REGULAR SKEW POLYHEDRA IN HYPERBOLIC THREE-SPACE

CYRIL W. L. GARNER

To Professor H. S. M. Coxeter on his sixtieth birthday

1. Introduction. The study of regular skew polyhedra was initiated in 1926 by Petrie's discovery of two infinite polyhedra in Euclidean three-space E^3 which were free of false vertices; the only other regular skew polyhedron in E^3 was found by Coxeter (**1**, pp. 33–34). The simplest of these is denoted $\{4, 6 \mid 4\}$ and is derived from the space-filling of cubes by omitting half the faces. Each square face has four adjoining faces inclined at 90° to the given face (Fig. 1). At any vertex there are six faces; the vertex figure is a regular skew hexagon, the Petrie polygon of a regular octahedron (**6**, p. 24). Figure 2 shows the arrangement of faces at a vertex; the dotted lines indicate the vertex figure.

A polygon is said to be *regular* if it possesses a symmetry operation cyclically permuting its vertices, and therefore also its sides. For a regular plane p-gon, this symmetry operation is a rotatory reflection, involving rotation through the

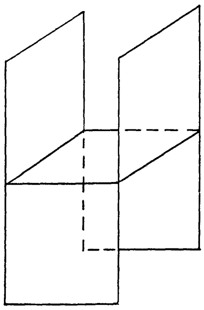

FIGURE 1

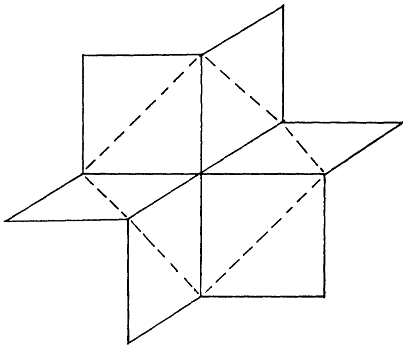

FIGURE 2

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angle π/p and reflection in a plane perpendicular to the axis of rotation (**5**, p. 99). Since the square of this rotatory reflection is a rotation through the angle $2\pi/p$, this regular finite skew polygon is a $2p$-gon whose vertices lie alternately on two circles which reflect into each other in a plane perpendicularly bisecting the line segment joining their centres. The sides of such a polygon are thus the lateral edges of an antiprism. By an *antiprism* we understand a polyhedron whose faces are two regular $\{p\}$'s and $2p$ isosceles triangles (**5**, p. 149).

A *polyhedron* is defined as a connected set of ordinary plane polygons, such that every side of each polygon belongs also to just one other polygon, and the polygons surrounding each vertex form a single circuit. Defining the *vertex figure* of a polyhedron as the polygon formed by the vertices of the polyhedron which are joined to a specific vertex by edges, we say a polyhedron is *regular* if its faces and vertex figures are regular polygons (**6**, p. 16). If these regular vertex figures are plane polygons, we have an ordinary *regular polyhedron*; if they are skew polygons, we have a *regular skew polyhedron* (or r.s.p.).

A regular polyhedron may also be defined as a polyhedron possessing two particular symmetry operations: one, say R, which cyclically permutes the vertices of any face, and another, say S, which cyclically permutes the faces that meet at any vertex of this face. These two symmetry operations generate a group which is transitive on the vertices, on the edges, and on the faces. For an ordinary regular polyhedron, S is a rotation; for an r.s.p., S is a rotatory reflection.

Consider an r.s.p. in hyperbolic three-space H^3. Although it divides space into two parts, an "inside" and an "outside," these are interchanged by R and S, so that they are alike. Hence the polyhedron must be infinite. Since the vertex figure is a skew polygon, the faces adjoining a given "horizontal" face are alternately "above" and "below" it, and so R is also a rotatory reflection.

As in Euclidean three-space (**1**, p. 38), it can be shown that each r.s.p. determines a plane regular polygon, called a "hole," which may be described as a path along edges of the r.s.p. such that at the end of each edge we leave two faces on, say, the left. That is, the edges to be selected are not adjacent but alternate.

The symbol $\{p, q|r\}$ will be used to denote an r.s.p. uniquely defined by

p, the number of vertices or edges of a face,
q, the number of edges or faces at a vertex,
r, the number of vertices or edges of a hole.

We remark at this point that p and q are even. For, since R is a rotatory reflection, its period is even. But R cyclically permutes the p edges of a face, and so its period is p. Thus, p is even. Similarly S, cyclically permuting the q faces at a vertex, is a rotatory reflection, and so q is even.

2. Regular skew polyhedra and associated honeycombs. Since the p and q of $\{p, q|r\}$ are both even, we are interested only in finding the values of

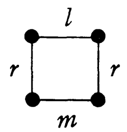
l, m, r for which $\{2l, 2m|r\}$ is an r.s.p. in H^3. Consider such an r.s.p. If each hole $\{r\}$ is considered as a face, we have a honeycomb whose faces are $\{2l\}$'s and $\{r\}$'s. If $r = 2l$, the faces are of one type only, and we shall see that the honeycomb is then regular in all cases but one. Moreover, the honeycomb obtained from $\{2l, 2m|r\}$ by "filling in" the holes must have, as vertex figure, an antiprism whose lateral edges form a skew $2m$-gon, the vertex figure of the r.s.p.

It is necessary at this point to place a slight restriction on the honeycombs we shall consider. Only those honeycombs having cells which are inscribed in finite spheres, or horospheres, or both, will be considered. Thus we are excluding honeycombs with cells inscribed in equidistant surfaces. Since the geometry of an equidistant surface is hyperbolic (**7**, p. 252), this means we are excluding honeycombs whose cells are hyperbolic tessellations, of which there are infinitely many (**3**, p. 156). Such honeycombs are very difficult to deal with and are of minimal interest, since their fundamental tetrahedra have faces which are ultraparallel.

In E^3, it can be shown that an r.s.p. $\{2l, 2m|r\}$ determines a tetrahedron bounded by planes h, h', k, k' whose dihedral angles are

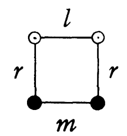
$$\begin{aligned}(hk) = (h'k') &= \pi/2,\\ (hh') &= \pi/l,\\ (kk') &= \pi/m,\\ (hk') = (h'k) &= \pi/r\end{aligned}$$

(**1**, p. 39). It can be seen easily that the argument used is an absolute one, and so is valid in H^3. This tetrahedron is also the fundamental region represented by the graph



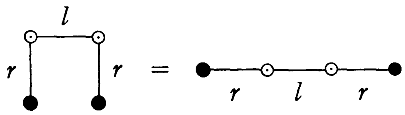
The dots represent the four planes of the tetrahedron, and a link is drawn between two dots whenever the corresponding planes are not perpendicular. Each link is marked with a number n to indicate that the dihedral angle between the two planes is π/n, except $n = 3$ which is regularly omitted. This graph also represents the symmetry group which is generated by reflections in these planes (**2**, pp. 382–387).

The honeycomb with antiprismatic vertex figure which is derived from this graph is



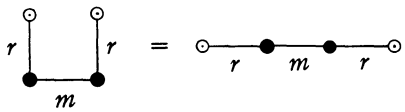
Its cells are $\{r, m\}$'s and $t\{l, r\}$'s with 2 of the former and $2m$ of the latter at each vertex. (See **(2**, pp. 400–404**)** for a thorough discussion of these graphs.) The faces of this honeycomb are then $\{r\}$'s and $\{2l\}$'s; the vertex figures of these faces are, respectively, the basal and lateral edges of the antiprismatic vertex figure of the honeycomb. Thus by omitting the r-gonal faces, we derive the r.s.p. $\{2l, 2m|r\}$.

Some reductions of this honeycomb are possible. If $m = 2$, the honeycomb becomes



or $t_{1,2}\{r, l, r\}$ **(4**, p. 70**)**, a honeycomb of $t\{l, r\}$'s, four at each vertex.

If $l = 2$ and so $m \neq 2$, the honeycomb becomes



or $t_{0,3}\{r, m, r\}$ **(4**, p. 70**)**, a honeycomb of $\{r, m\}$'s and $t\{2, r\}$'s or r-gonal prisms, with 2 of the former and $2m$ of the latter at each vertex.

The only other possible reduction occurs when $r = 2l$ and $m = 2$ or 3. The cell $t\{l, r\}$ now becomes $t\{l, 2l\} = \{2l, 3\}$. For example, when we truncate $\{3, 6\}$ (Fig. 3), the triangular faces become $\{6\}$'s and about each original vertex there is now a $\{6\}$. Since the general $t\{p, q\}$ has two $\{2p\}$'s and one $\{q\}$ at each vertex, $t\{3, 6\}$ has three $\{6\}$'s at each vertex and so is a $\{6, 3\}$. Figure 4 shows $t\{3, 6\}$ with the shaded hexagons indicating those replacing the vertices of the original $\{3, 6\}$.

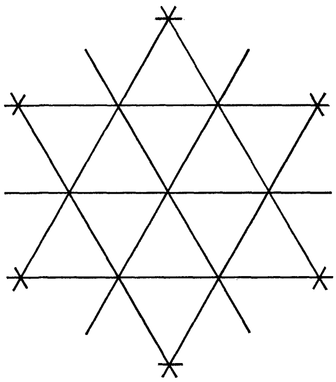


FIGURE 3

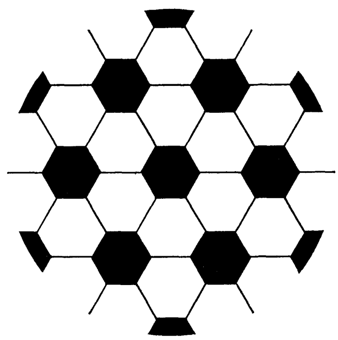
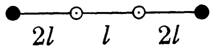


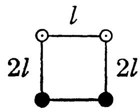
FIGURE 4

Thus, when $m = 2$,



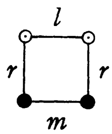
represents a honeycomb whose cells are $\{2l, 3\}$'s, four at a vertex, that is, the regular honeycomb $\{2l, 3, 3\}$ or

If, however, $m = 3$, the two types of cells $\{r, m\}$ and $t\{l, r\}$ both become $\{2l, 3\}$'s, and there are now eight at a vertex. Thus



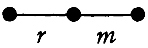
is the regular honeycomb $\{2l, 3, 4\}$ or

An important restriction on l, m, r for the existence of the tetrahedron



in H^3 is that the triangles

and

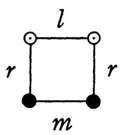


must be spherical or Euclidean; that is,

$$(1) \qquad (l-2)(r-2) \leqslant 4 \quad \text{and} \quad (r-2)(m-2) \leqslant 4$$

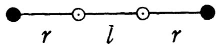
(**5**, pp. 62 and 153); for then $\{l, r\}$ and $\{r, m\}$ are regular polyhedra or "infinite regular polyhedra," and not hyperbolic tessellations.

3. Enumeration of regular skew polyhedra in hyperbolic three-space. We have seen that the general r.s.p. $\{2l, 2m|r\}$ is derived from the honeycomb

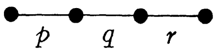


In Table I at the end we list 21 r.s.p. which are thus derived (with l and $m \neq 2$).

From the honeycombs whose graphs are reduced from the above by setting $m = 2$,

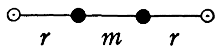


we derive the r.s.p. $\{2l, 4|r\}$. Five r.s.p. of this type are listed in Table I. Since the graphs



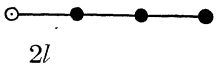
represent the groups from which the regular honeycomb $\{p, q, r\}$ are derived and since we know there are only five regular honeycombs $\{p, q, r\}$ in H^3 with $p = r$ (**3**, pp. 157–158}, there can be no more r.s.p. of this type.

The honeycombs represented by the next reduction ($l = 2$) are

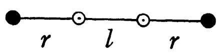


and the r.s.p. derived from them are $\{4, 2m|r\}$. Again there are exactly five groups from which honeycombs of this type are derived. The five r.s.p. corresponding to them are listed in Table I.

When $r = 2l$ and $m = 2$,

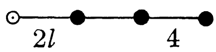


yields the r.s.p. $\{2l, 4|2l\}$ and the vertex figure is a regular skew quadrilateral, the lateral edges of a *regular* tetrahedron, for the honeycomb is regular. The only honeycomb of this form, $\{6, 3, 3\}$, yields the r.s.p. $\{6, 4|6\}$. This, however, has already been derived from



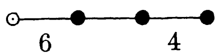
with $r = 6$ and $l = 3$; for the cells, $t\{l, r\}$ are then $t\{3, 6\} = \{6, 3\}$.

Finally, if $r = 2l$ and $m = 3$, the honeycomb is

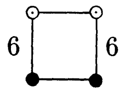


or $\{2l, 3, 4\}$, and the r.s.p. derived from it is $\{2l, 6|2l\}$. The vertex figure is a regular skew hexagon, the Petrie polygon of a regular octahedron. Thus the arrangement of the $2l$-gonal faces of this r.s.p. at a vertex is very like that of the Euclidean $\{4, 6|4\}$ described at the beginning of this paper.

The only such honeycomb is $\{6, 3, 4\}$ or



yielding the r.s.p. $\{6, 6|6\}$. This has already been derived from the honeycomb

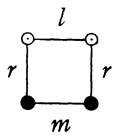
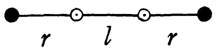
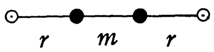


for then all the cells are $\{6, 3\}$'s and there are eight at a vertex.

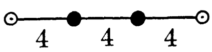
To justify the assertion that there are no more r.s.p. in H^3, consider the integral solutions of the inequalities (1) for the l, m, and r of $\{2l, 2m|r\}$. We need not allow l or m to take the improper value 2, for we have already seen that we have all the r.s.p. in these cases. The only r.s.p. admitted by the inequalities not already listed in Table I is $\{6, 6|3\}$, and this is an r.s.p. in E^3 (**1**, p. 40). Thus we can state:

In H^3 there are exactly 32 regular skew polyhedra which are derived from honeycombs whose cells and vertex figures are not inscribed in equidistant surfaces. They are of three types, depending on the kind of honeycomb from which they are derived. Table I lists these regular skew polyhedra and their associated honeycombs.

TABLE I

1. Honeycomb			$(l, m \neq 2)$	
R.S.P.		$\{2l, 2m\|r\}$		
Individual cases:		$\{6,\ 8\|3\}$,	$\{\ 8,\ 6\|3\}$,	$\{\ 6, 10\|3\}$,
$\{10,\ 6\|3\}$,	$\{8, 8\|3\}$,	$\{6,\ 6\|4\}$,	$\{\ 8, 10\|3\}$,	$\{10,\ 8\|3\}$,
$\{10, 10\|3\}$,	$\{6, 6\|5\}$,	$\{6, 12\|3\}$,	$\{12,\ 6\|3\}$,	$\{\ 8, 12\|3\}$,
$\{12,\ 8\|3\}$,	$\{6, 8\|4\}$,	$\{8,\ 6\|4\}$,	$\{10, 12\|3\}$,	$\{12, 10\|3\}$,
$\{12, 12\|3\}$,	$\{6, 6\|6\}$,	$\{8,\ 8\|4\}$		
2. Honeycomb			$(l \neq 2)$	
R.S.P.		$\{2l, 4\|r\}$		
Individual cases:				
$\{6, 4\|5\}$,	$\{10, 4\|3\}$,	$\{6, 4\|6\}$,	$\{12, 4\|3\}$,	$\{8, 4\|4\}$
3. Honeycomb				
R.S.P.		$\{4, 2m\|r\}$		
Individual cases:				
$\{4, 6\|5\}$,	$\{4, 10\|3\}$,	$\{4, 6\|6\}$,	$\{4, 12\|3\}$,	$\{4, 8\|4\}$.

It is interesting to note that the honeycomb associated with $\{4, 8|4\}$ is not regular, even though its faces are all regular quadrilaterals. For the honeycomb



has, as cells, Euclidean tessellations $\{4, 4\}$ and 4-gonal prisms, or hexahedra, $\{4, 3\}$.

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Carleton University,
Ottawa 1, Ontario