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Equilibrium states for piecewise monotonic transformations

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Abstract. We show that equilibrium states μ of a function ϕ on ([0, 1], T), where T is piecewise monotonic, have strong ergodic properties in the following three cases:

(i) $\sup \phi - \inf \phi < h_{top}(T)$ and ϕ is of bounded variation.

(ii) ϕ satisfies a variation condition and T has a local specification property.

(iii) $\phi = -\log |T'|$, which gives an absolutely continuous μ, T is C^2, the orbits of the critical points of T are finite, and all periodic orbits of T are uniformly repelling.

0. *Introduction*

In this paper we deal with piecewise monotonic transformations (X, T), i.e.

$$X = [0, 1] = \bigcup_{i=1}^{N} J_i,$$

where the J_i are disjoint intervals and T/J_i is continuous and strictly monotone. We consider two related problems for such dynamical systems. The first one is to find an equilibrium state μ for a given function $\phi : X \to \mathbb{R}$ of bounded variation, i.e. a T-invariant Borel probability measure μ on X, for which the map

$$\nu \to h(\nu) + \int \phi \, d\nu$$

attains its supremum (h denotes entropy). The second one is to find a T-invariant probability measure μ, which is absolutely continuous with respect to some given atom-free probability measure m with

$$m \circ T^{-1} \ll m$$

($\ll$ denotes absolute continuity).

A useful method to attack both of these problems and to show ergodic properties of the required measure μ is the investigation of the Perron–Frobenius-operator P, as it is done in [3] and other papers (cf. the references of [3]). Let $\mathcal{F}$ be the set of all real – or complex – valued bounded measurable functions on X. For a given bounded measurable

$$g : X \to (0, \infty),$$

the operator $P:\mathscr{F}\to\mathscr{F}$ is defined by

$$Pf(x)=\sum_{y\in T^{-1}x} g(y)f(y).$$

The following theorem is basically proved in [3] (cf. § 1).

THEOREM. *Let $g:X\to(0,\infty)$ be of bounded variation such that $\|g_n\|_\infty<1$ for some n, where*

$$g_n(x)=g(x)g(Tx)\cdots g(T^{n-1}x).$$

Suppose there is a Borel probability measure m on X such that

$$\int Pf\,dm=\int f\,dm$$

for all $f\in\mathscr{F}$. Then we have

(i) *There is a function $h:X\to[0,\infty)$ of bounded variation such that $\int h\,dm=1$ and $Ph=h$, which implies that the measure $\mu=hm$ is T-invariant and an equilibrium state for* $\log g$.

(ii) *For some $k\geq 1$, the measure μ on (X, T^k) splits up into finitely many ergodic components, on each of which T^k is weakly Bernoulli with exponential mixing rate. This implies central limit theorems and almost sure invariance principles for stochastic processes*

$$(f\circ T^{nk})_{n\geq 1} \text{ on } (X,\mu),$$

where f is of bounded variation.

The aim of this paper is to apply this theorem, in order to solve the problems mentioned above. First we consider equilibrium states. For a given continuous ϕ on X, P. Walters [13] gives a useful method to find a Borel probability measure m on X and a $\lambda>0$ such that

$$\int Pf\,dm=\int f\,dm$$

for all $f\in\mathscr{F}$, where $g(x)=e^{\phi(x)}/\lambda$. In § 1 we adapt this method to our situation, where ϕ is of bounded variation. Then $g=e^{\phi}/\lambda$ is also of bounded variation and an equilibrium state of $\log g$ is also one of ϕ. The above theorem gives us the existence and ergodic properties of an equilibrium state of ϕ, if we find an n with $\|g_n\|_\infty<1$. This problem is considered in § 2 and § 3. Lemma 1 in § 1 is a useful tool for this.

In order to show $\|g_n\|_\infty<1$, one needs results about the orbits of (X, T). Methods to prove such results are developed in [6]. In this paper the orbits of (X, T) are represented as one-sided paths of an oriented graph, which we shall call the Markov diagram of T.

In § 2 we show the existence of an n with $\|g_n\|_\infty<1$ for all ϕ of bounded variation with

$$\sup\phi-\inf\phi<h_{\text{top}}(T)$$

generalizing the example of III, § 5 in [3] (h_{top} denotes topological entropy). The proof of this result relies heavily on the first half of the following estimate which can be proved using the Markov diagram of T:

$$0<\liminf_{n\to\infty}\,(e^{-nh}\inf_{x\in\Omega}\operatorname{card}T^{-n}\{x\})\le\limsup_{n\to\infty}\,(e^{-nh}\sup_{x\in\Omega}\operatorname{card}T^{-n}\{x\})<\infty,$$

where Ω is a topologically transitive subset of (X, T) and $h=h_{\text{top}}(T/\Omega)$.

In § 3 we consider ϕ on X with $\sum_{i=1}^{\infty}\operatorname{var}_i\phi<\infty$, where

$$\operatorname{var}_i\phi=\sup\{|\phi(x)-\phi(y)|: x, y \text{ are in the same interval, on which } T^i \text{ is monotone}\}.$$

Generalizing an idea of P. Walters [13], we find an n with $\|g_n\|_\infty<1$, if (X, T) satisfies a local specification property (for definition see (ii) of theorem 3). This property can be proved for certain (X, T) with the aid of the Markov diagram of T. It is shared by the β-transformation, so that we generalize the example of III, § 4 in [3].

In § 4 we deal with the second problem mentioned at the beginning of § 0. We look for absolutely continuous (with respect to Lebesgue measure) invariant measures μ on (X, T), where T is C^2 and the orbits of the critical points of T become periodic (cf. [9], [12]). Under additional assumptions we are able to find a function g and a measure m equivalent to Lebesgue measure such that the requirements of the above theorem are satisfied. The measure μ, we get by (i) of this theorem, is then the desired one. It would be interesting to know, whether this approach also can be used to study the case, where one only requires the orbits of the critical points to be bounded away from the critical points (cf. [10], [11], [12]).

1. *The existence of m*

Let T be a piecewise monotonic transformation on $X=[0, 1]$ and $\phi: X\to\mathbb{R}$ be of bounded variation. It is not difficult to show that

$$\phi(x+)=\lim_{t\downarrow x}\phi(t)\quad\text{and}\quad\phi(x-)=\lim_{t\uparrow x}\phi(t)$$

exist for all $x\in X$ and that $\phi(x+)\neq\phi(x-)$ can happen only for countably many x. Furthermore we suppose that $\phi(x)$ is either $\phi(x+)$ or $\phi(x-)$.

As in [13] we want to introduce a finer topology in X such that ϕ and T become continuous and that the Perron–Frobenius-operator P becomes a continuous operator on the Banach space of continuous functions. To this end let

$$c_1<c_2<\cdots<c_{N-1}$$

be the points, which separate the J_i, set

$$V=\{x:\phi(x+)\neq\phi(x-)\}\cup\{0, 1\}$$

and

$$W=\Big(\bigcup_{i=0}^{\infty}T^{-i}\Big(\bigcup_{j=0}^{\infty}T^jV\cup\bigcup_{j=1}^{\infty}\{T^j(c_k+), T^j(c_k-): 1<k\le N-1\}\Big)\Big)\Big\backslash\{0, 1\},$$

W is T-invariant and countable. If $x \in W$, substitute x by $x+$ and $x-$ in X. We denote the set we get by $\bar{X}$. The order relation is extended to $\bar{X}$ by

$$y < x- < x+ < z,$$

if $y < x < z$ holds in X. The order topology makes $\bar{X}$ compact. Because $X \backslash W$ is dense in $\bar{X}$ and $\phi(x+)$, $\phi(x-)$, $T(x+)$, $T(x-)$ exist (T is piecewise monotone), ϕ and T can be extended continuously from $X \backslash W$ to $\bar{X}$. They are then continuous functions on $\bar{X}$, because their discontinuities in X are contained in W. The definition of W implies also, that $T(\bar{J}_i)$ is an open and closed interval in $\bar{X}$ (the bar denotes closure in $\bar{X}$). This implies that the operator $\tilde{P}$ defined by

$$\tilde{P}f(x) = \sum_{y \in T^{-1}x} e^{\phi(y)} f(y)$$

maps $C(\bar{X})$ into $C(\bar{X})$. Furthermore $\tilde{P}$ is continuous on $(C(\bar{X}), \| \, \|_\infty)$, hence the dual operator $\tilde{P}^*$ is continuous on the dual space of $C(\bar{X})$, that is the Banach space of finite Borel measures on $\bar{X}$, with respect to the w^*-topology. Applying the Schauder–Tychonoff-theorem (cf. [2]) to the continuous map

$$\nu \to \tilde{P}^*\nu / \tilde{P}^*\nu(1)$$

on the compact convex set of all positive Borel probability measures on $\bar{X}$, one gets a fixed point m, *i.e.*

$$m(\tilde{P}f) = \lambda m(f)$$

for all $f \in C(\bar{X})$, where $\lambda = \tilde{P}^* m(1) = m(\tilde{P}1)$.

Set $g(x) = e^{\phi(x)}/\lambda$. Then $Pf(x) = \tilde{P}f(x)/\lambda$, and $\int Pf\, dm = \int f\, dm$ for all $f \in \mathscr{F}$. In the following sections we shall show under different circumstances, that

$$\|g_n\|_\infty = \sup_{x \in \bar{X}} g_n(x) < 1$$

for some n.

Set $d = (\|g_n\|_\infty)^{1/n} < 1$. One easily checks that

$$P^n f(x) = \sum_{y \in T^{-n}x} g_n(y) f(y).$$

Hence it follows from lemma 2 of [3] that $m(\bar{A}) \le d^{nk}$ for all $A \in \mathscr{P}_0^{nk}$, where

$$\mathscr{P}_0^j = \bigvee_{i=0}^{j-1} T^{-i}\mathscr{P} \quad \text{and} \quad \mathscr{P} = \{J_1, \ldots, J_N\}.$$

Remark that $\mathscr{P}_0^n$ is the partition of X into intervals, on which T^n is monotone. Setting $C = d^{-n}$, it follows that $m(\bar{A}) \le Cd^j$ for all $A \in \mathscr{P}_0^j$, because A is a subset of some element of $\mathscr{P}_0^{nk}$, where k is such that

$$n(k-1) < j \le nk.$$

In particular, $m(\{x\}) = 0$ for all $x \in \bar{X}$. As W is countable, m is concentrated on $X \backslash W \subset \bar{X}$ and we can return from $\bar{X}$ to X. Hence all requirements of the theorem in § 0 are satisfied, if we can show $\|g_n\|_\infty < 1$ for some n.

These arguments show also that the theorem of § 0 is actually proved in [3], because all proofs of [3] are still valid if one uses the result $m(A) \le Cd^j$ instead of $m(A) \le d^j$ for $A \in \mathscr{P}_0^j$ (cf. lemma 2 of [3]) and $\|g_n\|_\infty < 1$ instead of $\|g\|_\infty < 1$.

We conclude § 1, showing in lemma 2 that the condition $\|g_n\|_\infty < 1$ can be weakened further. To this end we need lemma 1, which will be also useful in the following sections.

LEMMA 1. (i) *Let $A_1, \ldots, A_k$ be subsets of $\bar{X}$ such that*

$$\bigcup_{i=1}^{k} A_i = \bar{X}.$$

Suppose there are integers $n_1, n_2, \ldots, n_k$ such that

$$\sup_{A_i} g_{n_i} < 1$$

for $1 \le i \le k$. Then there is an n with $\|g_n\|_\infty < 1$.

(ii) *Suppose for every $x \in \bar{X}$ there is a neighbourhood U_x of x in $\bar{X}$ and an integer n_x such that*

$$\sup_{U_x} g_{n_x} < 1.$$

Then there is an n with $\|g_n\|_\infty < 1$.

Proof. (i) Set

$$\tilde{n} = \max\{n_1, \ldots, n_k\} \quad \text{and} \quad \gamma = \max\{\sup_{A_i} g_{n_i} : 1 \le i \le k\} < 1.$$

Then choose an integer t such that

$$(\sup_{\bar{X}} g)^{\tilde{n}} \gamma^t < 1$$

and set $n = t\tilde{n}$.

For a fixed $x \in \bar{X}$ let $j_1 \in \{1, \ldots, k\}$ be such that $x \in A_{j_1}$. Set $r_1 = n_{j_1}$. If $j_1, \ldots, j_l$ and $r_1, \ldots, r_l$ are defined, choose j_{l+1} such that

$$T^{r_1 + \cdots + r_l} x \in A_{j_{l+1}}$$

and set $r_{l+1} = n_{j_{l+1}}$. Finally let $s \in \mathbb{N}$ be such that

$$r_1 + \cdots + r_s \le n < r_1 + \cdots + r_{s+1}.$$

As $n_i \le \tilde{n}$ for all i and $n = \tilde{n}t$, we have $s \ge t$ and

$$n - r_1 - \cdots - r_s < r_{s+1} = n_{j_{s+1}} \le \tilde{n}.$$

Now we get

$$\begin{aligned} g_n(x) &= g_{r_1}(x) g_{r_2}(T^{r_1}x) \cdots g_{r_s}(T^{r_1 + \cdots + r_{s-1}}x) g_{n - r_1 - \cdots - r_s}(T^{r_1 + \cdots + r_s}x) \\ &\le \gamma \cdot \gamma \cdot \cdots \cdot \gamma \cdot (\sup g)^{n - r_1 - \cdots - r_s} \\ &\le \gamma^t (\sup g)^{\tilde{n}}, \end{aligned}$$

because $\gamma < 1$, $s \ge t$, $n - r_1 - \cdots - r_s \le \tilde{n}$ and $\sup g \ge 1$ (for $\sup g < 1$ nothing is to show).

This can be done for all $x \in \bar{X}$, hence

$$\|g_n\|_\infty \le \gamma^t (\sup g)^{\tilde{n}} < 1.$$

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(ii) As $\bar{X}$ is compact, there are $x_1, \ldots, x_k \in X$ such that

$$\bigcup_{i=1}^{k} U_{x_i} = \bar{X}.$$

Now apply (i). ☐

LEMMA 2. *Let $g: X \to (0, \infty)$ be of bounded variation, such that $g(x)$ is either $g(x+)$ or $g(x-)$. Suppose that*

$$\liminf_{k\to\infty} g_k(x+) = 0 \quad \textit{and} \quad \liminf_{k\to\infty} g_k(x-) = 0 \quad \textit{for all } x \in X.$$

Then there is an n with $\|g_n\|_\infty < 1$.

Proof. As g on $\bar{X}$ is continuous and an extension of g on $X\backslash W$, it follows from the density of $X\backslash W$ in $\bar{X}$ that

$$\liminf g_k(x) = 0 \quad \text{for all } x \in \bar{X}.$$

Hence for a fixed $x \in \bar{X}$, there is a k with $g_k(x) \le \frac{1}{2}$ and, by continuity of g_k on $\bar{X}$, there is a neighbourhood U of x with $g_k(y) \le \frac{3}{4}$ for all $y \in U$. The existence of an n with $\|g_n\|_\infty < 1$ follows now from (ii) of lemma 1. ☐

Of special interest is the case where $g = 1/|T'|$, because this gives absolutely continuous invariant measures μ.

2. *Equilibrium states for ϕ with sup ϕ − inf $\phi < h_{\text{top}}(T)$*

In this section we consider piecewise monotonic transformations T on $X = [0, 1]$ and functions

$$\phi : [0, 1] \to \mathbb{R}$$

of bounded variation satisfying

$$\sup \phi - \inf \phi < h_{\text{top}}(T).$$

In order to show $\|g_n\|_\infty < 1$ for such ϕ we need some results of [4], [5] and [6] for the transformation T. Recall that

$$\mathcal{P} = \{J_1, \ldots, J_N\}$$

is the partition of $X = [0, 1]$ into intervals on which T is monotone. If C is a subinterval of some J_k $(1 \le k \le N)$, then we call the non-empty sets among $TC \cap J_r$ for $1 \le r \le N$ the successors of C. They again are intervals contained in some $J_k \in \mathcal{P}$. Let $\mathcal{D}$ be the following set of subintervals of X: $\mathcal{D}$ contains $\mathcal{P}$, and if $C \in \mathcal{D}$ then all successors of C belong to $\mathcal{D}$, too. (In [4]–[6] the letter D is used for the set $\mathcal{D}$. Furthermore an isomorphic shift space is considered instead of (X, T), but this makes no essential difference.) Let M be the oriented graph whose vertices are the elements of $\mathcal{D}$ and which has an arrow $C \to D$ $(C, D \in \mathcal{D})$ iff D is a successor of C. We call M the Markov diagram of T. Alternatively M can be considered as a 0–1 matrix with $M_{CD} = 1$ iff there is an arrow $C \to D$. It is shown in [4] that the successors of different elements of $\mathcal{D}$ often coincide, such that the oriented graph M contains closed paths (cf. § 3, where M is explicitely determined for a special class of T). In [6] the Markov diagram is used to investigate the non-wandering

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set of T. If the topological entropy $h_{\text{top}}(T)$ of T, which is equal to the logarithm of the spectral radius $r(M)$ of M, is greater than 0, one always finds an irreducible submatrix $\tilde{M} = M/\tilde{\mathcal{D}}$ ($\tilde{\mathcal{D}} \subset \mathcal{D}$) of M with $r(\tilde{M}) = r(M)$. Set

$$\bar{\mathcal{D}} = \{D \in \mathcal{D} : \text{there is a path in } M \text{ from some } C \in \tilde{\mathcal{D}} \text{ to } D\}.$$

It is shown in [6] that the sets

$$F = \bigcup \{D : D \in \bar{\mathcal{D}}\} \text{ and } G = \bigcup \{D : D \in \bar{\mathcal{D}} \backslash \tilde{\mathcal{D}}\}$$

are finite unions of intervals satsifying $G \subseteq F$, $TF \subseteq F$ and $TG \subseteq G$. The set

$$\Omega := \bigcap_{i=0}^{\infty} \overline{T^{-i}(F \backslash G)}$$

is a T-invariant, closed, topologically transitive subset of (X, T).

As F is a finite union of intervals, T/F is piecewise monotonic again. Hence we can set $X = F$. We consider the Markov diagram of $(F, T/F)$ and call it again M and its index set again $\mathcal{D}$. Let $\tilde{\mathcal{D}}$ denote again the irreducible subset of $\mathcal{D}$, which gives rise to the topologically transitive subset Ω of F (in [6] it is shown that there is a 1–1 correspondence between irreducible submatrices $\tilde{M}$ of M with $r(\tilde{M}) > 1$ and topologically transitive subsets Ω of X with $h_{\text{top}}(\Omega) > 0$). The proof of lemma 1 of [6] gives information about the structure of F. From this it follows that one can choose the partition $\mathcal{P}$ of F into intervals where T is monotone in such a way that $\mathcal{P} \subset \tilde{\mathcal{D}}$. In particular, this implies that every element of $\mathcal{D}$ can be reached on a path in M which begins in $\tilde{\mathcal{D}}$. This will be used in the proof of theorem 1 below.

LEMMA 3. (i) *Let*

$$Z = J_1 \cap T^{-1} J_2 \cap \cdots \cap T^{-(n-1)} J_n \in \mathcal{P}_0^n \quad (J_r \in \mathcal{P}).$$

Then

$$D_i := T^{i-1} J_1 \cap \cdots \cap T J_{i-1} \cap J_i$$

is an element of $\mathcal{D}$ *and* D_{i+1} *is a successor of* D_i. *Furthermore* $T^{n-1} Z = D_n$.

(ii) *If* D_i *is in* $\mathcal{D}$ *for* $1 \le i \le n$, $D_1 \in \mathcal{P}$, *and* D_{i+1} *is a successor of* D_i, *determine* $J_i \in \mathcal{P}$ *such that* $D_i \subseteq J_i$. *Then*

$$D_i = T^{i-1} J_1 \cap \cdots \cap T J_{i-1} \cap J_i \quad \text{for } 1 \le i \le n.$$

This implies that there is a 1–1 *correspondence between elements* $Z \in \mathcal{P}_0^n$ *with* $T^{n-1} Z = D \in \mathcal{D}$ *and paths of length* n *beginning at some element of* $\mathcal{P}$ *and ending at* D.

Proof. (i) As T is monotone on every J_k, we have

$$\begin{aligned} D_{i+1} &= T^i J_1 \cap \cdots \cap J_{i+1} \\ &= T(T^{i-1} J_1 \cap \cdots \cap J_i) \cap J_{i+1} \\ &= T D_i \cap J_{i+1}. \end{aligned}$$

Because of $T^i Z \subseteq D_{i+1}$, we have $D_{i+1} \neq \varnothing$ and hence D_{i+1} is a successor of D_i. As $D_1 = J_1 \in \mathcal{D}$ this gives that all D_i are in $\mathcal{D}$. The equation $T^{n-1} Z = D_n$ is a special case of the formula

$$D_i = T^{i-1}(J_1 \cap \cdots \cap T^{-(i-1)} J_i),$$

which we prove by induction. For $i=1$ we have $D_1=J_1$, which is the definition of D_1. The induction step is as follows.

$$D_{i+1}=TD_i\cap J_{i+1}=T(T^{i-1}(J_1\cap\cdots\cap T^{-(i-1)}J_i))\cap J_{i+1}$$
$$=T^i(J_1\cap\cdots\cap T^{-(i-1)}J_i\cap T^{-i}J_{i+1})$$

using the formula $T^i(A\cap T^{-i}B)=T^iA\cap B$.

(ii) As $D_1\in\mathcal{P}$, we have $D_1=J_1$. Suppose we have shown that

$$D_i=T^{i-1}J_1\cap\cdots\cap J_i.$$

As D_{i+1} is a successor of D_i, there is a $J_{i+1}\in\mathcal{P}$ with

$$D_{i+1}=TD_i\cap J_{i+1}\neq\varnothing.$$

But this gives (cf. (i))

$$D_{i+1}=T^iJ_1\cap\cdots\cap TJ_i\cap J_{i+1}$$

proving (ii) by induction. □

The following lemma summarizes results of [4], [5] and [6].

LEMMA 4. (i) *$h_{\text{top}}(\Omega, T)=\log\alpha$, where $\alpha=r(\tilde M)$.*

(ii) *$\tilde M$ has a left eigenvector $(u_D)_{D\in\tilde{\mathcal{D}}}$ and a right eigenvector $(v_D)_{D\in\tilde{\mathcal{D}}}$ for the eigenvalue α such that $\sum u_Dv_D=1$, $u_D>0$, $v_D>0$. The matrix $(P_{CD})_{C,D\in\tilde{\mathcal{D}}}$ given by $P_{CD}=\tilde M_{CD}v_D/\alpha v_C$ is then a stochastic matrix with stationary probability vector $(\pi_D)_{D\in\tilde{\mathcal{D}}}$ given by $\pi_D=u_Dv_D$, i.e.*

$$\sum_{D\in\tilde{\mathcal{D}}}P_{CD}=1,\qquad \sum_{C\in\tilde{\mathcal{D}}}\pi_CP_{CD}=\pi_D,\qquad \sum_{D\in\tilde{\mathcal{D}}}\pi_D=1.$$

(iii) *As $\tilde M$ is irreducible, P is also irreducible. There is a $q\geq 1$ such that P^q is aperiodic.*

Now we can prove

THEOREM 1. *Set $\alpha=\exp(h_{\text{top}}(\Omega, T))$. Then*

(i) *for every $D\in\mathcal{D}$ we have*

$$\liminf_{n\to\infty}(\alpha^{-n}\operatorname{card}\{Z\in\mathcal{P}_0^n|T^{n-1}Z=D\})>0;$$

(ii)

$$\liminf_{n\to\infty}(\alpha^{-n}\inf_{x\in F}\operatorname{card}T^{-n}\{x\})=c\quad\textit{for some } c>0.$$

Proof. (cf. the proof of theorem 4 in [6]). It follows from lemma 3, that

$$\operatorname{card}\{Z\in\mathcal{P}_0^n|T^{n-1}Z=D\}=\sum_{C\in\mathcal{P}}M_{CD}^{(n-1)},$$

because

$$M_{CD}^{(n-1)}=\sum_{C_2\in\mathcal{D}}\cdots\sum_{C_{n-1}\in\mathcal{D}}M_{CC_2}M_{C_2C_3}\cdots M_{C_{n-1}D}$$

is the number of paths in M, which begin at C and end at D. Suppose first that $D\in\tilde{\mathcal{D}}$. Then $M_{CD}=0$ for all $C\notin\tilde{\mathcal{D}}$, because $\tilde M$ is a maximal irreducible submatrix of M and from $D \in \tilde{\mathscr{D}}$ there is a path to all $C \in \mathscr{D}$ in M. Hence

$$\sum_{C \in \mathscr{P}} M_{CD}^{(n-1)} = \sum_{C \in \mathscr{P} \cap \tilde{\mathscr{D}}} \tilde{M}_{CD}^{(n-1)}.$$

By lemma 4 we have $\tilde{M}_{CD}^{(n-1)} = P_{CD}^{(n-1)} v_C \alpha^{n-1}/v_D$ for $C, D \in \tilde{\mathscr{D}}$. By standard results of probability we have $P_{CD}^{(qk)}$ converges to π_D $(k \to \infty)$ for all $C, D \in \tilde{\mathscr{D}}$, beause $P^{(q)}$ is an aperiodic Markov chain with stationary probability vector π. This implies

$$\alpha^{-qk} \tilde{M}_{CD}^{(qk)} \to \pi_D v_C / v_D = u_D v_C \quad \text{for } k \to \infty.$$

Choose now a fixed j with $0 \le j \le q-1$. Then

$$\sum_{C \in \mathscr{P}} \tilde{M}_{CD}^{(qk+j)} \alpha^{-(qk+j)} = \sum_{C \in \mathscr{P} \cap \tilde{\mathscr{D}}} \alpha^{-j} \sum_{E \in \tilde{\mathscr{D}}} \tilde{M}_{CE}^{(j)} \alpha^{-qk} \tilde{M}_{ED}^{(qk)}$$

$$\to \sum_{C \in \mathscr{P} \cap \tilde{\mathscr{D}}} \alpha^{-j} \sum_{E \in \tilde{\mathscr{D}}} \tilde{M}_{CE}^{(j)} u_D v_E$$

which is a positive constant for every j, because $\tilde{M}_{CE}^{(j)} \neq 0$ holds for finitely many $E \in \tilde{\mathscr{D}}$ (every element of $\mathscr{D}$ has finitely many successors). Hence

$$\liminf(\alpha^{-n} \operatorname{card}\{Z \in \mathscr{P}_0^n | T^{n-1}Z = D\}) =: c_D > 0.$$

If $D \notin \tilde{\mathscr{D}}$, there is a $D' \in \tilde{\mathscr{D}}$ and a j such that $M_{D'D}^{(j)} \ge 1$. Then

$$\liminf \alpha^{-n} \sum_{C \in \mathscr{P}} M_{CD}^{(n)} \ge \liminf \alpha^{-j} \alpha^{-(n-j)} \sum_{C \in \mathscr{P} \cap \tilde{\mathscr{D}}} \tilde{M}_{CD'}^{(n-j)} M_{D'D}^{(j)}$$

$$\ge \alpha^{-j} c_{D'} =: c_D.$$

In order to show (ii) we remark that

$$\operatorname{card} T^{-n}\{x\} \ge \operatorname{card}\{Z \in \mathscr{P}_0^{n+1} | T^n Z = D\}$$

where $D \in \mathscr{P}$ is such that $x \in D$, because T^n is bijective on such a Z. Hence

$$\liminf (\alpha^{-n} \inf_{x \in F} \operatorname{card} T^{-n}\{x\}) \ge \min_{D \in \mathscr{P}} c_D > 0, \quad \text{as } \bigcup_{D \in \mathscr{P}} D = F. \qquad \square$$

Remark. Theorem 4 of [6] shows that $\operatorname{card} \mathscr{P}_0^n \le d\alpha^n$ for some $d < \infty$ and hence

$$\limsup (\alpha^{-n} \sup_{x \in F} \operatorname{card} T^{-n}\{x\}) < \infty,$$

using that $\sum_{D \in \tilde{\mathscr{D}}} u_D < \infty$.

Now we can show

THEOREM 2. *Let T be a piecewise monotonic transformation on* [0, 1] *and let* $\phi : [0, 1] \to \mathbb{R}$ *be of bounded variation such that* $\sup \phi - \inf \phi < h_{\text{top}}(T)$. *Then ϕ has an equilibrium state μ satisfying* (ii) *of the theorem quoted in* § 0.

Proof. We consider a T-invariant $F \subset [0, 1]$ as above with $h_{\text{top}}(F, T) = h_{\text{top}}(T)$ and restrict ϕ and T to F. By the results of § 1, for $g = e^{\phi}/\lambda$, it suffices to find an n with $\|g_n\|_\infty < 1$. Since g_n is continuous on $\bar{F}$ and F is dense in $\bar{F}$, we have $\|g_n\|_\infty = \sup_F g_n$.

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Set $\alpha = \exp h_{\text{top}}(F, T)$ and $\beta = \sup g/\inf g = \exp(\sup \phi - \inf \phi)$. Then $\beta < \alpha$ and

$$\begin{aligned} 1 = m(P^n 1) = m\Big(\sum_{y \in T^{-n}x} g_n(y)\Big) \\ &\geq \inf_{x \in F} \operatorname{card} T^{-n}\{x\} \cdot \inf_F g_n \\ &\geq c\alpha^n \sup_F g_n \cdot (\inf_F g/\sup_F g)^n \quad \text{(by theorem 1)} \\ &= c\left(\frac{\alpha}{\beta}\right)^n \|g_n\|_\infty. \end{aligned}$$

Hence

$$\|g_n\|_\infty \leq \frac{1}{c}\left(\frac{\beta}{\alpha}\right)^n.$$

As $\beta/\alpha < 1$ one finds an n with

$$\frac{1}{c}\left(\frac{\beta}{\alpha}\right)^n < 1.$$

We show now that the bound $h_{\text{top}}(T)$ for $\sup \phi - \inf \phi$ is sharp. Take $T(x) = 2x \pmod 1$ on $X = [0, 1]$. For every

$$b > h_{\text{top}}(T) = \log 2$$

we find a ϕ with $\sup \phi - \inf \phi = b$, such that P does not have the properties stated in the theorem in § 0.

Set

$$\phi = \sum_{k=0}^{\infty} a_k \cdot 1_{(2^{-k-1}, 2^{-k}]},$$

where a_k is a sequence of real numbers converging to 0. It is shown in [8] (the two-shift is used there instead of T) that the operator P does not satisfy $Ph = h$ for a bounded h, if

$$\sum_{n=0}^{\infty} e^{s_n} \leq 1, \quad \text{where } s_n = a_0 + \cdots + a_n.$$

For a fixed K set

$$a_k = -2 \log \frac{k+2}{k+1} \quad \text{if } k \geq K$$

and

$$a_k = -b \quad \text{if } 0 \leq k \leq K-1.$$

Then ϕ is of bounded variation and

$$\sup \phi - \inf \phi = b \quad \text{for } K \geq 3.$$

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Furthermore

$$\sum_{k=0}^{\infty} e^{s_k} = \sum_{k=0}^{K-1} e^{-(k+1)b} + e^{-Kb} \sum_{k=K}^{\infty} \frac{(K+1)^2}{(k+2)^2}$$

$$\leq \frac{e^{-b}}{1-e^{-b}} + e^{-Kb}(K+1)^2 \sum_{k=1}^{\infty} \frac{1}{(k+2)^2}$$

$$\to \frac{e^{-b}}{1-e^{-b}} < 1, \quad \text{for} \quad K \to \infty, \quad \text{because } b > \log 2.$$

Hence for every $b > \log 2$ there is a K such that

$$\sum_{k=0}^{\infty} e^{s_k} < 1. \qquad \square$$

3. *Local specification of T and a variation condition of ϕ*

In this section we consider piecewise monotonic transformations satisfying the property stated in (ii) of theorem 3 below. It can be called a local specification property, because one gets specification as it is defined in § 21 of [**1**], if the n_i and k do not depend on $(Z_i)_{i\geq 1}$ and the sequence $(n_{j+1}-n_j)_{j\geq 1}$ is bounded. In order to show $\|g_n\|_\infty < 1$ we only need a local property of ϕ and not a global one as in § 2. Define

$$\operatorname{var}_i \phi := \sup \{|\phi(x) - \phi(y)| : x, y \in Z, Z \in \mathcal{P}_0^i\}.$$

Then we have

THEOREM 3. *Let $\phi: X \to \mathbb{R}$ be of bounded variation. Suppose*

(i) $\sum_{i=1}^{\infty} \operatorname{var}_i \phi =: c < \infty$,

(ii) *for every sequence $Z_1 \supseteq Z_2 \supseteq Z_3 \supseteq \cdots$ with $Z_i \in \mathcal{P}_0^i$ there are integers $n_1 < n_2 < n_3 < \cdots$ and k, which may depend on $(Z_i)_{i\geq 1}$, such that*

$$\bigcup_{j=1}^{k} T^{n_i+j} Z_{n_i} = X \quad \textit{for all } i \geq 1.$$

Then there is an n with $\|g_n\|_\infty < 1$.

Proof. (cf. Walters [**13**]). For $A \subseteq X$ denote by $\bar{A}$ the closure of A in $\bar{X}$. For each sequence $(Z_i)_{i\geq 1}$ as in (ii) we may consider $\bigcap_{i=1}^{\infty} \bar{Z}_i$ as one single point. (If $I = \bigcap_{i=1}^{\infty} \bar{Z}_i$ is a non-trivial interval, it follows from (i) that ϕ is constant on I. Furthermore $T^k I$ for $k \geq 1$ is also a subset of some $\bar{A}$ with $A \in \mathcal{P}_0^n$ for all n, which implies that ϕ is constant on $T^k I$, too. This gives that g_k is constant on I for all k.)

We show that m has no atoms: For $x \in \bar{X}$ we have

$$m(\{x\}) = m(P^k 1_{\{x\}}) = m(g_k(x) \cdot 1_{\{T^k x\}}) = g_k(x) m(\{T^k x\}). \tag{1}$$

Suppose first that x is not periodic. Choose $Z_i \in \mathcal{P}_0^i$ such that $x \in \bar{Z}_i$. Then $Z_{i+1} \subseteq Z_i$. As X is dense in $\bar{X}$ and $\bar{Z}_i$ is compact, it follows from (ii) that

$$\bigcup_{j=1}^{k} T^{n_i+j} \bar{Z}_{n_i} = \bar{X}.$$

Hence for $i \geq 1$ we can find a $y_i \in \bar{Z}_{n_i}$ with $T^{n_i+j_i}(y_i) = x$, where $j_i \leq k$. Now it follows from (1) that

$$m(\{y_i\}) \geq m(\{y_i\}) m(\{T^{n_i+j_i}x\}) = m(\{x\})^2 g_{n_i+j_i}(y_i)/g_{n_i+j_i}(x)$$

$$\geq \exp\Big(-\sum_{r=1}^{n_i} \operatorname{var}_r \phi - 2\|\phi\|_\infty j_i\Big) m(\{x\})^2$$

$$\geq \exp(-c - 2k\|\phi\|_\infty) m(\{x\})^2,$$

because x and y_i are both in $\bar{Z}_{n_i}$ and $\operatorname{var}_r \phi$ equals

$$\sup\{|\phi(x) - \phi(y)|: x, y \in \bar{Z}, Z \in \mathcal{P}_0^r\}$$

(ϕ is continuous on $\bar{X}$ and X is dense in $\bar{X}$). As x is not periodic and $j_i \leq k$, only finitely many y_i can be equal to a fixed y_n. This implies $m(\{y_i\}) \to 0$ for $i \to \infty$ and the above inequalities give $m(\{x\}) = 0$.

If $x \in \bar{X}$ is periodic with period p, we find a $Z \in \mathcal{P}_0^r$ for some r such that $T^j(x) \notin \bar{Z}$ for $0 \leq j \leq p-1$. Choose some sequence $(Z_i)_{i \geq 1}$ as in (ii) with $Z_r = Z$. Then $T^j(x) \notin \bar{Z}_i$ for $0 \leq j \leq p-1$ and all $i \geq r$. By (ii) we find an $n > r$ with

$$\bigcup_{s=1}^{k} T^{n+s} \bar{Z}_n = \bar{X} \ni x.$$

Hence there is a $y \in \bar{Z}_n$, i.e. $y \neq T^j x$ for $0 \leq j \leq p-1$, and $T^{n+s} y = x$ for some s. As y is not periodic, we know that $m(\{y\}) = 0$. Now it follows from (1) that also $m(\{x\}) = 0$.

Now set $q_l = \max\{m(\bar{Z}) = m(Z) : Z \in \mathcal{P}_0^l\}$. We have $q_l \to 0$ for $l \to \infty$, because m has no atoms and X is compact. Fix some $x \in \bar{X}$ again. We show that there is a neighbourhood U_x for x and an integer n_x with $\sup_{U_x} g_{n_x} < 1$. First we show that for a measurable subset A of $\bar{X}$

$$\bigcup_{j=1}^{k} T^j A = \bar{X} \Rightarrow m(A) \geq d(k) := \frac{1}{k} (\inf g)^k > 0. \tag{2}$$

As $\bigcup_{j=1}^{k} T^j A = \bar{X}$, there is a j with $m(T^j A) \geq 1/k$. Hence

$$m(A) = m(P^j 1_A) = m\Big(\sum_{y \in T^{-j}x} g_j(y) \cdot 1_A(y)\Big)$$

$$\geq (\inf g_j) m\Big(\sum_{y \in T^{-j}x} 1_A(y)\Big)$$

$$\geq (\inf g)^j m(T^j A)$$

$$\geq (\inf g)^j \frac{1}{k} \geq d(k).$$

As above, choose $(Z_l)_{l \geq 1}$, $Z_l \in \mathcal{P}_0^l$, such that $x \in \bar{Z}_l$. Let k be the integer from (ii) for these Z_l,

$$c = \sum_l \operatorname{var}_l \phi,$$

and let n be one of the n_i's of (ii) such that

$$q_n e^c d(k)^{-1} < 1.$$

Set $U = \bar{Z}_n$, which is a neighbourhood of x in $\bar{X}$. As

$$\bigcup_{j=1}^{k} T^{n+j} U = \bar{X},$$

(2) implies that $m(T^n U) \geq d(k)$, and one can estimate

$$\begin{aligned} q_n \geq m(U) = m(P^n 1_U) &= m\Big(\sum_{y \in T^{-n}x} g_n(y) 1_U(y)\Big) \\ &\geq (\inf_U g_n) m(T^n U) \geq (\sup_U g_n) \exp\Big(-\sum_{i=1}^{n} \mathrm{var}_i \,\phi\Big) d(k) \\ &\geq (\sup_U g_n) e^{-c} d(k). \end{aligned}$$

Hence $\sup_U g_n \leq q_n e^c d(k)^{-1} < 1$. This means that for every $x \in \bar{X}$ we can find a neighbourhood U and an integer n with $\sup_U g_n < 1$. Now apply (ii) of lemma 1 to get the desired result. □

In order to show $\|g_n\|_\infty < 1$ for the class of ϕ's specified by (i), one has to prove that the piecewise monotonic transformation T satisfies (ii). We consider this problem for the following class of transformations T on $[0, 1]$: Let $f: [0, 1] \to \mathbb{R}$ be continuous and monotonically increasing. Define T by $T(x) = f(x)$ (mod 1). We call such a transformation monotonic mod one. For the investigation of (ii) of theorem 3 we use the Markov diagram M introduced in §2. We want to determine M for $T(x) = f(x)$ (mod 1).

Recall $\mathcal{P} = (J_1, \ldots, J_N)$ the partition of $[0, 1]$ into intervals of monotonicity of T, and suppose $N \geq 2$. $\mathcal{P}$ is a subset of $\mathcal{D}$. For $2 \leq i \leq N-1$ we have $TJ_i = [0, 1]$, hence the successors of such a J_i are all J_j for $1 \leq j \leq N$. Next we define intervals A_i, B_i for $i \geq 1$. Set $A_1 = J_1$, $B_1 = J_N$. Define $r_1 \geq 1$ as that integer such that $T^k A_1$ is a subset of an element of $\mathcal{P}$ for $0 \leq k \leq r_1 - 1$ and $T^{r_1} A_1$ is not. Set

$$A_i = T^{i-1} A_1 = T A_{i-1} \quad \text{for } 2 \leq i \leq r_1$$

and

$$A_{r_1+1} = T A_{r_1} \cap J_p,$$

where p is the smallest integer j with $T A_{r_1} \cap J_j \neq \varnothing$. If $r_1, \cdots, r_m$ and A_i for $1 \leq i \leq r_1 + \cdots + r_m + 1$ are defined, set $R_m = r_1 + \cdots + r_m$, define $r_{m+1} \geq 1$ such that $T^k A_{R_m+1}$ is a subset of some element of $\mathcal{P}$ for $0 \leq k \leq r_{m+1} - 1$ and

$$T^{r_{m+1}} A_{R_m+1} \cap J_j \neq \varnothing$$

for $p \leq j \leq q$ and $p < q$. Define

$$A_{R_m+i} = T^{i-1} A_{R_m+1} \quad \text{for } 2 \leq i \leq r_{m+1}$$

and

$$A_{R_{m+1}+1} = T A_{R_{m+1}} \cap J_p, \quad \text{where } R_{m+1} = R_m + r_{m+1}.$$

Similarly define $s_m \geq 1$ and $S_m = s_1 + \cdots + s_m$ ($S_0 = 0$) inductively such that

$$B_{S_m+i} := T^{i-1} B_{S_m+1} \subseteq J_k \in \mathcal{P} \quad \text{for } 2 \leq i \leq s_{m+1}$$

and

$$B_{S_{m+1}+1} = TB_{S_{m+1}} \cap J_q,$$

where q is the largest, but not the only integer k with

$$T^{s_{m+1}} B_{S_m+1} \cap J_k \neq \varnothing.$$

The successors of A_i for $i \neq R_m$ and B_i for $i \neq S_m$ are then A_{i+1} and B_{i+1} respectively. A_{R_m} has the successors A_{R_m+1}, all J_k with $J_k \subseteq TA_{R_m}$, and B_{r_m} because one can show that

$$TA_{R_m} \cap J_q = B_{r_m},$$

where q is the largest integer k such that

$$TA_{R_m} \cap J_k \neq \varnothing.$$

Similarly B_{S_m} has the successors B_{S_m+1}, all J_k with $J_k \subseteq TB_{S_m}$, and A_{s_m}. Hence

$$\mathscr{D} = \{J_2, \ldots, J_{N-1}\} \cup \{A_i, B_i : i \geq 1\}$$

(cf. chapter II of [7]).

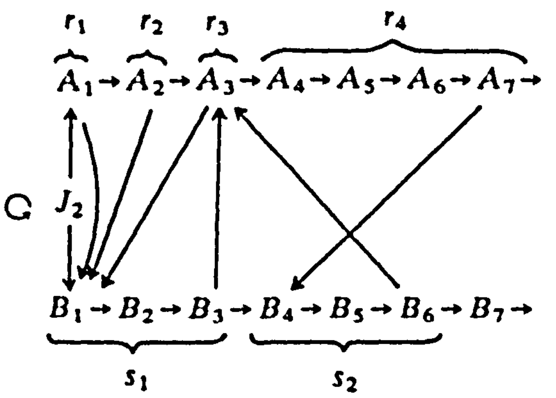

FIGURE 1

In order to show (ii) of theorem 3 one can proceed as follows: We call $\mathscr{B} = \{A_i, B_j\} \subseteq \mathscr{D}$ a *K-barrier*, if from both elements of $\mathscr{B}$ there is a path in M of length less than K to each element of $\mathscr{P} \subseteq \mathscr{D}$ and if

$$\bar{\mathscr{B}} = \{A_k, B_m \mid k \leq i, m \leq j\}$$

can be left only via an arrow

$$A_i \to A_{i+1} \quad \text{or} \quad B_j \to B_{j+1}.$$

Then we have

LEMMA 5. *Suppose that for the Markov diagram M of a monotonic mod one transformation T there is a sequence $\mathscr{B}_i \subseteq \mathscr{D}$ of K-barriers for some constant K. Then T satisfies* (ii) *of theorem* 3.

Proof. For $i \geq 1$ let $Z_i \in \mathscr{P}_0^i$ be such that $Z_{i+1} \subseteq Z_i$. By lemma 3 there is a path $D_1 \to D_2 \to \cdots \to D_i$ in M with $D_1 \in \mathscr{P}$ such that

$$Z_i = D_1 \cap T^{-1} D_2 \cap \cdots \cap T^{-(i-1)} D_i \tag{3}$$

Furthermore, as $Z_{i+1} \subseteq Z_i$, the path for Z_{i+1} is the same as that for Z_i, only with a D_{i+1} added. Hence we have an infinite path $D_1 \to D_2 \to D_3 \to \dots$ in M with $D_1 \in \mathcal{P}$ such that (3) holds for all i. We consider two cases:

If the set $\{D_i | i \geq 1\} \subseteq \mathcal{D}$ is finite, then there is a constant k such that for every n and every $E \in \mathcal{P}$ one can find a path

$$D_n \to C_1 \to \cdots \to C_{j+1} = E \quad \text{in } M \text{ with } j \leq k.$$

This gives

$$T^{n+j} Z_n \supseteq T^{n+j}(Z_n \cap T^{-n} C_1 \cap \cdots \cap T^{-n-j} C_{j+1}) = C_{j+1} = E \quad \text{(cf. lemma 3).}$$

Hence

$$[0, 1] = \bigcup_{E \in \mathcal{P}} E \subseteq \bigcup_{j=1}^{k} T^{n+j} Z_n.$$

If $\{D_i | i \geq 1\}$ is an infinite subset of $\mathcal{D}$ then, by definition of a barrier, there is a sequence n_i with $D_{n_i} \in \mathcal{B}_i$. Otherwise $\{D_j | j \geq 1\} \subseteq \bar{\mathcal{B}}_i$ for some i and hence would be finite. As there are paths from $D_{n_i} \in \mathcal{B}_i$ to every element of $\mathcal{P} \subseteq \mathcal{D}$ of length less than K, one gets as above that

$$\bigcup_{j=1}^{K} T^{n_i+j} Z_{n_i} = [0, 1].$$

This proves the lemma. □

Now theorem 3 and lemma 5 imply:

THEOREM 4. *Let T be a monotonic* mod 1 *transformation and suppose the Markov diagram of T possesses a sequence of K-barriers for some K. Let $\phi : [0, 1] \to \mathbb{R}$ be of bounded variation with*

$$\sum_{i=1}^{\infty} \operatorname{var}_i \phi < \infty.$$

Then ϕ has an equilibrium state μ satisfying (ii) *of the theorem quoted in* § 0.

We consider first the β-transformation $x \to \beta x \pmod 1$, $\beta > 1$. For this transformation the Markov diagram found above for monotonic mod 1 transformations satisfies $A_i = A_1$ for all i and $r_i = 1$ for all i. One can identify all A_i's and the successors of A_1 are the elements of $\mathcal{P}$. In this case set $\mathcal{B}_i = \{A_1, B_{R_i}\}$, which is a barrier because $A_{r_i} = A_1$ is a successor of B_{R_i}. Hence theorem 4 can be applied to the β-transformation $x \to \beta x \pmod 1$ (cf. Walters [**13**] and III, § 4 of [**3**]).

Now we consider $T(x) = \beta x + \alpha \pmod 1$. For $\alpha = 0$ we have $r_i = 1$ for all i. If β and α are such that the sequence (r_i) or the sequence (s_i) is bounded, a slight generalization of the above argument shows that the barrier property of lemma 5 is satisfied. The sequence (r_i) is bounded, if and only if

$$1 \notin \overline{\{T^k(0) | k \geq 0\}}.$$

Let $\varepsilon > 0$ be such that $T^k(0) < 1 - \varepsilon$ for all $k \geq 0$. One easily checks that

$$TA_{R_m+1} = [T^{R_m+1}(0), 1] \quad \text{for all } m,$$

which then has the length larger then ε. If k is such that $\beta^k\varepsilon > 1$ then at least $T^k A_{R_m+1}$ cannot be contained in some element of $\mathcal{P}$, hence $r_{m+1} \le k$. In particular,

$$1 \notin \overline{\{T^k(0) | k \ge 0\}}$$

holds, if the orbit of 0 is periodic. Hence theorem 4 can be applied to $T(x) = \beta x + \alpha \pmod 1$ if either the orbit of 0 or that of 1 is periodic.

For every fixed $\alpha \in (0, 1)$, the set of β such that $x \mapsto \beta x + \alpha \pmod 1$ does not satisfy the barrier property of lemma 5 seems to be dense in $(1, \infty)$. It would be interesting, to find sufficient conditions for

$$(\alpha, \beta) \in (0, 1) \times (1, \infty),$$

such that this property is satisfied for

$$x \mapsto \beta x + \alpha \pmod 1.$$

Another class of piecewise monotonic transformations (X, T) which satisfy (ii) of theorem 3, is that where the initial points of all J_i are mapped to the initial point of X by T and T/J_1 is expanding. One can prove this with similar methods as lemma 5 using the Markov diagram of T.

4. *Absolutely continuous invariant measures*

Misiurewicz (**[9]**, **[10]**) and Szlenk (**[11]**, **[12]**) have given sufficient conditions for a piecewise monotonic C^2-mapping T of $[0, 1]$ to admit an absolutely continuous, invariant measure μ. Their main assumptions are that the critical points of T (i.e. those for which $T'(x) = 0$) are not contained in the closures of their orbits and that all periodic points are repellers. For transformations with negative Schwartzian derivative Misiurewicz **[10]** also has given a description of the asymptotic σ-algebra of (T, μ).

Here we must restrict ourselves to the case where the critical orbits eventually become periodic repellers. Using Szlenk's conditions (cf. **[12]**), it will be relatively easy to show that the results of **[3]** summarized in § 0 apply.

Let $T: [0, 1] \to [0, 1]$ be of class C^2. Denote by C_0 the set of critical points of T, i.e. $C_0 = \{x | T'(x) = 0\}$, and assume that

(T1) card $(\bigcup_{n>0} T^n C_0) < \infty$,
(T2) $T''(c) \ne 0$ and $|T''(c) - T''(x)| = O(|x - c|)$ for $c \in C_0$ and $x \in [0, 1]$,
(T3) $C_0 \cap \bigcup_{n\ge 1} T^n C_0 = \varnothing$, and
(T4) There exist $d_1, d_2 > 1$, $n_0 \in \mathbb{N}$ such that if

$$T_n(x) \in D := \{y : |T'(y)| < d_1\}$$

for some $n > n_0$, then $|T^n(x)| \ge d_2$.

The last condition is exactly the same as A.6 in **[11]** and **[12]**. From (T1) it follows that the orbits of critical points eventually become periodic, while (T4) forces all periodic points to be repellers. Hence, by passing to an iterate T^m if necessary, one may assume further:

(T5) For each $c \in C_0$ there is an $H(c) \in \mathbb{N}$ such that $c, Tc, \dots T^{H(c)}c$ are distinct, $z = T^{H(c)}c$ is a fixed point for T with $T'(z)$ positive, and

$$|T'(T^ic)| \geq d_3 > 1$$

for some constant d_3 and all $i \geq 1$.

Obviously (T2) and (T3) remain valid by passing from T to T^m, and it follows from lemma 6 in [**11**] that, with some new constants, T^m also satisfies (T4). Furthermore, in [**12**] Szlenk has shown that under (T1), (T2) and (T5), the following condition (T6) is equivalent to (T4):

(T6) There are constants $\gamma > 0$ and $\alpha > 1$ such that $T^p(x) = x$ implies $|T^{p'}(x)| \geq \gamma\alpha^p$ for all $p \in \mathbb{N}$, $x \in [0, 1]$.

The idea of the following construction is to start with a 'good guess' of the invariant density, i.e. with a density f which already has the expected singularities... The measure $f\lambda$ will play the role of the reference measure m, i.e. $P^*m = m$, where the Perron–Frobenius-operator P is defined by means of the function

$$g(x) = \frac{f(x)}{f(Tx)|T'(x)|}.$$

The problem is to choose f in such a way that $\|g_n\|_\infty < 1$ is satisfied for some n and g is of bounded variation.

Let $C_+ := \{T^ic \mid c \in C_0,\ 1 \leq i \leq H(c) - 1\}$ and $C_\infty := \{T^{H(c)}c \mid c \in C_0\}$. By (T1) and (T3) the sets C_0, C_+, and C_∞ are finite and pairwise disjoint. Set

$$C := C_0 \cup C_+ \cup C_\infty.$$

For $\varepsilon > 0$ and $z \in C$ define the following sets $U_\varepsilon(z)$ (one- or two-sided neighbourhoods of z):

(i) If $z \in C_0$ set

$$U_\varepsilon(z) := \{x : |x - z| < \varepsilon\}.$$

(ii) If $z \in C_+$ set

$$U_\varepsilon(z) := \bigcup_{i \geq 1} \bigcup_{\substack{c \in C_0 \\ T^ic = z}} T^iU_\varepsilon(c).$$

(iii) If $z \in C_\infty$ set

$$U_\varepsilon(z) := \bigcup_{\substack{c \in C_0 \\ T^{H(c)}c = z}} T^{H(c)}U_\varepsilon(c).$$

Furthermore, for $i = 1, \dots, n_0$ (n_0 from (T4)) we define

$$M_{\varepsilon,i} := \bigcup_{c \in C_0} T^{-i}U_\varepsilon(c) \quad \text{and} \quad M_\varepsilon := \bigcup_{i=1}^{n_0} M_{\varepsilon,i}.$$

Set $d := d_3^{1/4}$ (d_3 from (T5)).

Choosing $\varepsilon_0 > 0$ small enough we may assume:

(E1) The sets $U_{\varepsilon_0}(z)$ $(z \in C)$ are pairwise disjoint intervals and disjoint from M_{ε_0}.

(E2) The sets $M_{\varepsilon,i}$ $(i = 1, \cdots, n_0)$ are pairwise disjoint finite unions of intervals if $\varepsilon \le \varepsilon_0$.

(E3) $0 < T'x/T'y \le d$ and $T''x \cdot T''y \ge 0$ for $x, y \in U_{\varepsilon_0}(z)$, $z \in C_+ \cup C_\infty$.

(E4) $0 < T''x/T''y \le \frac{3}{2}$ for $x, y \in U_{\varepsilon_0}(c)$, $c \in C_0$.

(E5) $TU_{\varepsilon_0}(z) \cap (M_{\varepsilon_0} \cup \bigcup_{y \in C_0 \cup C_+} U_{\varepsilon_0}(y)) = \emptyset$ for all $z \in C_\infty$.

Set $K := \min_{c \in C_0} \frac{1}{2}(2|T''(c)|)^{1/2}$, and define the density f by

$$f(x) := \begin{cases} |x-z|^{-1/2} & \text{if } x \in U_\varepsilon(z), z \in C_+ \\ \alpha_z |x-z|^{-1/2} & \text{if } x \in U_{\varepsilon_0}(z), z \in C_\infty \\ K_i & \text{if } x \in M_{\varepsilon,i}, i = 1, \ldots, n_0 \\ K & \text{otherwise,} \end{cases}$$

where the constants α_z, $K_{i,\varepsilon}$ and a further constant $n_1 \in \mathbb{N}$ are chosen such that

(a) f becomes continuous at the endpoints of the $U_{\varepsilon_0}(z)$ different from z $(z \in C_\infty)$, assuming the value K there (by choice of the α_z),

(b) $g(x) = \dfrac{f(x)}{f(Tx)|T'(x)|} \le \frac{1}{2}$ for $x \in M_{\varepsilon_0}$ (by choice of the K_i),

(c) $\alpha_z d^{n_1} \ge 2$ for all $z \in C_\infty$ (by choice of n_1),

(d) (i) $T^{H(c)+i}(x) \in U_{\varepsilon_0}(T^{H(c)}c)$ for $c \in C_0$, $x \in U_\varepsilon(c)$, $i \le n_1$, (ii) $f(x) \ge K$ for $x \in \bigcup_{c \in C} U_\varepsilon(c)$, (iii) $|T'(x)| < d_1$ for $x \in U_\varepsilon(c)$, $c \in C_0$. (This can be achieved choosing $\varepsilon \le \varepsilon_0$ small enough.)

(e) Finally set $\gamma := \inf |T'(x)| > 0$, where the infinum extends over all

$$x \notin \bigcup_{c \in C_0} U_\varepsilon(c).$$

According to the definition of f we now decompose [0, 1] into a finite number of intervals on each of which g will be shown to be of bounded variation. Furthermore, one will see that there are constants $\delta < 1$ and $N \in \mathbb{N}$ such that for each $x \in [0, 1]$ there is an $n = n(x) \le N$ with

$$g_n(x) = \frac{f(x)}{f(T^n x)|(T^{n})'(x)|} < \delta.$$

Hence g is of bounded variation, and, using lemma 1 with

$$A_i = \{x \mid n(x) = i\},$$

there is an $n \in \mathbb{N}$ with $\sup_{x \in [0,1]} g_n(x) < 1$.

(1) $x \in U_\varepsilon(c)$, $c \in C_0$:

(i) $Tz \in C_+$:

$$g(x) = \frac{K}{|T'(x)|}|Tx - Tc|^{1/2} = K\frac{(\frac{1}{2}|T''(y_1)|)^{1/2}}{|T''(y_2)|}$$

for some y_1, y_2 between x and c. Hence $g|_{U_\varepsilon(c)}$ is continuous and $g(x) \le \frac{3}{2}K(2|T''(c)|)^{-1/2} \le \frac{3}{4}$ ((E4) and definition of K).

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(ii) $Tz \in C_\infty$: With n_1 from (c) we have

$$g_{n_1+1}(x) = g(x)g_{n_1}(Tx) = g(x)\left|\frac{T^{n_1+1}x - Tz}{Tx - Tz}\right|^{1/2}\frac{1}{|(T^{n_1})'(Tx)|}$$

$$= g(x)\frac{|(T^{n_1})'(y)|^{1/2}}{|(T^{n_1})'(Tx)|} \quad \text{for some } y \in U_\varepsilon(Tz)$$

$$\leq g(x)d^{-n_1} \quad \text{by (E3), (T5) and the choice of } d,$$

$$\leq \tfrac{3}{8} \quad \text{by } (c), \text{ since } g(x) \leq \tfrac{3}{4}\min_{z \in C_\infty} \alpha_z^{-1} \text{ in analogy to (i).}$$

Assuming w.l.o.g. that $T''(c) > 0$ and $x \geq c$, it is easy to show (using (E4), (T1), (T2)) that

$$g'(x) = \begin{Bmatrix} \alpha_z^{-1} & \text{if } Tz \in C_\infty \\ 1 & \text{if } Tz \in C_+ \end{Bmatrix} \frac{K}{2}(Tx - Tc)^{-1/2}\left(1 - 2\frac{T''(x)}{T'(x)^2}(Tx - Tc)\right)$$

is bounded on $U_\varepsilon(c)$ such that g has finite variation on $U_\varepsilon(c)$.

(2) $x \in U_\varepsilon(z)$, $z \in C_+$:

(i) $Tz \in C_+$:

$$g(x) = \left|\frac{Tx - Tz}{x - z}\right|^{1/2}\frac{1}{|T'(x)|} = \frac{|T'(y)|^{1/2}}{|T'(x)|}$$

for some y between z and x. Hence $g|_{U_\varepsilon(z)}$ is continuous and $g(x) \leq d^{-1}$ by (E3) and (T5).

(ii) $Tz \in C_\infty$: As in (ii) of (1) one shows that

$$g_{n_1}(x) = \alpha_{Tz}^{-1}\left|\frac{(T^{n_1})'(y)}{(T^{n_1})'(x)}\right| \quad \text{for some } y \in U_\varepsilon(z)$$

$$\leq \alpha_{Tz}^{-1}d^{-n_1} \leq \frac{1}{2}.$$

In both cases

$$\left|\left(\left|\frac{Tx - Tz}{x - z}\right|^{1/2}\right)'\right| = \tfrac{1}{4}\,|T''(y_1)|\,|T'(y_2)|^{-1/2} \leq \tfrac{1}{4}\|T''\|_\infty \gamma^{-1/2} < \infty$$

for some $y_1, y_2 \in U_\varepsilon(z)$, such that g is of bounded variation on $U_\varepsilon(z)$ as

$$\left|\frac{Tx - Tz}{x - z}\right|^{1/2} \quad \text{and} \quad \frac{1}{|T'(x)|}$$

are.

(3) $x \in U_{\varepsilon_0}(z)$, $z \in C_\infty$:

(i) $Tx \in U_{\varepsilon_0}(z)$: analogously to (i) of (2).

(ii) $Tx \notin U_{\varepsilon_0}(z)$:

$$g(x) = \frac{\alpha_z|z - x|^{-1/2}}{K|T'(x)|} \leq d^{-1} \quad \text{(as in (i) of (2))},$$

as $K \geq \alpha_z|Tx - z|^{-1/2}$ and $Tz = z$. It can be shown as in (2) that g is of bounded variation, observing that if $Tx \notin U_{\varepsilon_0}(z)$ then $x \notin U_\varepsilon(z)$ by (i) of (d).

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(4) $x \in M_{\varepsilon,i}$, $i = 1, \ldots, n_0$:

$$g(x) = \frac{K_i}{K_{i-1}} \le \tfrac{1}{2} \text{ by } (b), \quad \text{where } K_0 := K.$$

(5) $x \in \{f(x) = K\}$:
$g|\{f(x) = K\}$ is of bounded variation as the set $\{f(x) = K\}$ is bounded away from the set C_0, and $f(Tx)$ is glued together from pieces of K, K_i, $|Tx - z|^{1/2}$ ($z \in C_0 \cup C_+$), and $\alpha_z^{-1}|Tx - z|^{1/2}$ ($z \in C_\infty$). Furthermore

$$x \in \{f(x) = K\} \Rightarrow x \notin M_\varepsilon \Rightarrow |(T^{n_0})'(x)| \ge \gamma^{n_0}.$$

Fix $n_2 \in \mathbb{N}$ such that

$$d_1^{n_2 - n_0} \gamma^{n_0} \ge 2 \frac{K}{\min\{K, K_i\}}.$$

From (T4) one has the following alternative: either $\exists n$, $n_0 \le n < n_2$: $|(T^n)'(x)| \ge d_2$ or $|(T^{n_2})'(x)| \ge d_1^{n_2 - n_0} |(T^{n_0})'(x)| \ge d_1^{n_2 - n_0} \gamma^{n_0}$.

We consider three cases:

(i) $|(T^{n_2})'(x)| \ge d_1^{n_2 - n_0} \gamma^{n_0} \ge 2 \dfrac{K}{\min\{K, K_i\}}$. Hence $g_{n_2}(x) = \dfrac{K}{f(T^{n_2}x)|(T^{n_2})'(x)|} \le \frac{1}{2}$ since $f \ge \min\{K, K_i\}$ by (ii) of (d).

(ii) $|(T^n)'(x)| \ge d_2$ and $T^n x \notin M_\varepsilon$ for some $n_0 \le n < n_2$. Hence $g_n(x) \le d_2^{-1}$ by (ii) of (d) since $f(T^n x) \ge K$.

(iii) $|(T^n)'(x)| \ge d_2$ and $T^n x \in M_\varepsilon$ for some $n_0 \le n < n_2$. Hence $T^m x \in U_\varepsilon(c)$ for some $m < n_2 + n_0$ and $c \in C_0$ therefore $|(T^m)'(x)| \ge d_2$ by (ii) of (d) and (T4) and we conclude that $g_m(x) \le d_2^{-1}$.

We have shown that the hypotheses of the theorem quoted in § 0 are satisfied. Hence we have

THEOREM 5. *Let* $T: [0, 1] \to [0, 1]$ *be piecewise monotonic and of class* C^2, *such that* (T1), (T2), (T3) and (T4) *are satisfied. Then there is an absolutely continuous* T*-invariant measure* μ *on* $[0, 1]$ *satisfying* (ii) *of the theorem quoted in* § 0.

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