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SYMMETRIC ALGEBRAS OVER RINGS AND FIELDS

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Abstract

Connections between annihilators and ideals in Frobenius and symmetric algebras are used to provide a new proof of a result of Nakayama on quotient algebras, and an application is given to central symmetric algebras.

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1. Introduction

Throughout this paper, R will denote a commutative ring with identity element 1 and A will denote an associative unital R-algebra which is a finitely generated projective R-module. For any algebra A, the centre of A will be denoted by $Z(A)$.

Definition 1.1. The algebra A is called a *Frobenius algebra* if it is finitely generated and projective as an R-module and there exists a left A-module isomorphism $\varphi : A \cong A^*$, where A^* denotes the ring $\hom_R(A, R)$ as a right R-module. Note that A^* is an (A, A)-bimodule via $(a \cdot \lambda)(b) = \lambda(ba)$ and $(\lambda \cdot a)(b) = \lambda(ab)$ for any $a, b \in A$, $\lambda \in A^*$. If there exists a two-sided A-module isomorphism $\varphi : A \cong A^*$, then A is called *symmetric* [1, 3].

There has been a resurgence of interest in Frobenius algebras in recent years due to applications in coding theory (see, for example, [6, 7]). In this note we consider the connections between hyperplanes and ideals in Frobenius and symmetric algebras over commutative rings. This allows us to develop a succinct, coordinate-free proof of a result of Nakayama [4] that determines when the quotient of a symmetric algebra over a field is again symmetric. As a corollary, we show the class of central symmetric algebras is identical to the class of central simple algebras.

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Definition 1.2. A *form* f on A is a bilinear mapping $f : A \times A \to R$. The form is called *associative* if $f(ac, b) = f(a, cb)$ for all $a, b, c \in A$. It is called *right nonsingular* if $b \mapsto f(\cdot, b)$ is an R-module isomorphism from A to A^* and *left nonsingular* if $a \mapsto f(a, \cdot)$ is an R-module isomorphism. When both conditions hold, we say the form is *nonsingular*.

Remark 1.3. Some references will refer to such forms as 'nondegenerate'. We shall need a distinction between this and a weaker condition also called nondegenerate, so we shall follow the convention in [5]. The weaker condition states that f is *nondegenerate* if $f(a, b) = 0$ for all $a \in A$ implies $b = 0$ and $f(a, b) = 0$ for all $b \in A$ implies $a = 0$. Over a field these conditions are easily seen to be equivalent using a dimension argument.

Proposition 1.4. *A finitely generated projective R-algebra A is Frobenius if and only if there exists a nonsingular associative bilinear form $f : A \times A \to R$, and is symmetric if and only if there exists such a form which is also symmetric.*

Proof. Assume first that A is a Frobenius algebra. Since A is Frobenius there exists a left A-module isomorphism $\varphi : A \cong A^*$. We define $f(a, b) = \varphi(b)(a)$. The form f is right nonsingular since φ is an isomorphism. Left nonsingularity holds for the same reason applied to the transpose mapping φ' defined in [2, page 2]. Since $f(a, cb) = \varphi(cb)(a) = [c\varphi(b)](a) = \varphi(b)(ac) = f(ac, b)$, we see that f is associative.

For the converse, assume there is an associative nonsingular form f. Define $\varphi : A \to A^*$ by $\varphi(b)(a) = f(a, b)$. Then, by definition, φ is an isomorphism as desired.

If φ is a two-sided A-module isomorphism, then $f(a, b) = \varphi(b)(a) = (b \cdot \varphi(1))(a) = \varphi(1)(ab)$ using the left module isomorphism, and $f(b, a) = \varphi(a)(b) = (\varphi(1) \cdot a)(b) = \varphi(1)(ab)$ using the right module isomorphism, so $f(a, b) = f(b, a)$. Conversely, if such a form is symmetric, the left module structure of $\varphi : A \cong A^*$ is straightforward from associativity and the right module structure is shown by $(\varphi(b) \cdot a)(x) = \varphi(b)(ax) = f(ax, b) = f(b, ax) = f(ba, x) = f(x, ba) = \varphi(ba)(x)$ for all $a, b, x \in A$. □

2. Hyperplanes and ideals

We begin by defining the notion of a 'hyperplane' with respect to an associative bilinear form on an R-algebra A. In the case where A is an algebra over a field, this notion coincides with the usual notion of a hyperplane as a subspace of codimension one. We then determine the maximal left and right ideals in a hyperplane. We conclude by exploring the connections between annihilators of ideals and intersections of hyperplanes. In particular, we see why results for fields only partially generalise due to different versions of the nondegeneracy hypothesis for forms on a module over a ring.

Definition 2.1. Given an R-algebra A with associative bilinear form f, we can associate two (not necessarily distinct) hyperplanes to each nonzero $c \in A$. A (*left*)

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hyperplane with respect to f is a set

$$_cH = \{x \in A : f(x, c) = 0\} \quad (c \neq 0, c \in A),$$

and a (*right*) *hyperplane* with respect to f is a set

$$H_c = \{x \in A : f(c, x) = 0\} \quad (c \neq 0, c \in A).$$

Hyperplanes are R-submodules of A. If the form f is symmetric, then $_cH = H_c$ for all $c \in A$. In general, $_1H = H_1$ since associativity implies $f(1, a) = f(a, 1)$ for all $a \in A$.

More generally, if $S \subseteq A$, we may define $H_S = \{x \in A : f(s, x) = 0 \ \forall s \in S\}$. The set $_SH$ is defined similarly. These are intersections of hyperplanes. If S is a right ideal in A, then H_S will be a left ideal, and if S is a left ideal, then $_SH$ will be a right ideal.

Definition 2.2. The hyperplane H_c (respectively, $_cH$) is *nondegenerate* if it contains no nontrivial left (respectively, right) ideals. It is *symmetric* if it contains all commutators of A.

Proposition 2.3. *If an R-algebra A is Frobenius, then A admits an associative bilinear form f having a pair of nondegenerate hyperplanes $_cH$ and H_c, for some $c \in A$. If the algebra A is symmetric, then A admits an associative bilinear form f having a pair of symmetric nondegenerate hyperplanes $_cH$ and H_c, for some $c \in Z(A)$.*

Proof. Let f be an associative nonsingular bilinear form as guaranteed by Proposition 1.4. Consider the hyperplane H_1. If $Ax \subseteq H_1$, we have $0 = f(1, Ax) = f(A, x)$ by associativity, whence $x = 0$ since f is nonsingular. Thus H_1 contains no nontrivial left ideals. Also $xA \subseteq {_1H}$ implies that $0 = f(xA, 1) = f(x, A)$, so that $x = 0$. Therefore $_1H = H_1$ is nondegenerate.

Now assume that A is a symmetric algebra and again consider H_1. Then

$$f(1, yx - xy) = f(1, yx) - f(1, xy) = f(y, x) - f(x, y) = 0$$

since f is symmetric, and so H_1 is symmetric. □

As the following proposition shows, the condition $c \in Z(A)$ is inextricably linked to the condition that H_c (or $_cH$) is symmetric.

Proposition 2.4. *Let A be a symmetric R-algebra, f a nonsingular associative symmetric form on A and $c \in A$. Then H_c or $_cH$ is symmetric if and only if $c \in Z(A)$.*

Proof. For all $x, y \in A$,

$$\begin{aligned} f(c, yx - xy) = 0 &\iff f(c, yx) = f(c, xy) \\ &\iff f(cy, x) = f(xy, c) = f(x, yc) = f(yc, x) \\ &\iff f(cy - yc, x) = 0 \\ &\iff cy - yc = 0 \quad (\forall y \in A) \\ &\iff c \in Z(A). \end{aligned}$$

The proof for $_cH$ is analogous. □

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Notice that the proof of Proposition 2.3 did not use the full force of nonsingularity, but rather only nondegeneracy of the form f. We next obtain partial converses to the statements in Proposition 2.3; these are not full converses when nondegeneracy is not equivalent to nonsingularity.

PROPOSITION 2.5. *Let A be an R-algebra with an associative bilinear form f. Let H_c and ${}_cH$ be a pair of nondegenerate hyperplanes with respect to f, for some $c \in A$. Then f is nondegenerate. If the hyperplanes are also symmetric and $c \in Z(A)$, then there exists an associative nondegenerate symmetric form on A.*

PROOF. If $x \in A$ is such that $f(a, x) = 0$ for all $a \in A$, then $f(c, Ax) = f(cA, x) = 0$; since H_c is nondegenerate, $Ax = 0$, so $x = 0$, which is to say that $b \mapsto f(\cdot, b)$ is injective. Also if $x \in A$ is such that $f(x, a) = 0$ for all $a \in A$, then $f(xA, c) = f(x, Ac) = 0$; since ${}_cH$ is nondegenerate, $xA = 0$, so $x = 0$. Thus the form f is nondegenerate (but may not be nonsingular).

Now assume that the nondegenerate hyperplanes H_c and ${}_cH$ are also symmetric and $c \in Z(A)$. Then f is nondegenerate as above. We first check that these hyperplanes are equal. Since $c \in Z(A)$, we have $f(x, c) = 0$ if and only if $f(xc, 1) = f(cx, 1) = f(c, x) = 0$, and so we obtain ${}_cH = H_c$. For all $x, y \in A$, define $g(x, y) = f(x, yc)$. Then g is also associative. We next check that g is nondegenerate. If $g(x, y) = 0$ for all $y \in A$, then $xA \subseteq H_c$ and $x = 0$ since $H_c = {}_cH$ is nondegenerate. Similarly, if $g(x, y) = 0$ for all $x \in A$, then $Ay \subseteq {}_cH = H_c$ and so $y = 0$. Finally, we must check that g is symmetric. Since H_c is symmetric, we have $f(xy - yx, c) = 0$, from which we obtain $f(xy, c) = f(yx, c)$, or $f(x, yc) = f(y, xc)$. Now by definition of g, we have $g(x, y) = g(y, x)$ and thus g is symmetric. □

EXAMPLE 2.6. As an example where nondegeneracy is weaker than nonsingularity, we let $R = \mathbb{Z}$ and A be the group ring $\mathbb{Z}[\mathbb{Z}_2] = \mathbb{Z} \oplus \mathbb{Z}g$, where $g^2 = 1$. Using $\{1, g\}$ as a basis, the form given by the matrix $\left(\begin{smallmatrix}1 & 2\\ 2 & 3\end{smallmatrix}\right)$ is nonsingular since its determinant is a unit and so it induces an isomorphism with A^*. On the other hand, the form given by the matrix $\left(\begin{smallmatrix}2 & 2\\ 2 & 3\end{smallmatrix}\right)$ is nondegenerate since its determinant is 2, a nonunit (but also nonzero), just as it is if viewed as a form over $\mathbb{Q}$. But it is not nonsingular. Specifically, the linear functional $\psi(x + yg) = x$ is not in the image of the mapping induced by the form f since $f(x + yg, b_1 + b_2g) = 2b_1x + 2b_2x + 2b_1y + 3b_2y$ cannot equal x for any choice of $b_1, b_2 \in \mathbb{Z}$.

COROLLARY 2.7. *For a field K, a K-algebra A is Frobenius if and only if A admits an associative bilinear form f having a pair of nondegenerate hyperplanes ${}_cH$ and H_c, for some $c \in A$. The algebra A is symmetric if and only A admits an associative bilinear form f having a pair of symmetric nondegenerate hyperplanes ${}_cH$ and H_c, for some $c \in Z(A)$.*

LEMMA 2.8. *Let A be an R-algebra with a nondegenerate associative bilinear form f and let $c \in A$. The hyperplane H_c contains H_{cA} as its largest left ideal. Assume further that f is symmetric. Then H_{cA} is nontrivial if and only if c is a (right) zero divisor.*

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Proof. Suppose J is a left ideal contained in $H_c = \{x : f(c, x) = 0\}$. For all $a \in A$, we have $f(c, aJ) = f(c, J) = 0$, so $f(cA, J) = 0$ and $J \subseteq H_{cA}$. Since H_{cA} is itself a left ideal, it is the largest one.

Suppose now that f is symmetric and $0 \neq x \in H_{cA}$. Then for all $a \in A$, we have $f(ca, x) = 0$, which implies $f(xc, a) = f(x, ca) = 0$ since f is symmetric. By nondegeneracy, we obtain $xc = 0$ and c is a right zero divisor. Conversely, if $bc = 0$ with $b \neq 0$, then for any $a \in A$, $f(a, bc) = 0$, which implies $f(Ab, c) = 0$, so that $Ab \subseteq H_c$. As above, this left ideal is contained in H_{cA}, showing that H_{cA} is nontrivial. □

Let J be a two-sided ideal in a Frobenius algebra A. By associativity, we can write $f(x, J) = f(xJ, A)$, so that, using the nondegeneracy of f, we have $xJ = 0$ if and only if $f(x, J) = 0$. Similarly, $Jx = 0$ if and only if $f(J, x) = 0$. Thus we can use the form f to express the right annihilator $\mathrm{r}(J)$ and left annihilator $\mathrm{l}(J)$ of J as follows:

$$\mathrm{r}(J) = \{x \in A : f(J, x) = 0\} = H_J,$$
$$\mathrm{l}(J) = \{x \in A : f(x, J) = 0\} = {}_J H.$$

These will play a crucial role in determining when a quotient of a symmetric algebra over a field is again symmetric.

Lemma 2.9. *Let J be a two-sided ideal in a symmetric algebra A. Then the left and right annihilators are equal; that is,* $\mathrm{l}(J) = \mathrm{r}(J)$.

Proof. We have $x \in \mathrm{r}(J)$ if and only if $f(J, x) = 0$ if and only if $f(x, J) = 0$ (by symmetry) if and only if $x \in \mathrm{l}(J)$. □

3. Quotients of algebras

We ask under what conditions a quotient of a symmetric algebra over a field is Frobenius or symmetric. This was done by Nakayama in the 1930s using matrix arguments [4], but we present a coordinate-free approach here.

Throughout this section, we shall assume that A is a Frobenius algebra over a field K with nondegenerate associative bilinear form f. We again point out that nonsingularity and nondegeneracy are equivalent concepts when the base ring is a field.

Lemma 3.1. *Let H be a maximal proper K-submodule of A. Then H is a right hyperplane H_c for some $c \in A$. Conversely, any right hyperplane H_c is a maximal proper submodule. The analogous result holds for left hyperplanes as well.*

Proof. Let H be a maximal proper K-submodule of A and set $H^{\perp} = \{x \in A : f(x, H) = 0\}$. Choose $0 \neq c \in H^{\perp}$. Then $A \neq H_c = \{x \in A : f(c, x) = 0\} \supseteq H$. By maximality of H, we have $H_c = H$. Conversely, let $y \notin H_c$. Then $H_c + Ky = A$ since given any $\alpha \in A$, there exists $r \in K$ such that $\alpha - ry \in H_c$. Specifically, if $r = f(c, \alpha)/f(c, y)$, then $f(c, \alpha - ry) = 0$. Therefore H_c is maximal. □

Theorem 3.2. *Let A be a symmetric K-algebra and let J be a two-sided ideal of A. Then A/J is Frobenius if and only if* $\mathrm{r}(J)$ *is a principal ideal generated by some $c \in A$, where $Ac = cA$. This quotient is also symmetric if and only if* $\mathrm{r}(J) = cA$ *where $c \in Z(A)$.*

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Proof. Suppose first that $\bar{A} = A/J$ is a Frobenius algebra. Let $\bar{H}$ be a nondegenerate hyperplane in $\bar{A}$. Let $H' = \{x \in A : \bar{x} \in \bar{H}\}$. Then H' is a hyperplane in A, for if there exists a proper submodule $\tilde{H}$ properly containing H', then $\tilde{H}/J$ would be a proper submodule of $\bar{A}$ properly containing $\bar{H}$, contradicting the maximality of $\bar{H}$. We claim that J is the largest left or right ideal in H'. If not, assume that I is a (left or right) ideal in H' not contained in J. Then $I + J \subseteq H'$ so we may assume that I properly contains J. But then I/J is a nonzero (left or right) ideal in $\bar{H}$, contradicting the nondegeneracy of $\bar{H}$. Now $H' = H_c$ for some $c \in A$ by Lemma 3.1, and by Lemma 2.8, this implies that $J = H_{cA}$. From this it follows that $J = r(cA)$. But since A is symmetric, we have $J = \{x \in A : f(x, cA) = 0\}$ as well, and so also $J = \mathrm{l}(cA)$. Then we know that we have $\mathrm{r}(\mathrm{l}(cA)) = cA = \mathrm{r}(J)$ since a Frobenius algebra over a Frobenius ring is a quasi-Frobenius ring [2, Corollary 20]. Since $\mathrm{r}(J)$ is a two-sided ideal, we have $Ac \subseteq AcA = cA$. By an analogous argument, the left annihilator $\mathrm{l}(J)$ is the principal left ideal Ac, which is also two-sided, giving the reverse inclusion. Therefore $Ac = cA$.

Conversely, assume that $\mathrm{r}(J) = \mathrm{l}(J) = Ac = cA$. Let f be the associative, nondegenerate, symmetric bilinear form on A given by Proposition 1.4. Define $\bar{f} : \bar{A} \times \bar{A} \to K$ by

$$\bar{f}(\bar{x}, \bar{y}) = f(xy, c) = f(x, yc).$$

To see that $\bar{f}$ is well defined, let $\tilde{x} = x + j_1 \in x + J$, $\tilde{y} = y + j_2 \in y + J$. Then

$$f(\tilde{x}\tilde{y}, c) = f((x + j_1)(y + j_2), c) = f(xy, c) + f(j_1 y, c) + f(x j_2, c) + f(j_1 j_2, c). \quad (3.1)$$

Since c annihilates the two-sided ideal J on both the right and left, and $j_1, x j_2, j_1 j_2$ are all in J, we have $f(\tilde{x}\tilde{y}, c) = f(xy, c)$ as desired.

To complete the proof that $\bar{A}$ is Frobenius, we check that $\bar{f}$ is nondegenerate. Indeed,

$$\begin{aligned} \bar{f}(\bar{x}, \bar{y}) = 0\ \forall \bar{y} &\iff f(x, yc) = 0\ \forall y \\ &\iff x \in H_{Ac} = J \iff \bar{x} = 0 \end{aligned}$$

and

$$\begin{aligned} \bar{f}(\bar{x}, \bar{y}) = 0\ \forall \bar{x} &\iff f(x, yc) = 0\ \forall x \\ &\iff yc = 0 \iff y \in H_{cA} = J \iff \bar{y} = 0. \end{aligned}$$

Finally, we have that $\bar{A}$ is symmetric if and only if $\bar{f}(\bar{x}, \bar{y}) = \bar{f}(\bar{y}, \bar{x})$. We note that

$$\begin{aligned} \bar{f}(\bar{x}, \bar{y}) = \bar{f}(\bar{y}, \bar{x})\ \forall x\ \forall y &\iff f(xy, c) = f(yx, c)\ \forall x\ \forall y \\ &\iff xy - yx \in H_c\ \forall x\ \forall y \\ &\iff c \in Z(A) \quad \text{by Proposition 2.4.} \end{aligned}$$

Now the proof is complete. □

This theorem then allows the following characterisation of symmetric K-algebras A, where K is a field, and $Z(A) = K$. To the best of the authors' knowledge, this result has not previously been observed.

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COROLLARY 3.3. *Central symmetric algebras over a field are simple.*

PROOF. By Theorem 3.2, if A is a symmetric algebra and J is an ideal of A such that A/J is symmetric, then $\mathrm{r}(J) = \mathrm{l}(J) = Ac$ for some $c \in Z(A)$. Let $J = \mathrm{rad}(A)$. Then if A is an artinian K-algebra, A/J is a semisimple K-algebra, so is necessarily symmetric. Combining these two facts, we see that if A is central symmetric, then $\mathrm{r}(J) = \mathrm{l}(J) = Ac$ for some $c \in K$. In particular, $\mathrm{r}(J) = \mathrm{l}(J) = 0$ or A. Then we must have $\mathrm{r}(J) = A$, since $\mathrm{l}(\mathrm{r}(J)) = J$, and $\mathrm{l}(0) = A$, and of course $J \neq A$.

Thus A is semisimple, so we may write A as a direct sum of simple K-algebras, $A = \bigoplus M_{n_i}(D_i)$, where the D_i are K-division algebras. The centre of A is $\bigoplus Z(D_i) \cong K$, so $A \cong M_n(D)$ where D is a K-central division algebra. □

EXAMPLE 3.4. The results of this section do not generally hold for rings since, as pointed out in Example 2.6, a nondegenerate form need not be nonsingular. For example, we can generalise the previous example by taking $R = \mathbb{Z}$ and A to be the group ring $\mathbb{Z}[\mathbb{Z}_2 \times \mathbb{Z}_2]$, where the copies of $\mathbb{Z}_2$ are generated by g and h. Let J be the ideal generated by $(1, h)$, so that A/J is isomorphic to the algebra of Example 2.6. To make the proof of Theorem 3.2 work, we need to check that the induced bilinear form $\bar{f}$ is nonsingular (that is, it has a determinant which is a unit). It is easy to make examples where this does or does not happen, depending on the given symmetric form on A, but this does allow some application of the results of Section 2 outside of the field case.

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