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AN INDEX THEORY FOR SEMIGROUPS OF *-ENDOMORPHISMS OF $\mathscr{B}(\mathscr{H})$ AND TYPE II_1 FACTORS.

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Introduction. In this paper we study unit preserving *-endomorphisms of $\mathscr{B}(\mathscr{H})$ and type II_1 factors. A *-endomorphism α which has the property that the intersection of the ranges of α^n for $n = 1, 2, \dots$ consists solely of multiples of the unit are called shifts. In Section 2 it is shown that shifts of $\mathscr{B}(\mathscr{H})$ can be characterized up to outer conjugacy by an index $n = \infty\ 1, 2, \dots$. In Section 3 shifts of R the hyperfinite II_1 factor are studied. An outer conjugacy invariant of a shift of R is the Jones index $[R: \alpha(R)]$. In Section 3 a class of shifts of index 2 are studied. These are called binary shifts. It is shown that there are uncountably many binary shifts which are pairwise non conjugate and among the binary shifts there are at least a countable infinity of shifts which are pairwise not outer conjugate.

In Section 4 continuous one parameter semigroups of *-endomorphisms of $\mathscr{B}(\mathscr{H})$ are studied. It is shown that one can define a *-representation associated with each such semigroup. The multiplicity of this representation is defined as the index of the semigroup. It is shown that the index is subadditive under taking tensor products. In Section 5 it is shown one can define such an index for semigroups of *-endomorphisms of type II_1 factors.

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1. Shifts.

Definition 1.1. Suppose $\mathscr{U}$ is a C^*-algebra with unit I. We say α is a *shift* of $\mathscr{U}$ if α is a *-endomorphism of $\mathscr{U}$ so that

$$\alpha(I) = I \quad \text{and} \quad \bigcap_{n=1}^{\infty} \alpha^n(\mathscr{U}) = \{\lambda I\}.$$

Simple examples of shifts are obtained as follows. Let $\mathscr{B}_0$ be the algebra of all complex $(n \times n)$-matrices. Let $\mathscr{B}_p$ be an isomorphic copy of $\mathscr{B}_0$ and let γ_p be an isomorphism of $\mathscr{B}_0$ with $\mathscr{B}_p$ for $p = 1, 2, \dots$. Let

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$$\mathcal{U}_m = \bigotimes_{p=1}^{m} \mathcal{B}_p$$

and let $\mathcal{U}$ be the completion of the inductive limit of the $\mathcal{U}_m$. More compactly we express this by writing

$$\mathcal{U} = \bigotimes_{p=1}^{\infty} \mathcal{B}_p.$$

This algebra is the well known UHF-algebra of type n^∞ introduced by Glimm [**6**]. The linear span of elements of the form

$$\gamma_{i_1}(A_1)\gamma_{i_2}(A_2) \ldots \gamma_{i_n}(A_n)$$

for $i_1 < i_2 < \ldots < i_n$, $A_k \in \mathcal{B}_0$ for $k = 1, \ldots, n$ are dense in $\mathcal{U}$. We define a shift α of $\mathcal{U}$ by the requirement

$$\alpha(\gamma_p(A)) = \gamma_{p+1}(A) \text{ for } A \in \mathcal{B}_0 \text{ and } p = 1, 2, \ldots.$$

Definition 1.2. Suppose $\mathcal{U}$ is a C^*-algebra with identity and α is a shift of $\mathcal{U}$. The normalizer of α, denoted $\mathcal{N}(\alpha)$, consists of those unitary elements $U \in \mathcal{U}$ so that

$$U\alpha^k(\mathcal{U})U^{-1} = \alpha^k(\mathcal{U}) \text{ for all } k = 1, 2, \ldots.$$

If α is the shift of the UHF-algebra $\mathcal{U}$ of type n^∞ previously discussed then one can show that a unitary $U \in \mathcal{N}(\alpha)$ if U is of the form

$$U = \gamma_{i_1}(U_1)\gamma_{i_2}(U_2) \ldots \gamma_{i_n}(U_n)$$

with $0 < i_1 < i_2 < \ldots < i_n$ and $U_k \in \mathcal{B}_0$ unitary for $k = 1, \ldots, n$.

Definition 1.3. We say a shift α of $\mathcal{U}$ is *regular* if the normalizer of α generates $\mathcal{U}$ as a C^*-algebra. If $\mathcal{U}$ is a von Neumann algebra we say α is *regular* if the normalizer of α generates $\mathcal{U}$ as a von Neumann algebra (i.e., $\mathcal{N}(\alpha)'' = \mathcal{U}$).

Note the shift α of the UHF-algebra $\mathcal{U}$ just described is regular.

Following [**3**] we define,

Definition 1.4. We say two *-endomorphisms α and β of a C^*-algebra $\mathcal{U}$ are *conjugate* if there is a *-automorphism γ of $\mathcal{U}$ so that

$$\alpha(A) = \gamma(\beta(\gamma^{-1}(A))) \text{ for all } A \in \mathcal{U}.$$

We say two *-endomorphisms α and β are *outer conjugate* if there is a *-automorphism γ of $\mathcal{U}$ and a unitary $U \in \mathcal{U}$ so that

$$\alpha(UAU^{-1}) = \gamma(\beta(\gamma^{-1}(A))) \text{ for all } A \in \mathcal{U}.$$

2. Shifts of $\mathcal{B}(\mathcal{H})$. Suppose α is a shift of $\mathcal{B}(\mathcal{H})$ (where $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded operators on a separable Hilbert space $\mathcal{H}$). Let

$N_1 = \alpha(\mathscr{B}(\mathscr{H}))'$ the set of operators in $\mathscr{B}(\mathscr{H})$ which commute with all operators in the range of α. Then N_1 is a factor of type I_n with $n = 2, 3, \ldots$ or $n = \infty$. We will call n the multiplicity of α. In this section we will show that a shift α of $\mathscr{B}(\mathscr{H})$ is determined up to outer conjugacy by its multiplicity.

Suppose α is a shift of $\mathscr{B}(\mathscr{H})$ and ω_0 is a pure normal state of $\mathscr{B}(\mathscr{H})$ which is invariant under α (i.e., $\omega_0(\alpha(A)) = \omega_0(A)$ for all $A \in \mathscr{B}(\mathscr{H})$). We will show in Theorem 2.3 that a shift β of $\mathscr{B}(\mathscr{H})$ is conjugate to α if and only if there is a pure normal β-invariant state of $\mathscr{B}(\mathscr{H})$ and α and β have the same multiplicity.

LEMMA 2.1. *Suppose α is a shift of $\mathscr{B}(\mathscr{H})$. Let*

$$N_1 = \alpha(\mathscr{B}(\mathscr{H}))' \cap \mathscr{B}(\mathscr{H})$$

and let

$$N_{k+1} = \alpha(N_k) \text{ for } k = 1, 2, \ldots.$$

Then the N_k are mutually commuting type I *factors and*

$$\{N_1, N_2, \ldots, N_p\}' = \alpha^p(\mathscr{B}(\mathscr{H})).$$

Proof. Let the N_k be as defined in the statement of the lemma. Clearly, the N_k are mutually commuting type I factors. We prove the lemma by induction. We have $N_1' = \alpha(\mathscr{B}(\mathscr{H}))$ so the lemma is true for $p = 1$. Suppose the lemma is true for p. We have

$$\begin{aligned}\{N_1, \ldots, N_p, N_{p+1}\}' &= \{N_1, \ldots, N_p\}' \cap N_{p+1}' \\ &= \alpha^p(\mathscr{B}(\mathscr{H})) \cap \alpha^p(N_1') = \alpha^p(\mathscr{B}(\mathscr{H}) \cap N_1') \\ &= \alpha^p(\alpha(\mathscr{B}(\mathscr{H}))) = \alpha^{p+1}(\mathscr{B}(\mathscr{H})).\end{aligned}$$

Hence, the lemma is true for $p + 1$ and so by induction the lemma is true for all p.

COROLLARY 2.2. *Suppose α is a shift of $\mathscr{B}(\mathscr{H})$ and the N_k are as defined in Lemma* 2.1. *Then $\{N_1, N_2, \ldots\}'' = \mathscr{B}(\mathscr{H})$.*

Proof. From Lemma 2.1 it follows that

$$\{N_1, N_2, \ldots\}' = \bigcap_{k=1}^{\infty} \alpha^k(\mathscr{B}(\mathscr{H}))$$

and since α is a shift it follows that the intersection of the $\alpha^k(\mathscr{B}(\mathscr{H}))$ contains only multiples of the identity. Hence, the corollary follows.

THEOREM 2.3. *Suppose α and β are shifts of $\mathscr{B}(\mathscr{H})$ and there is a pure normal state ω_0 of $\mathscr{B}(\mathscr{H})$ which is invariant under α (i.e., $\omega_0(\alpha(A)) = \omega_0(A)$ for all $A \in \mathscr{B}(\mathscr{H})$). Then α and β are conjugate if and only if there is a pure normal state ω_1 of $\mathscr{B}(\mathscr{H})$ which is invariant under β and α and β have the same multiplicity.*

Proof. It follows immediately from the definition of conjugacy that if α and β are conjugate then they have the same multiplicity and if there is a pure normal α-invariant state of $\mathscr{B}(\mathscr{H})$ then there must be a pure normal β-invariant state of $\mathscr{B}(\mathscr{H})$. Suppose then that α and β have the same multiplicity and they each have pure normal invariant states ω_0 and ω_1, respectively. We complete the proof by showing that α and β are conjugate.

Let

$$N_1 = \alpha(\mathscr{B}(\mathscr{H}))' \cap \mathscr{B}(\mathscr{H}) \text{ and}$$

$$N_{k+1} = \alpha(N_k) \text{ for } k = 1, 2, \ldots .$$

We show ω_0 is a product state with respect to the N_k. Suppose $A \in N_1$ and $0 \leqq A \leqq I$. Let

$$\rho(B) = \omega_0(AB) \text{ for all } B \in \alpha(\mathscr{B}(\mathscr{H})).$$

Since

$$\rho(B) = \omega_0(A^{1/2}BA^{1/2}) \text{ and}$$

$$\omega_0(B) - \rho(B) = \omega_0((I - A)^{1/2}B(I - A)^{1/2})$$

for all $B \in \alpha(\mathscr{B}(\mathscr{H}))$ it follows that $0 \leqq \rho \leqq \omega_0|\alpha(\mathscr{B}(\mathscr{H}))$. Since ω_0 is pure and α-invariant it follows that $\omega_0|\alpha(\mathscr{B}(\mathscr{H}))$ is pure. Hence, ρ is a multiple of $\omega_0|\alpha(\mathscr{B}(\mathscr{H}))$, in fact, we have

$$\rho(B) = \rho(I)\omega_0(B) \text{ for } B \in \alpha(\mathscr{B}(\mathscr{H})).$$

Hence, $\omega_0(AB) = \omega_0(A)\omega_0(B)$ for $A \in N_1$, $0 \leqq A \leqq I$ and $B \in \alpha(\mathscr{B}(\mathscr{H}))$. By linearity this relation extends to all of N_1. Thus,

$$\omega_0(AB) = \omega_0(A)\omega_0(B) \text{ for all } A \in N_1 \text{ and } B \in \alpha(\mathscr{B}(\mathscr{H})).$$

Since ω_0 is α-invariant it follows that

$$\omega_0(AB) = \omega_0(A)\omega_0(B) \text{ for } A \in N_2 \text{ and } B \in \alpha^2(\mathscr{B}(\mathscr{H})).$$

Continuing this argument we find if $A_k \in N_k$ for $k = 1, \ldots, n$ then

$$\omega_0(A_1A_2 \ldots A_n) = \omega_0(A_1)\omega_0(A_2) \ldots \omega_0(A_n).$$

Hence, ω_0 is a pure product state with respect to the N_k. Note $\omega_0|N_k$ is pure for each k else a decomposition of $\omega_0|N_k$ would yield a decomposition of the pure state ω_0.

Let $M_1 = \beta(\mathscr{B}(\mathscr{H}))' \cap \mathscr{B}(\mathscr{H})$ and $M_{k+1} = \beta(M_k)$ for $k = 1, 2, \ldots$. By the argument just given we have ω_1 is a pure product state of $\mathscr{B}(\mathscr{H})$ with respect to the M_k. Let $\{e_{ij}; i, j = 1, \ldots, n\}$ and $\{f_{ij}; i, j = 1, \ldots, n\}$ be families of matrix units of N_1 and M_1, respectively, chosen so that $\omega_0(e_{11}) = 1$ and $\omega_1(f_{11}) = 1$ (i.e.,

$$e_{ij^*} = e_{ji}, \; e_{ij}e_{kl} = \delta_{jk}e_{il} \text{ and } \sum_{i=1}^{n} e_{ii} = I$$

and the same relations hold for the f_{ij}). Let

$$e_{ij}^{(k)} = \alpha^{(k-1)}(e_{ij}) \text{ and } f_{ij}^{(k)} = \alpha^{(k-1)}(f_{ij})$$

for $k = 1, 2, \dots$ and let $\mathcal{U}_1$ and $\mathcal{U}_2$ be the C^*-algebras generated by the $\{e_{ij}^{(k)}\}$ and the $\{f_{ij}^{(k)}\}$, respectively. From Corollary 2.2 we have $\mathcal{U}_1'' = \mathcal{U}_2'' = \mathcal{B}(\mathcal{H})$. Let γ be the *-isomorphism of $\mathcal{U}_1$ onto $\mathcal{U}_2$ given by the requirements

$$\gamma(f_{ij}^{(k)}) = e_{ij}^{(k)} \text{ for } i, j = 1, \dots, n \text{ and } k = 1, 2, \dots.$$

We show γ extends to a *-automorphism of $\mathcal{B}(\mathcal{H})$.

Let f_0 and f_1 be unit vectors in $\mathcal{H}$ so that

$$\omega_0(A) = (f_0, Af_0) \text{ and } \omega_1(A) = (f_1, Af_1).$$

A straightforward computation shows that

$$\omega_0(\gamma(A)) = \omega_1(A) \text{ for } A \in \mathcal{U}_2$$

(one first shows this for polynomials in the $f_{ij}^{(k)}$ and then extends to $\mathcal{U}_2$ by norm continuity). Then we can define a unitary operator U by the relation

$$UAf_1 = \gamma(A)f_0 \text{ for all } A \in \mathcal{U}_2.$$

One checks that this defines a unitary operator with the property that $UAU^{-1} = \gamma(A)$ for all $A \in \mathcal{U}_2$. Hence, γ is weakly continuous and, therefore, it has a weakly continuous extension to $\mathcal{B}(\mathcal{H})$ = the weak closure of $\mathcal{U}_2$. We also denote this extension by γ. One checks that by the construction of γ we have $\alpha = \gamma\beta\gamma^{-1}$.

THEOREM 2.4. *Suppose α and β are shifts of $\mathcal{B}(\mathcal{H})$. Then α and β are outer conjugate if and only if they have the same multiplicity.*

Proof. One checks from the definition of outer conjugacy that if α and β are outer conjugate that they must have the same multiplicity. To show the reverse implication suppose α and β are shifts with the same multiplicity. Let $\{e_{ij}; i, j = 1, 2, \dots\}$ be a set of matrix units for $\mathcal{B}(\mathcal{H})$. One sees that the multiplicity of α is equal to $n = \dim(\alpha(e_{11}))$. Since α and β have the same multiplicity we have

$$\dim(\alpha(e_{11})) = \dim(\beta(e_{11})).$$

Hence, there is a partial isometry $W \in \mathcal{B}(\mathcal{H})$ so that $W^*W = \alpha(e_{11})$ and $WW^* = \beta(e_{11})$. Let

$$S = \sum_{i=1}^{\infty} \beta(e_{i1})W\alpha(e_{1i}).$$

A computation shows that S is unitary and

$$S\alpha(e_{ij})S^* = \beta(e_{ij}) \text{ for } i, j = 1, 2, \dots.$$

Hence,

$$\alpha(SAS^*) = \gamma(\beta(\gamma^{-1}(A))) \text{ for all } A \in \mathcal{B}(\mathcal{H})$$

where $\gamma(A) = S^*AS$.

3. Binary shifts of the hyperfinite II_1 factor. Throughout this section R will denote the unique injective II_1 factor ([**4**]), the hyperfinite II_1 factor. If α is a shift of R then $\alpha(R)$ is a subfactor of R. An outer conjugacy invariant of α is the Jones index $[R: \alpha(R)]$ which measures the relative size of $\alpha(R)$ in R. In this section we will restrict our attention to the simplest case where $[R: \alpha(R)] = 2$. Such shifts will be called shifts of index 2.

Here are some results of Jones' index theory we will need (see [**7**, **8**] for further details). Suppose N is a subfactor of R of index 2 and let Φ be the conditional expectation of R onto N via the trace (i.e., the linear mapping $A \rightarrow \Phi(A)$ from R to N is defined by the requirement that $\mathrm{tr}(AB) = \mathrm{tr}(\Phi(A)B)$ for all $B \in N$ where tr is the trace on R). Then $\theta(A) = 2\Phi(A) - A$ is an outer *-automorphism of R of period 2 with the property that an element $A \in R$ is contained in N if and only if $\theta(A) = A$. There is a unitary $S \in R$ so that $S^2 = I$ and $\theta(S) = -S$. (Note that S is not unique since if $U \in N$ is unitary and $S' = USU^{-1}$ then $S'^2 = I$ and $\theta(S') = -S'$.) Given such an element S then every element of R can be uniquely expressed in the form $A + SB$ with $A, B \in N$.

Definition 3.1. Suppose α is a shift of R of index 2. We define θ_α as the unique *-automorphism of R so that

$$\theta_\alpha(\theta_\alpha(A)) = A \quad \text{for all } A \in R$$

and $A \in \alpha(R)$ if and only if $\theta_\alpha(A) = A$.

Some other results of the Jones index theory are the following. If α is a shift of index 2 then the index of $\alpha^{k+n}(R)$ in $\alpha^k(R)$ is 2^n (*i.e.*, $[\alpha^k(R):\alpha^{k+n}(R)] = 2^n$). If N is a subfactor of R and $[R: N] < 4$ then the relative commutant of N in R is trivial, i.e., $R \cap N' = \{\lambda I\}$. If $[R: N] = 4$ and E is a projection in $N' \cap R$ then $\mathrm{tr}(E) = 0$, $1/2$ or 1.

Definition 3.2. A shift α of R is called a *binary shift* if there is a unitary $U \in R$ satisfying the requirements,

i) $U^2 = I$.

ii) $U\alpha^k(U)U^{-1} = \pm\alpha^k(U)$ for all $k = 1, 2, \dots.$

iii) $R = \{U, \alpha(U), \alpha^2(U), \dots\}''$.

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The unitary U is called an *α-generator* of R.

We will show that if α is a binary shift of R and U and V are α-generators of R then $U = \pm V$. It will follow that the set

$$S = \{k \in \mathbf{N};\ U\alpha^k(U) = -\alpha^k(U)U\}$$

is a conjugacy invariant of α.

LEMMA 3.3. *Suppose α is a binary shift of R and $U \in R$ is an α-generator. Suppose W is in the normalizer of α (i.e., $W \in \mathcal{N}(\alpha)$). Then*

$$W = \lambda\alpha^{k_1}(U)\alpha^{k_2}(U)\dots\alpha^{k_s}(U)$$

with $0 < k_1 < k_2 < \dots < k_s$.

Proof. Suppose α is a binary shift of R with an α-generator U. Suppose $W \in \mathcal{N}(\alpha)$. Let

$$U_k = \alpha^{(k-1)}(U) \text{ for } k = 1, 2, \dots .$$

For Q a finite subset of positive integers we define

$$\Gamma(Q) = U_{i_1}U_{i_2}\dots U_{i_s}$$

with $Q = \{i_1, i_2, \dots, i_s\}$ and $i_1 < i_2 < \dots < i_s$. If Q is the empty set we define $\Gamma(Q) = I$. From the commutation relations for the U_i it follows that

$$\Gamma(Q_1)\Gamma(Q_2) = \pm\Gamma(Q_1\Delta Q_2) \text{ and } \Gamma(Q_1)\Gamma(Q_2) = \pm\Gamma(Q_2)\Gamma(Q_1)$$

for all finite sets Q_1, Q_2 where $Q_1\Delta Q_2$ is the symmetric difference of Q_1 and Q_2. Since the U_i generate R it follows that the linear span of the $\Gamma(Q)$ is weakly dense in R.

Let tr denote the trace on R. We show that $\mathrm{tr}(\Gamma(Q)) = 0$ unless Q is the empty set. Suppose Q is a non-empty finite set of positive integers. We claim there is an integer k so that

$$U_k\Gamma(Q)U_k^{-1} = -\Gamma(Q).$$

Suppose no such k exists. Then $\Gamma(Q)$ is in the center of R so $\Gamma(Q) = \lambda I$. Then

$$U_{i_1}U_{i_2}\dots U_{i_s} = \lambda I$$

and, therefore, U_{i_s} can be expressed in terms of the U_k with $k < i_s$. Applying α to this expression for U_{i_s} we find U_{i_s+1} can be expressed in terms of the U_k with $k < i_s + 1$ and since we already have an expression for U_{i_s} we have that U_{i_s+1} can be expressed in terms of the U_k with $k < i_s$. Continuing by induction we find each U_i can be expressed in terms of the U_k with $k < i_s$. This leads to the conclusion that R is finite dimensional which is a contradiction. Hence, there is a positive k so that

$$U_k\Gamma(Q)U_k^{-1} = -\Gamma(Q).$$

Then, it follows that

$$\mathrm{tr}(\Gamma(Q)) = \mathrm{tr}(U_k\Gamma(Q)U_k^{-1}) = -\mathrm{tr}(\Gamma(Q)).$$

Hence, $\mathrm{tr}(\Gamma(Q)) = 0$ unless Q is the empty set. Hence, we have

$$\mathrm{tr}(\Gamma(Q_1)^*\Gamma(Q_2)) = 0$$

unless $Q_1 = Q_2$.

Let Φ be the conditional expectation of R onto $\alpha(R)$ via the trace and let $\theta(A) = 2\Phi(A) - A$ for $A \in R$. We have

$$\alpha(\Gamma(Q)) = \Gamma(Q + 1)$$

where $k \in Q + 1$ if and only if $k - 1 \in Q$. It follows that $\Gamma(Q) \in \alpha(R)$ if and only if $1 \notin Q$. Hence, $\theta(\Gamma(Q)) = \Gamma(Q)$ if $1 \notin Q$ and $\theta(\Gamma(Q)) = -\Gamma(Q)$ if $1 \in Q$.

In what follows we will need the following. If $Q \subset \{1, \ldots, n\}$ and Q is not empty then

$$\mathrm{tr}(\Gamma(Q)\alpha^n(A)) = 0 \text{ for all } A \in R.$$

This may be seen as follows. We have

$$\mathrm{tr}(\Gamma(Q)\alpha^n(\Gamma(Q'))) = 0$$

for all finite sets Q'. Since each $A \in R$ is the strong limit of linear combinations of $\Gamma(Q')$ the result follows.

Now consider $W \in \mathcal{N}(\alpha)$. Since $W\alpha(R)W^{-1} = \alpha(R)$ there is a *-automorphism γ of R so that

$$W\alpha(A)W^{-1} = \alpha(\gamma(A)) \text{ for all } A \in R.$$

Applying the automorphism θ to this equation we find

$$\theta(W)\alpha(A)\theta(W^{-1}) = \alpha(\gamma(A)) \text{ for } A \in R.$$

Combining these equations we find

$$W^{-1}\theta(W)\alpha(A)\theta(W)^{-1}W = \alpha(A).$$

Hence,

$$W^{-1}\theta(W) \in \alpha(R)' \cap R.$$

Since $[R: \alpha(R)] = 2 < 4$ we have $W^{-1}\theta(W) = \lambda I$ and, hence, $\theta(W) = \lambda W$. Since $\theta(\theta(W)) = W$ we have $\theta(W) = \pm W$.

We claim we can express W in the form

$$W = U_1^{k_1}\alpha(W_1)$$

where

$$\theta(W) = (-1)^{k_1}W \text{ and } W_1 \in \mathcal{N}(\alpha).$$

If $\theta(W) = W$ then $W \in \alpha(R)$ and there is a unique $W_1 \in \alpha(R)$ so that $W = \alpha(W_1)$. If $\theta(W) = -W$ then we have $\theta(U_1 W) = U_1 W$ so $U_1 W \in \alpha(R)$. Then there is a unique $W_1 \in R$ so that $U_1 W = \alpha(W_1)$. In either case we have

$$W = U_1^{k_1}\alpha(W_1)$$

and one easily checks that W_1 is in the normalizer of α. Hence, by the same argument W_1 can be expressed in the form

$$W_1 = U_1^{k_2}\alpha(W_2)$$

with

$$\theta(W_1) = (-1)^{k_2}W_1 \text{ and } W_2 \in \mathcal{N}(\alpha).$$

Then we have

$$W = U_1^{k_1}U_2^{k_2}\alpha^2(W_2).$$

Continuing by induction we have

$$W = U_1^{k_1}U_2^{k_2} \dots U_s^{k_s}\alpha^s(W_s)$$

with $W_s \in \mathcal{N}(\alpha)$. We will show that $W_s = \lambda I$ for sufficiently large s.

Let m be the supremum of the i so that k_i is odd. More precisely

$$m = \sup\{s;\ W = U_1^{k_1}U_2^{k_2} \dots U_s^{k_s}\alpha^s(W_s) \text{ with } k_s = 1\}.$$

We show m is finite. Suppose m is infinite. Suppose Q is a finite set of positive integers. Choose q so large that $Q \subset \{1, \dots, q\}$. Since m is infinite there is an integer $s > q$ so that

$$W = U_1^{k_1}U_2^{k_2} \dots U_s^{k_s}\alpha^s(W_s) \text{ with } k_s = 1.$$

Then

$$\Gamma(Q)^* W = \lambda\Gamma(S)U_s\alpha^s(W_s)$$

with $S \subset \{1, \dots, s - 1\}$. Hence,

$$\Gamma(Q)^* W = \Gamma(Q_1)\alpha^s(A)$$

with $Q_1 \subset \{1, \dots, s\}$ and Q_1 not empty and $A \in R$. As we have seen this implies

$$\mathrm{tr}(\Gamma(Q)^* W) = 0.$$

Hence, $\mathrm{tr}(\Gamma(Q)^* W) = 0$ for all finite sets Q. But this is impossible since linear combinations of the $\Gamma(Q)$ are dense in R. Hence, m is finite and we have

$$W = U_1^{k_1}U_2^{k_2} \dots U_m^{k_m}\alpha^n(W_n) \quad \text{for all } n > m.$$

Hence, $U_m^{k_m} \dots U_2^{k_2} U_1^{k_1} W \in \alpha^n(R)$ for all $n > m$. But since α is a shift we have

$$\bigcap_{n=1}^{\infty} \alpha^n(R) = \{\lambda I\}.$$

Hence,

$$U_m^{k_m} \dots U_2^{k_2} U_1^{k_1} W = \lambda I$$

and, thus,

$$W = \lambda U_1^{k_1} U_2^{k_2} \dots U_m^{k_m}.$$

THEOREM 3.4. *Suppose α is a binary shift of R and U and V are α-generators of R. Then $U = \pm V$.*

Proof. Suppose α is a binary shift of R and U and V are α-generators of R. Since $V \in \mathcal{N}(\alpha)$ we have from Lemma 3.3 that

$$V = \lambda_1 U^{k_1} \alpha(U)^{k_2} \dots \alpha^s(U)^{k_s}.$$

The same argument shows

$$U = \lambda_2 V^{j_1} \alpha(V)^{j_2} \dots \alpha^m(V)^{j_m}.$$

Substituting the first expression for V in the second expression gives us an equation for U in terms of U and $\alpha^k(U)$ for $k = 1, 2, \dots$ and this equation can only be true if $U = \lambda V$. Since U and V are hermitian unitaries we have $U = \pm V$.

Definition 3.5. Suppose α is a binary shift of R with an α-generator U. The *anticommutator set of* α, denoted $S(\alpha)$, is the set of positive integers k so that

$$U\alpha^k(U) + \alpha^k(U)U = 0.$$

THEOREM 3.6. *Two binary shifts α and β are conjugate if and only if their anticommutator sets coincide.*

Proof. It is clear that conjugate binary shifts have equal anticommutator sets. Conversely, suppose α and β are binary shifts with equal anticommutator sets. Suppose U is an α-generator of R and V is a β-generator of R. One easily checks that the linear mapping γ defined by the relations

$$\gamma(\beta^k(V)) = \alpha^k(U) \text{ for } k = 0, 1, 2, \dots$$

extends to a *-automorphism of R such that $\beta = \gamma\alpha\gamma^{-1}$.

Definition 3.7. A subset S of the positive integers is said to be *primary* if it is the anticommutator set of a binary shift.

Definition 3.8. Suppose S is a (possibly infinite) subset of the positive integers. The *binary shift algebra* $\mathscr{B}(S)$ over S is the *-algebra generated by elements U_i for $i = 1, 2, \dots$ satisfying the relations,

i) $U_i^* = U_i$

ii) $U_i^2 = I$

iii) $U_iU_j = \sigma(i, j)U_jU_i$

where $\sigma(i, j) = -1$ if $|i - j| \in S$ and $\sigma(i, j) = 1$ if $|i - j| \notin S$. If Q is a finite set of positive integers then we define the element $\Gamma(Q) \in \mathscr{B}(S)$ as

$$\Gamma(Q) = U_{i_1}U_{i_2} \dots U_{i_s}$$

with $Q = \{i_1, i_2, \dots, i_s\}$ and $i_1 < i_2 < \dots < i_s$.

If S is a set of positive integers and P is a finite set of positive integers then the subalgebra of the binary shift algebra $\mathscr{B}(S)$ generated by the U_i with $i \in P$ is spanned by the $\Gamma(Q)$ with $Q \subset P$. Hence, the algebra generated by the U_i with $i \in P$ is of dimension 2^n where n is the number of elements of P. It follows that the C^*-algebra completion of any binary shift algebra is an AF-algebra (see **[1]**) since it is the closure of the union of an ascending sequence of finite dimensional algebras.

THEOREM 3.9. *Suppose $\mathscr{B}(S)$ is the binary shift algebra over S and $\mathscr{U}(S)$ is the C^*-algebra completion of $\mathscr{B}(S)$. Then the following statements are equivalent.*

i) *S is primary.*

ii) *$\mathscr{B}(S)$ is simple.*

iii) *$\mathscr{U}(S)$ is simple.*

iv) *The center of $\mathscr{B}(S)$ consists of multiples of the unit.*

v) *The center of $\mathscr{U}(S)$ consists of multiples of the unit.*

vi) *$\mathscr{B}(S)$ has a unique trace.*

vii) *$\mathscr{U}(S)$ has a unique trace.*

viii) *For each non-empty finite set Q of positive integers there is an integer k so that $U_k\Gamma(Q) = -\Gamma(Q)U_k$.*

Proof. Suppose $\mathscr{B}(S)$ is the binary shift algebra over S and $\mathscr{U}(S)$ is the C^*-completion of $\mathscr{B}(S)$. Suppose statement viii) is false. Then there is a non-empty finite set Q of positive integers so that

$$U_k\Gamma(Q) = \Gamma(Q)U_k \text{ for all } k = 1, 2, \dots .$$

Hence, $\Gamma(Q) \neq \lambda I$ is in the center of $\mathscr{B}(S)$. Hence, statements ii) through vii) are false. Hence, any of the statements ii) through vii) imply viii).

We prove the reverse implications. Suppose viii) is true. There is a trace on $\mathscr{U}(S)$ given by $\mathrm{tr}(\Gamma(Q)) = 0$ unless Q is the empty set and $\mathrm{tr}(I) = 1$. We show this trace is unique. Suppose τ is a tracial state of $\mathscr{U}(S)$. Suppose

Q is a non-empty finite set of positive integers. Then there is a U_k so that

$$U_k \Gamma(Q) U_k = -\Gamma(Q).$$

Hence,

$$\tau(\Gamma(Q)) = \tau(U_k \Gamma(Q) U_k) = -\tau(\Gamma(Q)).$$

Hence, $\tau(\Gamma(Q)) = 0$. Hence, the trace is unique so we have viii) $\Rightarrow$ vii) and viii) $\Rightarrow$ vi).

We continue to assume viii) is true. Suppose A is in the center of $\mathcal{U}(S)$ and $\epsilon > 0$. Since the linear span of the $\Gamma(Q)$ is norm dense in $\mathcal{U}(S)$ there is a finite collection $\{Q_1, Q_2, \ldots, Q_n\}$ of non-empty finite sets of positive integers and complex numbers λ_i so that

$$\left\| A - \lambda_0 I - \sum_{i=1}^{n} \lambda_i \Gamma(Q_i) \right\| < \epsilon.$$

There are integers k_i so that

$$U_{k_i} \Gamma(Q_i) U_{k_i} = -\Gamma(Q_i) \text{ for } i = 1, \ldots, n.$$

Let

$$\Phi_i(B) = \frac{1}{2} B + \frac{1}{2} U_{k_i} B U_{k_i} \text{ for } B \in \mathcal{U}(S).$$

We have that Φ_i is norm decreasing and since A is in the center of $\mathcal{U}(S)$ we have

$$\Phi_i(A - \lambda_0 I) = A - \lambda_0 I.$$

Then we have

$$\Phi_1\left(\Phi_2\left(\ldots\left(\Phi_n\left(A - \lambda_0 I - \sum_{i=1}^{n} \lambda_i \Gamma(Q_i)\right)\right)\ldots\right)\right) = A - \lambda_0 I.$$

Since the Φ_i are norm decreasing we have

$$\|A - \lambda_0 I\| < \epsilon.$$

Since ϵ is arbitrary we have $A = \lambda I$. Hence, viii) $\Rightarrow$ v) and the same argument shows viii) $\Rightarrow$ iv).

We continue to assume viii) is true. Suppose $\mathcal{U}(S)$ is not simple. Then there is an A contained in a two sided ideal $\mathcal{I}$ so that $\operatorname{tr}(A^*A) = 1$. (Note we have already shown that viii) implies $\mathcal{U}(S)$ has a unique faithful trace.) Arguing as before we can find non-empty sets Q_i and complex numbers λ_i so that

$$\left\| A^*A - \lambda_0 I - \sum_{i=1}^{n} \lambda_i \Gamma(Q_i) \right\| < \frac{1}{4}.$$

Then arguing as before we define mappings Φ_i given by

$$\Phi_i(B) = \frac{1}{2}B + \frac{1}{2}U_{k_i}BU_{k_i}$$

so that

$$\Phi_1\left(\Phi_2\left(\ldots\left(\Phi_n\left(A^*A - \lambda_0 I - \sum_{i=1}^{n} \lambda_i \Gamma(Q_i)\right)\right)\ldots\right)\right) = B - \lambda_0 I$$

where

$$B = \Phi_1(\Phi_2(\ldots(\Phi_n(A^*A))\ldots)) \in \mathscr{I}.$$

Since the Φ_i are norm decreasing and preserve the trace we have

$$\|B - \lambda_0 I\| < \frac{1}{4} \text{ and } \operatorname{tr}(B) = 1.$$

Then

$$|\operatorname{tr}(B - \lambda_0 I)| < \frac{1}{4}$$

so $|\lambda_0 - 1| < \frac{1}{4}$. Hence $\|B - I\| < \frac{1}{2}$ so B is invertible. Hence, $\mathscr{I} = \mathscr{U}$. Hence, viii) $\Rightarrow$ iii) and the same argument shows viii) $\Rightarrow$ ii).

Thus, we have shown conditions ii) through viii) are equivalent. The argument in the second paragraph of the proof of Lemma 3.3 shows i) $\Rightarrow$ viii). To show the reverse implication suppose viii) is true. Let π be the cyclic *-representation of $\mathscr{U}(S)$ induced by the trace. Since the trace is unique $\pi(\mathscr{U}(S))''$ is a II_1-factor. One checks that the mapping $\alpha(\pi(U_k)) = \pi(U_{k+1})$ defines a shift of $\pi(\mathscr{U}(S))''$ with α-generator $\pi(U_1)$ and the anticommutator set of α is S. Hence, viii) $\Rightarrow$ i).

Condition viii) of Theorem 3.9 is the most useful for determining whether a given set S is primary. Here are a few remarks concerning primary sets. If S is a set of positive integers we define the signature function σ_s as a mapping of the integers into the integers $\{-1, 1\}$ so that $\sigma_s(i) = -1$ if $i \in S$ or $-i \in S$ and $\sigma_s(i) = 1$ otherwise. Then

$$U_iU_j = \sigma_s(i - j)U_jU_i.$$

From condition viii) it follows that if S is not primary then there are integers p and m so that

$$\sigma_s(p + n) = \sigma_s(n) \text{ for all } n > m$$

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(i.e., σ_s must be periodic beyond a certain point). A closer analysis shows that if S is not primary then σ_s must be periodic starting at zero (i.e., $\sigma_s(p + n) = \sigma_s(n)$ for all $n > 0$.

Conversely one can show that if σ_s is periodic for all of $\mathbf{Z}$ (i.e., $\sigma_s(n + p) = \sigma_s(n)$ for all $n = 0, \pm 1, \pm 2, \dots$) then S is not primary. The question arises if there are sets S which are not primary such that σ_s is not periodic for all of $\mathbf{Z}$. With an IBM PC (made available through the IBM Threshold program at the University of Pennsylvania) approximately 4000 cases were examined to try and find an example of such a set. None were found. Recently G. Price has shown that if S is not primary then σ_s must be periodic for all of $\mathbf{Z}$ (the result will appear elsewhere).

THEOREM 3.10. *There are uncountably many non-conjugate binary shifts of R and there are at least a countable infinity of outer conjugacy classes among the binary shifts.*

Proof. Since there are uncountably many sets S of positive integers and only countably many of them are not primary it follows that there are uncountably many non-conjugate binary shifts.

If α is a binary shift let $q(\alpha)$ be the first integer k so that $\alpha^k(R)' \cap R$ is not trivial. One easily checks that $q(\alpha)$ is an outer conjugacy invariant (i.e., if α and β are binary shifts which are outer conjugate then $q(\alpha) = q(\beta)$). A computation shows that if $S_n = \{n\}$ for $n = 1, 2, \dots$ then the S_n are all primary and if α_n is the binary shift associated with S_n then $q(\alpha_n) = n + 1$. Hence, there are at least a countable infinity of outer conjugacy classes of binary shifts.

Finally, we would like to end this section with some questions about binary shifts.

Question 3.1. Is the number of outer conjugacy classes of the binary shifts countable? If α and β are binary shifts so that $\alpha^k(R)' \cap R$ and $\beta^k(R)' \cap R$ are trivial for all $k = 1, 2, \dots$ are α and β outer conjugate?

Question 3.2. If α and β are binary shifts and $q(\alpha) = q(\beta) < \infty$ then are α and β outer conjugate? In approximately one hundred examples an IBM PC found that when $q(\alpha) = q(\beta)$ then α and β were outer conjugate.

4. Continuous semigroups of *-endomorphisms of $\mathscr{B}(\mathscr{H})$.

Definition 4.1. We say $\{\alpha_t; t \geqq 0\}$ is an E_0-semigroup of a von Neumann algebra M if the following conditions are satisfied.

i) α_t is a *-endomorphism of M for each $t \geqq 0$.

ii) α_0 is the identity endomorphism and $\alpha_t \circ \alpha_s = \alpha_{t+s}$ for all $t, s \geqq 0$.

iii) For each $f \in M_*$ (the predual of M) and $A \in M$ the function $f(\alpha_t(A))$ is a continuous function of t.

Definition 4.2. We say $\{\alpha_t;\ t \geqq 0\}$ is a *continuous flow of shifts of a von Neumann algebra M* if $\{\alpha_t\}$ is an E_0-semigroup of M and α_t is a shift of M for each $t > 0$.

We give an example of a flow of shifts of $\mathscr{B}(\mathscr{H})$ which we will call the CAR-flow. Let $\mathscr{U}$ be the CAR algebra over $L^2(-\infty, \infty)$. Specifically $\mathscr{U}$ is a C^*-algebra generated by elements $a(f)$, defined for each $f \in L^2(-\infty, \infty)$, and satisfying the CAR relations,

$$a(\alpha f + g) = \alpha a(f) + a(g)$$

$$a(f)a(g) + a(g)a(f) = 0$$

$$a(f)^*a(g) + a(g)a(f)^* = (f, g)I,$$

for $f, g \in L^2(-\infty, \infty)$ and where

$$(f, g) = \int_{-\infty}^{\infty} \overline{f(x)}g(x)dx.$$

Let ω_0 be the Fock state of $\mathscr{U}$. This state is determined by the requirements that

$$\omega_0(a(f)^*a(f)) = 0 \text{ for all } f \in L^2(-\infty, \infty).$$

Let $(\pi, \mathscr{H}, \Omega_0)$ be a cyclic $*$-representation induced by ω_0 with cyclic unit vector $\Omega_0 \in \mathscr{H}$ so that

$$\pi(a(f))\Omega_0 = 0 \text{ for all } f \in L^2(-\infty, \infty).$$

Since the Fock state is pure the representation π is irreducible. We define a one parameter unitary group of translations on $L^2(-\infty, \infty)$ given by

$$(S_tf)(x) = f(x - t) \text{ for } f \in L^2(-\infty, \infty) \text{ and } t \text{ real}.$$

Let $\{\beta_t\}$ be the group of $*$-automorphisms of $\mathscr{U}$ determined by the requirement that

$$\beta_t(a(f)) = a(S_tf) \text{ for all } f \in L^2(-\infty, \infty).$$

Note the Fock state ω_0 is invariant under β_t (i.e., $\omega_0(A) = \omega_0(\beta_t(A))$ for all $A \in \mathscr{U}$). Then on the representation space $\mathscr{H}$ we can define a strongly continuous one parameter unitary group $\{V(t)\}$ by the relation

$$V(t)\pi(A)\Omega_0 = \pi(\beta_t(A))\Omega_0 \text{ for } A \in \mathscr{U}.$$

Let

$$\alpha_t(A) = V(t)AV(t)^{-1} \text{ for all } A \in \mathscr{B}(\mathscr{H}).$$

Let $\mathscr{U}_+$ be the C^*-subalgebra of $\mathscr{U}$ generated by the $a(f)$ with f having support in $[0, \infty)$ and let $M_+ = \pi(\mathscr{U}_+)''$. Since $\beta_t(\mathscr{U}_+) \subset \mathscr{U}_+$ for $t > 0$ it follows that $\alpha_t(M_+) \subset M_+$. We show M_+ is a type I factor.

We begin by determining the commutant of M_+. Let E_+ be the projection of $L^2(-\infty, \infty)$ onto $L^2(0, \infty)$ given by

$$(E_+ f)(x) = f(x) \text{ for } x \geqq 0 \text{ and}$$

$$(E_+ f)(x) = 0 \text{ for } x < 0.$$

Let θ be the *-automorphism of $\mathcal{U}$ determined by the requirement that

$$\theta(a(f)) = a((I - 2E_+)f) \text{ for all } f \in L^2(-\infty, \infty).$$

Since θ leaves the Fock state ω_0 invariant there is a unitary operator W on $\mathcal{H}$ defined by the relation

$$W\pi(A)\Omega_0 = \pi(\theta(A))\Omega_0 \text{ for all } A \in \mathcal{U}.$$

Since θ^2 is the identity automorphism we have $W^2 = I$. We claim $W \in M_+$. This may be seen as follows. Let $\{f_i; i = 1, 2, \dots\}$ be an orthonormal basis for $L^2(0, \infty) = E_+ L^2(-\infty, \infty)$. Let

$$W_n = \pi((I - 2a(f_1)^*a(f_1))(I - 2a(f_2)^*a(f_2)) \dots (I - 2a(f_n)^*a(f_n))).$$

Let $\{g_i; i = 1, 2, \dots\}$ be an orthonormal basis for

$$L^2(-\infty, 0) = (I - E_+)L^2(-\infty, \infty).$$

From the commutation relations one can compute that if p is a polynomial in the $a(f_i)$, $a(g_i)$, $a(f_i)^*$ and $a(g_i)^*$ then for n sufficiently large

$$W_n\pi(p)\Omega_0 = W\pi(p)\Omega_0.$$

Since the set of such vectors $\pi(p)\Omega_0$ is dense in $\mathcal{H}$ and the W_n are uniformly bounded it follows that $W_n \to W$ strongly as $n \to \infty$. Since $W_n \in M_+$ it follows that $W \in M_+$.

For $f \in L^2(0, \infty)$ we define $A(f) = \pi(a(f))$ and for $f \in L^2(-\infty, 0)$ we define $B(f) = \pi(a(f))W$. Note that $A(f)$ and $B(f)$ each satisfy the CAR relations and note that the $A(f)$ and their adjoints commute with the $B(f)$ and their adjoints. Let M be the von Neumann algebra generated by the $A(f)$ and $B(f)$. Since $W \in M_+$ and $M_+ \subset M$ we have $W \in M$. Hence,

$$\pi(a(f)) = B(f)W \in M \text{ for all } f \in L^2(-\infty, 0)$$

and, hence,

$$\pi(a(f)) \in M \text{ for all } f \in L^2(-\infty, \infty).$$

Since the representation π is irreducible we have $M = \mathcal{B}(\mathcal{H})$. Hence, M_+ and its commutant generate $\mathcal{B}(\mathcal{H})$. Hence, M_+ is a factor and since the vector state $(\Omega_0, A\Omega_0)$ restricted to M_+ is pure it follows that M_+ is a type I_∞ factor.

Next we show α_t is a shift of M_+ for all $t > 0$. To this end suppose $t > 0$ and

$$Z = \bigcap_{n=1}^{\infty} \alpha_{nt}(M_+)$$

and $C \in Z$. Since the involutive *-automorphism $A \to WAW^*$ maps each $\alpha_{nt}(M_+)$ onto itself it follows this automorphism maps Z into itself. Let

$$C_1 = (C + WCW^*)/2 \text{ and } C_2 = (C - WCW^*)/2.$$

Then we have $C_1, C_2 \in Z$, $C = C_1 + C_2$ and $WC_1W^* = C_1$ and $WC_2W^* = -C_2$. Note that if p is an even polynomial in the $a(f)$ and $a(f)^*$ with $f \in L^2(0, \infty)$ then $W\pi(p)W^* = \pi(p)$ and if p is an odd polynomial then $W\pi(p)W^* = -\pi(p)$. It follows that for each positive integer n, C_1 can be approximated in the strong operator topology by even polynomials p in the $a(f)$ and $a(f)^*$ with $f \in L^2(nt, \infty)$ and C_2 can be similarly approximated by odd polynomials. Hence, it follows that if $f \in L^2(0, nt)$ then $\pi(a(f))$ commutes with C_1 and anticommutes with C_2. Since this is true for all positive n it follows $C_1 \in M'_+$ and, hence, $C_1 = \lambda I$. Note $WC_2 \in M_+$ commutes with all the $\pi(a(f))$ with $f \in L^2(0, nt)$ and since n is arbitrary we have $WC_2 \in M'_+$. Hence, $WC_2 = \lambda I$ so $C_2 = \lambda W$ but since $WC_2W^* = -C_2$ we must have $\lambda = 0$. Hence $C = \lambda I$, and α_t is a shift of M_+ for $t > 0$.

Thus, we have shown that $\{\alpha_t;\ t \geqq 0\}$ is a continuous flow of shifts of M_+ and M_+ is isomorphic to $\mathscr{B}(\mathscr{H})$. Note the state $(\Omega_0, A\Omega_0)$ for $A \in M_+$ is pure on M_+ and α_t-invariant. To obtain an irreducible representation of M_+ one need only restrict M_+ to the closed span of $\{M_+\Omega_0\}$. In this way we obtain a flow of shifts of $\mathscr{B}(\mathscr{H})$ with $\mathscr{H} =$ the closure of $\{M_+\Omega_0\}$. We will call this example the CAR-flow of $\mathscr{B}(\mathscr{H})$.

We note the CAR-flow α_t of $\mathscr{B}(\mathscr{H})$ has the property that there exists a strongly continuous one parameter semigroup $\{U(t);\ 0 \leqq t < \infty\}$ of isometries having the property that

$$U(t)A = \alpha_t(A)U(t) \text{ for all } A \in \mathscr{B}(\mathscr{H}).$$

Note the $U(t)$ are just the restrictions of the $V(t)$ previously constructed to the subspace spanned by $\{M_+\Omega_0\}$. If α_t is an E_0-semigroup of $\mathscr{B}(\mathscr{H})$ with a pure normal α_t invariant state $\omega_0(\omega_0(A) = (f_0, Af_0))$ then such a strongly continuous semigroup of isometries can be constructed by the defining relation,

$$U(t)Af_0 = \alpha_t(A)f_0.$$

Question 4.1. If $\{\alpha_t;\ t \geqq 0\}$ is an E_0-semigroup of $\mathscr{B}(\mathscr{H})$ does there always exist a strongly continuous one parameter semigroup $\{U(t);\ t \geqq 0\}$ so that

$$U(t)A = \alpha_t(A)U(t) \text{ for all } A \in \mathscr{B}(\mathscr{H}) \text{ and } t \geqq 0?$$

If α_t is an E_0-semigroup of $\mathscr{B}(\mathscr{H})$ then one can construct for each $t > 0$ an isometry $U(t)$ so that

$$U(t)A = \alpha_t(A)U(t) \text{ for all } A \in \mathscr{B}(\mathscr{H}).$$

The question is can one choose the $U(t)$ so as to have a strongly continuous one parameter semigroup. We remark without giving a proof that the existence of such a strongly continuous semigroup $U(t)$ is equivalent to the existence of a rank one projection e_0 having the property that

$$t^{-1}(\alpha_t(e_0)e_0 - e_0)$$

converges in norm to a bounded operator as $t \to 0^+$.

If such a semigroup $U(t)$ exists then one can define an index for α_t as the next theorem will show.

THEOREM 4.2. *Suppose $\{\alpha_t; 0 \leqq t < \infty\}$ is an E_0-semigroup of $\mathscr{B}(\mathscr{H})$ and there is a strongly continuous one parameter semigroup of isometries $U(t)$ of $\mathscr{H}$ so that*

$$U(t)A = \alpha_t(A)U(t) \text{ for all } t \geqq 0.$$

*Let $\mathscr{M}$ be the subspace of $\mathscr{H}$ of vectors f so that $U(t)^*f = e^{-t}f$ for $t \geqq 0$ and let E be the projection onto $\mathscr{M}$. Let δ be the *-derivation of $\mathscr{B}(\mathscr{H})$ defined by*

$$\delta(A) = \lim_{t \to 0^+} (\alpha_t(A) - A)/t$$

where the domain $\mathscr{D}(\delta)$ of δ is the set of $A \in \mathscr{B}(\mathscr{H})$ so that the above limit exists in the sense of norm convergence. Then the mapping

$$A \to \Gamma(A) = E(A + \frac{1}{2}\delta(A))E$$

*is a *-representation of $\mathscr{D}(\delta)$ on $\mathscr{M}$. Furthermore this *-representation has a unique norm continuous extension to $\mathscr{B}_\alpha$ the C^*-algebra of all $A \in \mathscr{B}(\mathscr{H})$ so that*

$$\|\alpha_t(A) - A\| \to 0 \text{ as } t \to 0^+.$$

Proof. Suppose α_t is an E_0-semigroup of $\mathscr{B}(\mathscr{H})$ satisfying the hypothesis of the theorem. Let $-d$ be the generator of $U(t)$. More specifically we define

$$df = \lim_{t \to 0^+} (f - U(t)f)/t$$

where the domain $\mathscr{D}(d)$ of d is the set of all $f \in \mathscr{H}$ so that the limit exists in the sense of norm convergence. Let d^* be the hermitian adjoint of d. Note that since the $U(t)$ are isometric d is skew-hermitian so $-d^*$ is an extension of d (i.e., $-d^* \supset d$).

Then it follows from the theory of hermitian operators (see e.g. [**5**] Chapter XII Section 4) that each $f \in \mathscr{D}(d^*)$ can be uniquely expressed in the form $f = f_0 + f_+ + f_-$ where $d^*f_\pm = \pm f_\pm$ and $f_0 \in \mathscr{D}(d)$. There are no solutions to the equation $d^*f_- = -f_-$ since this would imply for $g \in \mathscr{D}(d)$ that

$$\frac{d}{dt}(f_-, U(t)g) = -(f_-, dU(t)g) = -(d^*f_-, U(t)g)$$
$$= (f_-, U(t)g).$$

Hence,

$$(f_-, U(t)g) = e^t(f_-, g) \text{ for } g \in \mathscr{D}(d)$$

and this contradicts the fact that the $U(t)$ are isometric. Hence, each $f \in \mathscr{D}(d^*)$ can be uniquely expressed in the form $f = f_0 + f_+$ with $f_0 \in \mathscr{D}(d)$ and $f_+ \in \mathscr{D}(d^*)$ with $d^*f_+ = f_+$. Note that the space of such vectors f_+ is precisely the space $\mathscr{M}$ of vectors $f \in \mathscr{H}$ so that

$$U(t)^*f_+ = e^{-t}f_+ \text{ for } t > 0.$$

We define a bilinear form $\langle\ ,\ \rangle$ on $\mathscr{D}(d^*)$ as follows:

$$\langle f, g\rangle = \frac{1}{2}(f, d^*g) + \frac{1}{2}(d^*f, g).$$

A straightforward computation shows that if $f = f_0 + f_+$ and $g = g_0 + g_+$ with $f_0, g_0 \in \mathscr{D}(d)$ and $f_+, g_+ \in \mathscr{M}$ then $\langle f, g\rangle = (f_+, g_+)$. Hence, the bilinear form $\langle\ ,\ \rangle$ is positive on $\mathscr{D}(d^*)$.

Let δ and $\mathscr{D}(\delta)$ be as given in the statement of the theorem. First we will show that if $A \in \mathscr{D}(\delta)$ then

$$A\mathscr{D}(d) \subset \mathscr{D}(d) \text{ and } dAf = -\delta(A)f + Adf.$$

To this end suppose $A \in \mathscr{D}(\delta)$ and $f \in \mathscr{D}(d)$. Then we have

$$\begin{aligned} t^{-1}(I - U(t))Af &= t^{-1}(I - \alpha_t(A)U(t))f \\ &= -t^{-1}(\alpha_t(A) - A)U(t)f \\ &\quad + t^{-1}A(I - U(t))f. \\ &\to -\delta(A)f + Adf \text{ as } t \to 0^+. \end{aligned}$$

Hence, $Af \in D(d)$ and $dAf = -\delta(A)f + Adf$. Now if $f \in \mathscr{D}(d^*)$, $A \in \mathscr{D}(\delta)$ and $g \in \mathscr{D}(\delta)$ then we have

$$\begin{aligned} (Af, dg) &= (f, A^*dg) = (f, dA^*g) + (f, \delta(A^*)g) \\ &= ((Ad^*f + \delta(A)f), g). \end{aligned}$$

Hence, we have for $A \in \mathscr{D}(\delta)$ that $A\mathscr{D}(d^*) \subset \mathscr{D}(d^*)$ and for $f \in \mathscr{D}(d^*)$

(*) $\quad d^*Af = Ad^*f + \delta(A)f.$

Using (*) we will show that the mapping $A \to A$ gives us a *-representation of $\mathscr{D}(\delta)$ on $\mathscr{D}(d^*)$ with respect to the bilinear form $\langle \cdot , \cdot \rangle$. To this end suppose $f, g \in \mathscr{D}(d^*)$ and $A \in \mathscr{D}(\delta)$. Then we have

$$\begin{aligned}\langle f, Ag\rangle &= \frac{1}{2}(d^*f, Ag) + \frac{1}{2}(f, d^*Ag)\\ &= \frac{1}{2}(A^*d^*f, g) + \frac{1}{2}(f, Ad^*g) + \frac{1}{2}(f, \delta(A)g)\\ &= \frac{1}{2}(d^*A^*f, g) + \frac{1}{2}(A^*f, d^*g) = \langle A^*f, g\rangle.\end{aligned}$$

If $f \in \mathscr{D}(d^*)$ and $\langle f, f\rangle = 0$ then $f \in \mathscr{D}(d)$. Hence, the mapping $A \to A$ gives a *-representation of $\mathscr{D}(\delta)$ on the quotient space $\mathscr{D}(d^*)$ mod $\mathscr{D}(d)$ with inner product $\langle \cdot , \cdot \rangle$. Given an $f \in \mathscr{D}(d^*)$ it has a unique decomposition $f = f_0 + f_+$ with $f_0 \in \mathscr{D}(d)$, $f_+ \in \mathscr{D}(d^*)$ and $d^*f_+ = f_+$. The vector f_+ uniquely determines the image of f in the quotient space $\mathscr{D}(d^*)$ mod $\mathscr{D}(d)$. Given $f \in \mathscr{D}(d^*)$ then f_+ is given by

$$f_+ = \frac{1}{2}E(f + d^*f).$$

Since $A \to A$ is a *-representation of $\mathscr{D}(\delta)$ on the quotient space and

$$f = \frac{1}{2}E(f + d^*f) \text{ mod } \mathscr{D}(d)$$

we have

$$A \to \frac{1}{2}E(I + d^*)AE$$

is a *-representation of $\mathscr{D}(\delta)$ on $\mathscr{M}$. Hence, from equation (*) we have

$$A \to \frac{1}{2}E(I + d^*)AE = E\left(A + \frac{1}{2}\delta(A)\right)E = \pi_\alpha(A)$$

is a *-representation of $\mathscr{D}(\delta)$ on $\mathscr{M}$.

Since $\mathscr{D}(\delta)$ is not norm closed we can not immediately conclude that π_α is norm continuous. We show π_α is norm continuous. Suppose $A = A^* \in \mathscr{D}(\delta)$ and $\|A\| \leqq 2/3$. Since $f(t) = t + i(1 - t^2)^{1/2}$ is twice differentiable in the interval $[-2/3, 2/3]$ it follows from the functional calculus of the domain of a *-derivation (see **[2]** Theorem 3.3.32 page 239) that

$$U = A + i(I - A^2)^{1/2} \in \mathscr{D}(\delta).$$

Since U is unitary and π_α is a *-representation we have

$$\|\pi_\alpha(U)\|^2 = \|\pi_\alpha(U^*U)\| = \|\pi_\alpha(I)\| = 1.$$

Hence,

$$\|\pi_\alpha(A)\| = \frac{1}{2}\|\pi_\alpha(U) + \pi_\alpha(U^*)\| \leqq \frac{1}{2} + \frac{1}{2} = 1.$$

Hence, $\|\pi_\alpha(A)\| \leqq 3/2\, \|A\|$ for all hermitian $A \in \mathscr{D}(\delta)$. Hence, π_α has a unique norm continuous extension to the norm closure of $\mathscr{D}(\delta)$. (Note the existence of this extension shows us $\|\pi_\alpha(A)\| \leqq \|A\|$ for $A \in \mathscr{D}(\delta)$.)

We claim $\mathscr{B}_\alpha$ is the norm closure of $\mathscr{D}(\delta)$. Since $\mathscr{D}(\delta) \subset \mathscr{B}_\alpha$ we have that $\mathscr{B}_\alpha$ contains the norm closure of $\mathscr{D}(\delta)$. Conversely, suppose $A \in \mathscr{B}_\alpha$. Let

$$A_n = n\int_0^{1/n} \alpha_t(A)dt \quad \text{for } n = 1, 2, \dots .$$

We have $A_n \in \mathscr{D}(\delta)$ and $A_n \to A$ in norm as $n \to \infty$. Hence, $\mathscr{B}_\alpha$ is the norm closure of $\mathscr{D}(\delta)$.

Definition 4.3. Suppose $\{\alpha_t;\ 0 \leqq t < \infty\}$ is an E_0-semigroup of $\mathscr{B}(\mathscr{H})$ and there is a strongly continuous one parameter semigroup $\{U(t);\ 0 \leqq t < \infty\}$ of isometries so that

$$U(t)A = \alpha_t(A)U(t) \text{ for all } A \in \mathscr{B}(\mathscr{H}) \text{ and } t \geqq 0.$$

Let π_α be the *-representation of $\mathscr{B}_\alpha$ constructed in Theorem 4.2. The *index* $i(\alpha_t)$ is defined as the multiplicity of π_α (i.e., $i(\alpha_t)$ is the maximal number of non-zero mutually orthogonal projections in the commutant of $\pi_\alpha(\mathscr{B}_\alpha)$).

Note the representation π_α is unchanged if the generator δ is perturbed by a bounded derivation, i.e., if β_t is a second E_0-semigroup of $\mathscr{B}(\mathscr{H})$ with generator δ_1 and

$$\delta_1(A) = \delta(A) + i[H, A] \text{ for } A \in \mathscr{D}(\delta)$$

and H is a bounded hermitian operator then π_α and π_β will be unitarily equivalent.

LEMMA 4.4. *Suppose for $k = 1, 2$ $\{U_i(t);\ 0 \leqq t < \infty\}$ are strongly continuous one parameter semigroups of $\mathscr{H}_k$ and $U(t) = U_1(t) \otimes U_2(t)$ is the tensor product of these semigroups acting on $\mathscr{H}_1 \otimes \mathscr{H}_2$. Let $-d_k$ be the generator of $U_k(t)$ and $-d$ be the generator of $U(t)$. Then*

$$\mathscr{D}(d_1^*) \otimes \mathscr{D}(d_2) \oplus \mathscr{D}(d_1) \otimes \mathscr{D}(d_2^*)$$

is a core for d^*.

Proof. Suppose the hypothesis and notation of the lemma are satisfied. Since $U_k(t)$ is a strongly continuous one parameter group of isometries the Hilbert space $\mathscr{H}_k$ and the $U_k(t)$ can be decomposed as follows (see e.g. page 328 of **[9]**):

(4.1) $\mathscr{H}_k = \mathscr{H}_k^0 \oplus (\mathscr{K}_k \otimes L^2(0, \infty))$, $U_k(t) = V_k(t) \oplus S_k(t)$

where $V_k(t)$ is a unitary group on $\mathscr{H}_k^0$ and

$$(S_k(t)f)(x) = f(x - t) \text{ for } x \geqq t \text{ and}$$

$$(S_k(t)f)(x) = 0 \text{ for } x < t$$

for $f(x)$ an $\mathscr{K}_k$ valued function so that

$$\int_0^\infty (f(x), f(x))dx < \infty.$$

We now show

$$\mathscr{D}_0 = \mathscr{D}(d_1^*) \otimes \mathscr{D}(d_2) \oplus \mathscr{D}(d_1) \otimes \mathscr{D}(d_2^*)$$

is a core for d^*. Suppose this is not the case. Then there is a point $\{F_0, d^*F_0\}$ in the graph of d^* which is orthogonal to $\{G, d^*G\}$ for all $G \in \mathscr{D}_0$. Then we have

(4.2) $(F_0, g_1 \otimes g_2) + (d^*F_0, (d_1^*g_1 \otimes g_2 + g_1 \otimes d_2^*g_2)) = 0$

for $g_1 \in \mathscr{D}(d_1^*)$, $g_2 \in \mathscr{D}(d_2)$ or for $g_1 \in \mathscr{D}(d_1)$, $g_2 \in \mathscr{D}(d_2^*)$. Since $-d^* \supset d$ we have

$$(F_0, g_1 \otimes g_2) - (d^*F_0, d(g_1 \otimes g_2)) = 0$$

for $g_1 \in \mathscr{D}(d_1)$ and $g_2 \in \mathscr{D}(d_2)$. Since

$$U(t)\mathscr{D}(d_1) \otimes \mathscr{D}(d_2) \subset \mathscr{D}(d_1) \otimes \mathscr{D}(d_2)$$

we have $\mathscr{D}(d_1) \otimes \mathscr{D}(d_2)$ is a core for d (see e.g. Corollary 3.1.7 page 167 of **[2]**). We have $F_0 \in \mathscr{D}(d^*)$ and

$$(F_0, g_1 \otimes g_2) = (d^*d^*F_0, g_1 \otimes g_2)$$

for all $g_1 \in \mathscr{D}(d_1)$ and $g_2 \in \mathscr{D}(d_2)$.

Hence, $d^*d^*F_0 = F_0$. Let

$$F_\pm = \frac{1}{2}(F_0 \pm d^*F_0).$$

We have

$$d^*F_\pm = \pm F_\pm, F_0 = F_+ + F_-.$$

Since the equation $d^*F = -F$ has no solution we have $F_- = 0$. Hence, $d^*F_0 = F_0$.

We can express F_0 in the form

$$F_0 = F_{00} + F_{01} + F_{10} + F_{11}$$

with

$$F_{00} \in \mathscr{H}_1^0 \otimes \mathscr{H}_2^0, F_{01} \in \mathscr{H}_1^0 \otimes \mathscr{K}_2 \otimes L^2(0, \infty),$$

$$F_{10} \in \mathscr{K}_1 \otimes \mathscr{H}_2^0 \otimes L^2(0, \infty) \text{ and}$$

$$F_{11} \in \mathscr{K}_1 \otimes \mathscr{K}_2 \otimes L^2((0, \infty)^2).$$

Since $U(t)^*F_0 = e^{-t}F_0$ for $t \geqq 0$ we have

$$F_{00} = 0,\ F_{01}(x) = e^{-x}(V_1(x) \otimes I)F_{01}(0),$$
$$F_{10}(x) = e^{-x}(I \otimes V_2(x))F_{10}(0) \text{ and}$$
$$F_{11} = F_{11}(x, y)$$

with

$$\text{(4.3)} \quad \text{and} \quad \begin{aligned} F_{11}(x, y) &= e^{-y}F_{11}(x - y, 0) \text{ for } x \geqq y \\ F_{11}(x, y) &= e^{-x}F_{11}(0, y - x) \text{ for } x \leqq y. \end{aligned}$$

Suppose $G = g_1 \otimes g_2$ with

$$g_1 \in \mathscr{H}_1^0 \cap \mathscr{D}(d_1) \text{ and } g_2 \in \mathscr{D}(d_2^*)$$

with $d_2^*g_2 = g_2$. Then from equation (4.2) and the facts that $d^*F_0 = F_0$ and $d_1^*g_1 = -d_1g_1$ we have

$$2(F_0, g_1 \otimes g_2) = (F_0, d_1g_1 \otimes g_2).$$

Let L be the linear functional

$$L(f) = (F_0, f \otimes g_2)$$

defined for $f \in \mathscr{H}_0^1$. By the Riesz representation theorem we have $L(f) = (h, f)$ with $h \in \mathscr{H}_0^1$. From the equation for F_0 we have

$$2(h, g_1) = (h, d_1g_1).$$

This equation implies $h \in \mathscr{D}(d_1^*)$ and $d_1^*h = 2h$, but this is only possible if $h = 0$ since

$$\mathscr{D}(d_1^*) \cap \mathscr{H}_0^1 = \mathscr{D}(d_1) \cap \mathscr{H}_0^1$$

(i.e., d_1 restricted to $\mathscr{H}_0^1$ is skewadjoint). Hence, $h = 0$ and

$$(F_0, g_1 \otimes g_2) = 0$$

for all $g_1 \in \mathscr{H}_1^0$ and $g_2 \in \mathscr{D}(d_2^*)$ with $d_2^*g_2 = g_2$. As F_0 was decomposed into four vectors F_{00}, F_{01}, etc., $G = g_1 \otimes g_2$ can be decomposed into four vectors $G = G_{00} + G_{01} + G_{10} + G_{11}$ and for the form of G under consideration we have $G_{00} = G_{10} = G_{11} = 0$ and G_{01} is of the form

$$G_{01} = G_{01}(x) = e^{-x}(k_1 \otimes g_1)$$

with k_1 any vector in $\mathscr{K}_1$. Then we have

$$\begin{aligned}(F_0, G) &= (F_{01}, G_{01}) \\ &= \int_0^\infty e^{-2x}((V_1(x) \otimes I)F_{01}(0), k_1 \otimes g_1)dx = 0\end{aligned}$$

for all $k_1 \in \mathscr{K}_1$ and $g_1 \in \mathscr{H}_1^0$. Since the operator

$$\int_0^\infty e^{-2x} V_1(x)dx = ((2I + d_1)|\mathscr{H}_1^0)^{-1}$$

maps only the zero vector to zero and since k_1 and g_1 can be freely chosen in their respective spaces it follows that $F_{01} = 0$. A similar argument shows $F_{10} = 0$. Hence we have $F_{00} = F_{01} = F_{10} = 0$.

It remains only to show that $F_{11} = 0$. To see this suppose $g_1 \in \mathscr{D}(d_1)$ and g_1 is orthogonal to $\mathscr{H}_1^0$. Then in the decomposition (4.1) we can represent $g_1 = g_1(x)$ where $g_1(x) \in \mathscr{K}_1$ is a differentiable function of x (strictly speaking, g_1 is absolutely continuous and, thus, differentiable almost everywhere) whose derivative is square integrable and satisfies the boundary condition $g_1(0) = 0$. Suppose $g_2 \in \mathscr{D}(d_2^*)$ and g_2 is orthogonal to $\mathscr{H}_2^0$. Then g_2 can be represented in the form $g_2(x)$ where $g_2(x) \in \mathscr{K}_2$ is a differentiable function whose derivative is square integrable and g_2 need not satisfy a boundary condition at $x = 0$. Then from equation (4.2) and the fact that $d^*F_0 = F_0$ we have

$$\int_0^\infty (F_{11}(x, y), (g_1(x) \otimes g_2(y) - \frac{d}{dx} g_1(x) \otimes g_2(y) - g_1(x) \otimes \frac{d}{dy} g_2(y))dxdy = 0.$$

Making a change of variable to new variables $\xi = (x + y)/2, \eta = -x + y$ and noting that in the new variables

$$F_{11}(\xi + t, \eta) = e^{-t} F_{11}(\xi, \eta)$$

we find, integrating by parts, that the above expression becomes

$$\int_0^\infty (F_{11}(x, 0), g_1(x) \otimes g_2(0))dx = 0.$$

Hence, $F_{11}(x, 0) = 0$ almost everywhere. Interchanging the roles of g_1 and g_2 we find $F_{11}(0, y) = 0$ almost everywhere. The function F_{11} is determined by its values on the boundary lines $y = 0$ and $x = 0$. In fact, one may calculate from the form for $F_{11}(x, y)$ given in equations (4.3) that

$$||F_{11}||^2 = \frac{1}{2} \int_0^\infty ||F_{11}(t, 0)||^2 + ||F_{11}(0, t)||^2 dt.$$

Since $F_{11}(t, 0) = F_{11}(0, t) = 0$ almost everywhere it follows that $F_{11} = 0$. Hence, $F_0 = 0$ and, therefore, $\mathscr{D}_0$ is a core for d^*.

THEOREM 4.5. *Suppose* $\{\alpha_t^{(k)}; 0 \leqq t < \infty\}$ *are* E_0-*semigroups of* $\mathscr{B}(\mathscr{H}_k)$ *and suppose there are strongly continuous one parameter semigroups* $(U_k(t); 0 \leqq t < \infty\}$ *so that*

$$U_k(t)A = \alpha_t^{(k)}(A)U_k(t) \text{ for } k = 1, 2.$$

Then $\alpha_t^{(1)} \otimes \alpha_t^{(2)}$ *is an* E_0*-semigroup of* $\mathscr{B}(\mathscr{H}_1 \otimes \mathscr{H}_2)$ *of index* $i \leqq i_1 + i_2$ *where* i_k *is the index of* $\alpha_t^{(k)}$.

Proof. Suppose $\{\alpha_t^{(k)}\}$ satisfy the hypothesis and notation of the theorem for $k = 1, 2$. One easily checks that $\alpha_t = \alpha_t^{(1)} \otimes \alpha_t^{(2)}$ is an E_0-semigroup of $\mathscr{B}(\mathscr{H}_1 \otimes \mathscr{H}_2)$ and $U(t) = U_1(t) \otimes U_2(t)$ is a strongly continuous one parameter semigroup of isometries so that

$$U(t)A = \alpha_t(A)U(t) \text{ for all } A \in \mathscr{B}(\mathscr{H}_1 \otimes \mathscr{H}_2) \text{ and } t \geqq 0.$$

Let $\langle \cdot\ , \cdot \rangle$ be the bilinear form on $\mathscr{D}(d^*)$ constructed from d^* as in Theorem 4.2 and let $\langle \cdot\ , \cdot \rangle_1$ and $\langle \cdot\ , \cdot \rangle_2$ be the bilinear forms constructed from d_1^* and d_2^* as in Theorem 4.2. Suppose $F = f_1 \otimes f_2 \in \mathscr{H}_1 \otimes \mathscr{H}_2$ and $G = g_1 \otimes g_2$ with $f_1, g_1 \in \mathscr{D}(d_1^*)$ and $f_2, g_2 \in \mathscr{D}(d_2)$. Suppose $A \in \mathscr{D}(\delta_1)$ and $B \in \mathscr{D}(\delta_2)$ where δ_1 and δ_2 are the generators of $\alpha_t^{(1)}$ and $\alpha_t^{(2)}$, respectively. Then $A \otimes B \in \mathscr{D}(\delta)$ where δ is the generator of α_t and

$$\delta(A \otimes B) = \delta_1(A) \otimes B + A \otimes \delta_2(B).$$

Then we have

$$\begin{aligned} \langle F, A \otimes BG \rangle = & \frac{1}{2}(d_1^* f_1 \otimes g_1, Ag_1 \otimes Bg_2) \\ & + \frac{1}{2}(f_1 \otimes d_2^* f_2, Ag_1 \otimes Bg_2) \\ & + \frac{1}{2}(f_1 \otimes g_1, d_1^* Ag_1 \otimes Bg_2) \\ & + \frac{1}{2}(f_1 \otimes f_2, Ag_1 \otimes d_2^* Bg_2). \end{aligned}$$

Note that since $f_2 \in \mathscr{D}(d_2)$ the second and fourth terms cancel in the above equation. Hence, we have

$$\langle F, A \otimes BG \rangle = \langle f_1, Ag_1 \rangle_1 (f_2, Bg_2).$$

Similarly, if $F = f_1 \otimes f_2$ and $G = g_1 \otimes g_2$ with $f_1, g_1 \in \mathscr{D}(d_1)$ and $f_2, g_2 \in \mathscr{D}(d_2^*)$ then for $A \in \mathscr{D}(\delta_1)$ and $B \in \mathscr{D}(\delta_2)$ we have

$$\langle F, A \otimes BG \rangle = (f_1, Ag_1)\langle f_2, Bg_2 \rangle_2.$$

If $F = f_1 \otimes f_2$ and $G = g_1 \otimes g_2$ with $f_1 \in \mathscr{D}(d_1)$, $f_2 \in \mathscr{D}(d_2^*)$, $g_1 \in \mathscr{D}(d_1^*)$ and $g_2 \in \mathscr{D}(d_2)$ then for $A \in \mathscr{D}(\delta_1)$ and $B \in \mathscr{D}(\delta_2)$ we find

$$\langle F, A \otimes BG \rangle = 0.$$

Similarly, if $f_1 \in \mathscr{D}(d_1^*)$, $f_2 \in \mathscr{D}(d_2)$, $g_1 \in \mathscr{D}(d_1)$ and $g_2 \in \mathscr{D}(d_2^*)$ then again we find $\langle F, A \otimes BG \rangle = 0$. Let π_α be the *-representation constructed from $\{\alpha_t\}$ as in Theorem 4.2 and let $\mathscr{B}_0$ be the C^*-algebra generated by elements of the form $A \otimes B$ with $A \in \mathscr{D}(\delta_1)$ and $B \in \mathscr{D}(\delta_2)$. Since the linear span of elements of the form $f_1 \otimes f_2$ with $f_1 \in \mathscr{D}(d_1^*)$ and $f_2 \in \mathscr{D}(d_2)$ or $f_1 \in \mathscr{D}(d_1)$ and $f_2 \in \mathscr{D}(d_2^*)$ are dense in the representation space of π_α by Lemma 4.4 we have that the representation π_α restricted to $\mathscr{B}_0$ is the direct sum of the representations $\pi_1 \otimes \varphi_2$ and $\varphi_1 \otimes \pi_2$ where π_i is the *-representation constructed from $\{\alpha_t^{(i)}\}$ and φ_i is the identity representation of $\mathscr{B}(\mathscr{H}_i)$ for $i = 1, 2$. It follows that the multiplicity of $\pi_\alpha|\mathscr{B}_0$ is the sum of the multiplicities of π_1 and π_2 (i.e., $i = i_1 + i_2$). Since $\mathscr{B}_0 \subset \mathscr{B}_\alpha$ it follows that the multiplicity of π_α is less than or equal to $i_1 + i_2$.

We believe that $\pi_\alpha(\mathscr{B}_0)$ is weakly dense in $\pi_\alpha(\mathscr{B}_\alpha)$ so the index i of α_t is, in fact, the sum of $i_1 + i_2$. This would mean that the index of E_0-semigroups is additive rather than subadditive.

Suppose $\{\alpha_t\}$ is the CAR-flow constructed earlier. One finds that if $F_1 = \pi(a(f_0))^*\Omega_0$ with $f_0(x) = \sqrt{2}\, e^{-x}$ then $F_1 \in \mathscr{D}(d^*)$ and

$$\langle \pi(a(f))F_1, \pi(a(f))F_1\rangle = 0 \text{ for all } f \in L^2(0, \infty)$$

with $f' \in L^2(0, \infty)$ and $f(0) = 0$. Hence, we find

$$\langle F_1, AF_1\rangle = (\Omega_0, A\Omega_0) \text{ for } A \in \mathscr{D}(\delta).$$

Using the argument of Lemma 4.4 one can show F_1 is cyclic for the representation π_α. Hence, for the CAR-flow π_α is unitarily equivalent to the identity representation of $\mathscr{B}_\alpha$ so π_α is normal and irreducible. Hence, the CAR-flow is of index one. If one forms the tensor product of CAR-flows one finds that the index is additive.

Question 4.2. Suppose $\{\alpha_t\}$ is an E_0-semigroup of $\mathscr{B}(\mathscr{H})$ and π_α is the representation constructed from $\{\alpha_t\}$ as in Theorem 4.2. Does π_α always have a normal extension to $\mathscr{B}(\mathscr{H})$?

If the answer to this question is yes it means that E_0-semigroups can be essentially classified up to outer conjugacy by the index of $\{\alpha_t\}$ and the CAR-flows are up to outer conjugacy the only examples of E_0-semigroups. On the other hand if there are E_0-semigroups of $\mathscr{B}(\mathscr{H})$ so that π_α has no normal extension to $\mathscr{B}(\mathscr{H})$ it means there are E_0-semigroups which are not outer conjugate to CAR-flows. Note we have not defined outer conjugacy for E_0-semigroups. There are a number of possible definitions and we feel it would be wise not to choose among them until the theory of E_0-semigroups is better understood.

5. Semigroups of *-endomorphisms of the hyperfinite II_1 factor. In this section we consider E_0-semigroups of the hyperfinite II_1 factor R and show that one can define an index i for such semigroups. We begin by giving an example of a flow of shifts of R. Let $\mathscr{U}_c$ be the Clifford algebra over $L^2(0, \infty)$. Specifically $\mathscr{U}_c$ is generated by elements $u(f)$ defined for

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real functions f in $L^2(0, \infty)$. The $u(f)$ satisfy the Clifford algebra relations,

$$u(\alpha f + g) = \alpha u(f) + u(g), \ u(f)^* = u(f)$$

and

$$u(f)u(g) + u(g)u(f) = 2(f, g)I$$

for α real and f and g real functions in $L^2(0, \infty)$. It is well known that $\mathcal{U}_c$ is isomorphic to the CAR algebra $\mathcal{U}$ discussed before. Let τ be the unique trace on $\mathcal{U}_c$ and let $(\pi, \Omega_0, \mathcal{H})$ be a cyclic *-representation induced by τ on a Hilbert space $\mathcal{H}$ with cyclic vector Ω_0. We have $R = \pi(\mathcal{U}_c)''$ is the hyperfinite II_1 factor. We define S_t on real functions in $L^2(0, \infty)$ as in the last section (i.e., $(S_tf(x) = f(x - t)$ for $x \geqq t$ and $(S_tf)(x) = 0$ for $x < t$). If

$$p = p(u(f_1), u(f_2), \dots, u(f_n))$$

is a polynomial in the $u(f)$ we define

$$\alpha_t(\pi(p)) = \pi(p(u(S_tf_1), u(S_tf_2), \dots, u(S_tf_n))).$$

One checks that α_t is weakly continuous and, therefore, has a weakly continuous extension to $\pi(\mathcal{U}_c)$ and this extension (which we also denote by α_t) is a continuous flow of shifts of R. We will call this flow the Clifford flow of R.

For the case of E_0-semigroups of type II_1 factors the existence of a semigroup $\{U(t): t \geqq 0\}$ is assured since the trace is invariant under endomorphisms. Using the following theorem one can define an index for E_0-semigroups of type II_1 factors.

THEOREM 5.1. *Suppose $\{\alpha_t; t \geqq 0\}$ is an E_0-semigroup of a type II_1 factor M. Let τ be the normalized trace on M and $\mathcal{H} = L^2(M, \tau)$ be the completion of M with respect to the inner product $(A, B) = \tau(A^*B)$ coming from the trace. For $F \in \mathcal{H}$ and $A \in M$ we denote the left and right actions of A on F by AF and FA. Let $\{U(t); t \geqq 0\}$ be the strongly continuous semigroup of isometries of $\mathcal{H}$ given by*

$$U(t)A = \alpha_t(A) \text{ for all } A \in M \text{ and } t \geqq 0.$$

*Let $\mathcal{M}$ be the subspace of vectors $F \in \mathcal{H}$ so that $U(t)^*F = e^{-t}F$ for $t \geqq 0$ and let E be the projection onto $\mathcal{M}$. Let δ be the *-derivation of M given by*

$$\delta(A) = \lim_{t \to 0^+} (\alpha_t(A) - A)/t$$

where the domain $\mathcal{D}(\delta)$ is the set of $A \in M$ so that the limit exists in the sense of norm convergence. For $F \in \mathcal{M}$ and $A \in \mathcal{D}(\delta)$ let $\pi_\alpha(A)F$ and $F\pi_\alpha(A)$ be given by

$$\pi_\alpha(A)F = E\left(\left(A + \frac{1}{2}\delta(A)\right)F\right) \text{ and}$$

$$F\pi_\alpha(A) = E\left(F\left(A + \frac{1}{2}\delta(A)\right)\right).$$

*Then $A \to \pi_\alpha(A)$ gives a right and left *-representation of $\mathscr{D}(\delta)$ on $\mathscr{M}$. Furthermore these right and left representations have a unique extension to $\mathscr{B}_\alpha$ the C^*-subalgebra of M of elements $A \in M$ so that*

$$\|\alpha_t(A) - A\| \to 0 \text{ as } t \to 0^+.$$

Proof. The proof for the left action is the same as the proof of Theorem 4.2 and taking adjoints (or just going through the proof again) gives the proof for the right action.

Definition 5.2. Suppose $\{\alpha_t; t \geqq 0\}$ is an E_0-semigroup of a II_1 factor M. Let π_α be the right and left representations of $\mathscr{B}_\alpha$ as constructed in Theorem 5.1. Then the *index* i of $\{\alpha_t\}$ is the multiplicity of π_α as a birepresentation of $\mathscr{B}_\alpha$ (i.e., the index i is the maximum number of mutually orthogonal non-zero projections which commute with both the right and left action of π_α).

Note that as in the last section the representation π_α is unchanged if the generator δ of α_t is perturbed by an inner derivation.

Following the proof of Theorem 4.5 one obtains the following theorem.

THEOREM 5.3. *Suppose $\{\alpha_t^{(k)}; t \geqq 0)$ are E_0-semigroups of type II_1 factors M_k of index i_k for $k = 1, 2$. Then $\alpha_t = \alpha_t^{(1)} \otimes \alpha_t^{(2)}$ is an E_0-semigroup of $M_1 \otimes M_2$ of index $i \leqq i_1 + i_2$.*

We remark that the Clifford flow discussed at the beginning of this section has index one. In fact, one has for $F_0 = u(f_0)$ with $f_0 = \sqrt{2}e^{-x}$ and $\langle \cdot , \cdot \rangle$ defined from $U(t)$ as in Theorem 4.2

$$(5.1) \quad \langle F_0, AF_0B\rangle = \mathrm{tr}(A\theta(B))$$

where θ is the *-automorphism of R given by

$$\theta(\pi(u(f))) = -\pi(u(f)) \text{ for all } f \in L^2(0, \infty).$$

By using Lemma 4.4 recursively one can show that F_0 is cyclic in $\mathscr{D}(d^*)$ mod $\mathscr{D}(d)$ with respect to the $\langle \cdot , \cdot \rangle$ inner product for both the right and left actions of $\mathscr{D}(\delta)$.

Note that both the right and left actions of π_α are normal. This leads one to wonder.

Question 5.1. Suppose $\{\alpha_t; t \geqq 0\}$ is an E_0-semigroup of a II_1 factor. Is the left representation π_α always normal?

We remark that for the case of continuous flows of shifts of the hyperfinite II_1 factor R of index one there appear to be flows which are not outer conjugate (given any reasonable definition of outer conjugacy). This may be seen as follows. Consider the Clifford flow described at the beginning of this section. Let R_1 be the subfactor of R generated by even polynomials in the $\pi(u(f))$ for $f \in L^2(0, \infty)$. Restricting α_t to R_1 one obtains a continuous flow of shifts of R_1 which we will denote by α_t^1. One easily sees that α_t and α_t^1 are not conjugate since

$$A\alpha_t^1(B) - \alpha_t^1(B)A \rightarrow 0$$

strongly as $t \rightarrow \infty$ for all $A, B \in R_1$ and for $A = B = \pi(u(f))$ one sees the same statement is false for α_t. We suspect that α_t and α_t^1 are not outer conjugate given any reasonable definition of outer conjugacy.

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