Let $g_0 \in G$, $g_0 \notin M$. Define $\{(y_\alpha{}^i) \mid i = 1, 2, \ldots\}$ by

$$y_\alpha{}^i = \begin{cases} g_0, & \alpha \in A_i, \\ e, & \alpha \notin A_i. \end{cases}$$

Consider the $(y_\alpha{}^i)$ as coset representatives of NM^I in G^I. If we can show that the $(y_\alpha{}^i)$ yield distinct cosets, then G^I/NM^I would be infinite, and a contradiction reached. This is our aim.

Suppose that there exist i and j, $i \neq j$, such that $(y_\alpha{}^i)(y_\alpha{}^j)^{-1} \in NM^I$, where now

$$y_\alpha{}^i y_\alpha{}^{j^{-1}} = \begin{cases} g_0 \text{ or } g_0{}^{-1}, & \alpha \in A_i + A_j, \\ e, & \alpha \notin A_i + A_j. \end{cases}$$

Then there exists $(n_\alpha) \in N$ and there exists $(m_\alpha) \in M^I$ such that $(y_\alpha{}^i)(y_\alpha{}^j)^{-1} =$

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$(n_\alpha)(m_\alpha)$. Hence, if $\alpha \in A_i + A_j$, then $n_\alpha = g_0 m_\alpha^{-1}$ or $g_0^{-1} m_\alpha^{-1}$. Therefore, $n_\alpha \notin M$ for all $\alpha \in A_i + A_j$. By Lemma 2, $G^{A_i + A_j} \subseteq N$. However, this implies that $(x_\alpha^i)(x_\alpha^j)^{-1} \in N$. This contradicts the distinctness of the cosets $(x_\alpha^i)(N \cap M^I)$ and $(x_\alpha^j)(N \cap M^I)$. Therefore, G^I/NM^I is infinite and the desired contradiction is obtained.

An easy application of the fact that Abelian groups of finite exponent are a direct sum of cyclic groups shows that finite Abelian groups do not belong to P. However, for ease of reference, we list this fact as a separate lemma.

LEMMA 4. *If G is a finite Abelian group, then G does not belong to P.*

4. Proof of the Theorem. The proof of the Theorem will depend upon two theorems of Wielandt (**3**, p. 228, theorems (22) and (23)). These are necessary only for conciseness. By induction on the order of the group, we could give a valid but more lengthy proof of the desired result. We need the following definition.

Definition. Let G be a finite group. A subnormal subgroup H of G is called an *atom* if $H' = H$, and H has a unique maximal normal subgroup.

We now proceed to prove the Theorem.

Proof. Let G be a finite group such that $G \in P$. If G is not perfect, then G/G' is a non-trivial Abelian group and belongs to P by P_1. This is contrary to Lemma 4.

Let G be a finite perfect group. By a theorem of Wielandt (**3**, p. 228, theorem (22)), G is generated by its atoms. In addition, Wielandt showed that atoms and subnormal subgroups commute (**3**, p. 228, theorem (23)). Hence, if H_i $(i = 1, \ldots, k)$ are the atoms of G, then $G = H_1 H_2 \ldots H_k$. Since $H_i \in P$ $(i = 1, \ldots, k)$ by Lemma 3, the result follows by P_2.

There are a number of observations that can be made concerning possible generalizations of this theorem. If we remove the finiteness condition, then it is still true that groups must be perfect in order that their powers have no countably infinite factor groups. This follows from the fact that P_1 and Lemma 4 are still valid when G is infinite. However, the converse is false. We need only to take a countable perfect group with any indexing set I, and it is easy to obtain a countable factor group. The question still exists as to whether or not certain classes of uncountable perfect groups satisfy the Theorem.

Another, and I think, more interesting question is: "Is the Theorem true if the finite and perfect conditions remain, but the groups are allowed to vary?". That one can allow a finite number of distinct finite perfect groups to occur is not difficult to show. However, I have not been able to answer the general question.

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