THE MOTION OF A RIGID SPHERE EMBEDDED IN AN ADHERING ELASTIC OR VISCOELASTIC MEDIUM

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Abstract. The motion of a rigid sphere embedded in an adhering medium and subjected to an external force is analysed exactly in the context of classical elastodynamics. For the limiting case of an incompressible medium it is possible to write down a simple second order differential equation relating displacement of the ball to the external force.

The theory is generalised to the case of a viscoelastic solid and the results obtained are pertinent to recently developed experimental methods for testing the dynamical mechanical properties of very low modulus polymer gels.

1. Introduction

The present paper is concerned with the problem of a rigid sphere embedded in a sticky elastic or viscoelastic medium and subjected to an external time dependent force. The sticky nature of the medium is taken into account in the sense that the boundary conditions assumed at the surface of the ball ensure that material initially in contact with the surface remains in contact.

This problem was suggested to the author by Professor M. Gordon and arises from the following experimental situation. In the course of formation of any network polymer, a chemical " cross linking " reaction converts a liquid system to a gel at a critical (gel) point. Thereafter, the elastic shear modulus of the solid gel increases monotonically. The rate of increase of modulus, at first vanishingly slow, subsequently increases to a maximum and then finally dies away again as the chemical equilibrium of a glassy state is approached. Intermediately, the gel has rubbery-elastic type properties with shear modulus ranging from 10^2-10^5 dynes/cm^2. This region is of especial interest from the viewpoint of developing molecular theories of rubber elasticity (Dobson and Gordon (1965), (1966)) and it is desirable to have available experimental techniques for measurement of the relevant shear moduli. It is not possible to test such soft materials by conventional methods. At first it was proposed to use ball indentation techniques obtaining the shear modulus from the Hertz theory of contact stresses. Unfortunately the experiment is complicated by surface tension effects and this method was abandoned in favour of using a completely immersed metallic ball which is displaced in the medium by an external magnetic field. Measurement of the ratio (Magnetic force/ball displacement) then leads

directly to a value of the elastic shear modulus for a near incompressible medium (equations (1.1), (1.2) below). The general case for this method of testing as a rational geometry for the measurement of viscoelastic and elastic parameters of rubbery materials has been argued elsewhere (Gordon, Hunter and Ward (1967)).

It is an easy calculation in elastostatics to derive the formula

$$E = \frac{24\pi(1-v)}{(5-6v)} \mu a \xi \tag{1.1}$$

relating the magnitude of the applied force E to the ball displacement ξ; other parameters appearing in equation (1.1) are the radius of the ball a, and the shear modulus μ and Poisson's ration v for the medium. For the case of an incompressible solid $v = \frac{1}{2}$ and (1.1) reduces to

$$E = 6\pi\mu a\xi \tag{1.2}$$

which result is reminiscent of Stoke's formula for the steady slow motion of a sphere through a viscous liquid. In obtaining equation (1.1) we have assumed the sticky nature of the medium.*

Equation (1.1) is valid for small static displacements of the ball in an elastic medium (i.e. for $\xi/a \ll 1$). However, in view of the interest in measuring dynamic shear moduli (for which purpose it is imperative to use a ball in motion), the question arises as to whether (1.1) provides a valid first approximation for the dynamic case, and also the nature of correction terms accounting for inertial effects in the medium. A possible method for assessing first order inertial corrections is to use energy variational techniques in which kinetic and potential energies are evaluated using a quasi static displacement field. The latter is algebraically identical with the static field, but the parameter ξ is allowed to depend on time. This method fails for the present problem because the static displacement field is of order r^{-1} for large r and the resulting kinetic energy integral diverges. In fact the far field of the true dynamic problem is quite different from that of the static problem, and the only method of computing the dynamic analogue of equation (1.1) is to solve the equations of elastodynamics.

It is with the latter problem that this paper is concerned. Section 2 deals with the motion of a rigid sphere in an elastic medium and the extension to the viscoelastic case follows in Section 3. In both cases considerable simplifications ensue with the assumption of incompressibility; this is a highly valid approximation for the polymer solutions, for which typical values of the bulk modulus are 10^{10} dynes/cm^2. For both elastic and viscoelastic cases we consider periodic

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* The corresponding non-sticky problem, defined by boundary conditions of zero tangential shear stress and non-tensile normal stress at the surface $r = a$ of the sphere, is much more difficult. For this problem the surface $r = a$ is divided into a contact region with prescribed normal displacement and a stress free region. The resulting mixed boundary value problem leads to a pair of dual series equations in Legendre Polynomials.

motion and motion generated impulsively from rest by a delta function type force.

F. M. Leslie (1961) has solved the related problem of the slow steady motion of a rigid sphere in a viscoelastic liquid of the Oldroyd type. There are no points of comparison between the present work and that of Leslie. Leslie's analysis is not limited to small deformations as is the case here; on the other hand the present paper takes account of both inertial and unsteady effects for arbitrary linear viscoelastic materials.

2. Elastic analysis

Without loss of generality we assume motion of the sphere $\xi = \xi(t)$ to take place in the z direction. Choosing spherical polar coordinates, r, θ, ϕ defined by

$$z = r \cos \theta, \quad x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi$$

the boundary conditions at $r = a$ for a medium which adheres to the sphere are

$$u_r = \xi(t) \cos \theta, \quad u_\theta = -\xi(t) \sin \theta \tag{2.1}$$

where u_r and u_θ are respectively displacement components in the r and θ directions, and where for $\xi = 0$ the centre of the sphere lies at $r = 0$.

The equations of elastodynamics are satisfied by a displacement field

$$u = \operatorname{grad} \Phi + \operatorname{curl} A \tag{2.2}$$

where the scalar and vector potentials are respectively solutions of the wave equations

$$\nabla^2 \Phi = c_1^{-2} \partial^2 \Phi / \partial t^2, \quad \nabla^2 A = c_2^{-2} \partial^2 A / \partial t^2 \tag{2.3}$$

Here c_1 and c_2 are the speeds of dilatational and shear waves; in terms of the Lamé constants λ, μ and the density ρ

$$c_1^2 = (\lambda + 2\mu)/\rho, \quad c_2^2 = \mu/\rho \tag{2.4}$$

The axisymmetric problem defined by the boundary conditions (2.1) is met by assuming

$$\Phi = \psi(r, t) \cos \theta, \ A_r = A_\theta = 0, \ A_\phi = B(r, t) \sin \theta. \tag{2.5}$$

where A_r, A_θ and A_ϕ are the spherical polar components of A. Substituting (2.5) into equations (2.3) leads to the equations

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \frac{\partial \psi}{\partial r} - \frac{2\psi}{r^2} = c_1^{-2} \frac{\partial^2 \psi}{\partial t^2}, \quad \frac{\partial^2 B}{\partial r^2} + \frac{2}{r} \frac{\partial B}{\partial r} - \frac{2B}{r^2} = c_2^{-2} \frac{\partial^2 B}{\partial t^2} \tag{2.6}$$

with outgoing wave solutions

$$\psi = \frac{\partial}{\partial r} \left[r^{-1} \chi_1(t - (r-a)/c_1) \right], \quad B = \frac{\partial}{\partial r} \left[r^{-1} \chi_2(t - (r-a)/c_2) \right] \tag{2.7}$$

in which χ_1 and χ_2 are arbitrary functions of the indicated arguments. For

subsequent convenience we have incorporated the constants a/c_1 and a/c_2 into the arguments of χ_1 and χ_2.

From (2.2) and (2.7) we derive the following expressions for the displacement components u_r and u_θ,

$$u_r = \cos\theta(\ddot{\chi}_1/c_1^2 r + 2\dot{\chi}_1/c_1 r^2 + 2\chi_1/r^3 - 2\dot{\chi}_2/c_2 r^2 - 2\chi_2/r^3) \tag{2.8}$$

$$u_\theta = -\sin\theta(\ddot{\chi}_2/c_2^2 r + \dot{\chi}_2/c_2 r^2 + \chi_2/r^3 - \dot{\chi}_1/c_1 r^2 - \chi_1/r^3) \tag{2.9}$$

where the fluxion dots denote differentiation with respect to the arguments of χ_1 and χ_2 (and also differentiation with respect to time).

The boundary conditions (2.1) now yield the pair of coupled ordinary differential equations for $\chi_1(t)$ and $\chi_2(t)$,

$$\ddot{\chi}_1/c_1^2 a + 2\dot{\chi}_1/c_1 a^2 + 2\chi_1/a^3 - 2\dot{\chi}_2/c_2 a^2 - 2\chi_2/a^3 = \zeta(t) \tag{2.10}$$

$$\ddot{\chi}_2/c_2^2 a + \dot{\chi}_2/c_2 a^2 + \chi_2/a^3 - \dot{\chi}_1/c_1 a^2 - \chi_1/a^3 = \zeta(t) \tag{2.11}$$

For motion initiated at $t = 0$ the initial conditions on χ_1, χ_2 and their derivatives derive from the requirement of material continuity at the wave fronts $r = a + c_1 t$ and $r = a + c_2 t$. If no fracture occurs for all positions of the wave fronts equations (2.8) and (2.9) necessitate

$$\chi_1 = \dot{\chi}_1 = \ddot{\chi}_1 = \chi_2 = \dot{\chi}_2 = \ddot{\chi}_2 = 0 \text{ for } t = 0. \tag{2.12}$$

However, since for $t = 0$, $\xi = 0$, equations (2.10) and (2.11) imply that the conditions $\ddot{\chi}_1(0) = 0$, $\ddot{\chi}_2(0) = 0$ are redundant. It follows that equations (2.12) yield a necessary and sufficient number of initial conditions (4) to effect a complete solution of (2.10) and (2.11) for specified $\xi(t)$.

While (2.10), (2.11) and (2.12) solve the proposed problem for given $\xi(t)$, the present paper is concerned primarily with the determination of $\xi(t)$ for a given external force $E(t)$. For this purpose we require to compute the resistive force exerted by the medium on the sphere. At the surface $r = a$ we find from (2.8) and (2.9) the following values of the strain components

$$\varepsilon_{rr} = \cos\theta F(t), \quad \varepsilon_{\theta\theta} = \varepsilon_{\phi\phi} = 0, \quad \varepsilon_{r\theta} = \tfrac{1}{2}\sin\theta G(t) \tag{2.13}$$

where F and G are given by

$$F(t) = -\dddot{\chi}_1/c_1^3 a - 3\ddot{\chi}_1/c_1^2 a^2 - 6\dot{\chi}_1/c_1 a^3 - 6\chi_1/a^4$$
$$+ 2\ddot{\chi}_2/c_2^2 a^2 + 6\dot{\chi}_2/c_2 a^3 + 6\chi_2/a^4 \tag{2.14}$$

$$G(t) = \dddot{\chi}_2/c_2^3 a + 3\ddot{\chi}_2/c_2^2 a^2 + 6\dot{\chi}_2/c_2 a^3 + 6\chi_2/a^4$$
$$- 2\ddot{\chi}_1/c_1^2 a^2 - 6\dot{\chi}_1/c_1 a^3 - 6\chi_1/a^4 \tag{2.15}$$

The results that the hoop strains vanish at $r = a$ is a direct consequence of assuming the sticky boundary conditions (2.1). From Hooke's law and (2.13) the stresses σ_{rr} and $\sigma_{r\theta}$ at $r = a$ are

$$\sigma_{rr} \equiv \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\phi\phi}) + 2\mu\varepsilon_{rr} = (\lambda + 2\mu)\varepsilon_{rr} = \rho c_1^2 F(t)\cos\theta \tag{2.16}$$

$$\sigma_{r\theta} \equiv 2\mu\varepsilon_{r\theta} = \rho c_2^2 G(t)\sin\theta \tag{2.17}$$

The total internal force $I(t)$ exerted by the medium on the sphere lies in the z direction and is given by the integral

$$I(t) = \int_S (\cos \theta \, \sigma_{rr} - \sin \theta \, \sigma_{r\theta}) \, dS$$

evaluated over the surface of the sphere $r = a$. Substituting from (2.16) and (2.17) leads finally to

$$I(t) = \tfrac{4}{3}\pi\rho a^2(c_1^2 F - 2c_2^2 G)$$
$$\equiv (M/a)(c_1^2 F - 2c_2^2 G) \tag{2.18}$$

where M is the mass of media displaced by the sphere. Our equations are completed by Newton's law of accelerated motion for the sphere,

$$E(t) = m\ddot{\xi} - I(t) \tag{2.19}$$

where m is the mass of the sphere.

For given external force $E(t)$, the equations (2.10), (2.11), (2.14), (2.15), (2.18) and (2.19) may be combined to yield a single sixth order ordinary differential equation relating $\xi(t)$ to $E(t)$. The algebra is cumbersome and complicated by the necessity of expressing the initial conditions (2.12) entirely in terms of $\xi(t)$. A much more attractive approach is through the use of periodic solutions which may be used to generate general solutions by Fourier analysis techniques. Further the periodic solutions are (a) of direct interest (b) readily generalise to the viscoelastic case and (c) obviate the necessity of considering initial conditions.

We look for solutions ξ, χ_1, χ_2, E and I of our equations in the form

$$\xi = \bar{\xi}(\omega)e^{i\omega t}, \chi_1 = \bar{\chi}_1(\omega)e^{i\omega t}, I = \bar{I}(\omega)e^{i\omega t} \text{ etc.} \tag{2.20}$$

Substituting from (2.20) into (2.10), (2.11), (2.14), (2.15), (2.18) and eliminating χ_1, χ_2, F, G, there results (after some algebra)

$$\bar{I}(\omega) = -Q(\omega)\bar{\xi}(\omega) \tag{2.21}$$

with $Q(\omega)$ given by

$$Q = \frac{Mc_1^2[9 + 9i(1+\gamma)\Omega - (2+9\gamma+\gamma^2)\Omega^2 - i\gamma(2+\gamma)\Omega^3]}{a^2[1 + 2\gamma^2 + i\gamma(1+2\gamma)\Omega - \gamma^2\Omega^2]} \tag{2.22}$$

where

$$\gamma = c_1/c_2 = [2(1-v)/(1-2v)]^{\frac{1}{2}} \tag{2.23}$$
$$\Omega = a\omega/c_1 \tag{2.24}$$

Also from (2.19)

$$\bar{E}(\omega) = -(m\omega^2\bar{\xi} + \bar{I}) \equiv -[m\omega^2 - Q(\omega)]\bar{\xi} \tag{2.25}$$

which completes the solution of the periodic problem.

To obtain solutions for arbitrary $E(t)$ we write

$$E(t) = \int_{-\infty}^{\infty} \bar{E}(\omega)e^{i\omega t}d\omega$$

where $\bar{E}$ is obtained from Fourier's theorem. From (2.25) the corresponding $\xi(t)$ is given by

$$\xi(t) = -\int_{-\infty}^{\infty} \frac{\bar{E}(\omega)e^{i\omega t}d\omega}{m\omega^2 - Q(\omega)} \qquad (2.26)$$

The simplest non-periodic problem is that of a sphere set into motion impulsively at $t = 0$ by a force

$$E(t) = P_0\delta(t) \text{ i.e. } \bar{E}(\omega) = P_0/2\pi \qquad (2.27)$$

Substituting from (2.27) for $\bar{E}$ into (2.26) yields a Fourier integral whose inversion is straightforward in principle. However the practice entails solving the equation $m\omega^2 - Q(\omega) = 0$ and this is essentially a quartic equation in $i\omega$ with real coefficients which are functions of two parameters $(\gamma, M/m)$. The solution could only be effected by numerical methods and in view of the simplifications which ensue with the assumption of incompressibility, we indicate only the form the solution takes for the compressible problem. The solution is of the form

$$\xi = \sum_{i=1}^{4} A_i e^{\alpha_i t} \qquad (2.28)$$

where the α_i are solutions of a quartic equation with real coefficients. For all $\gamma(1 < \gamma < \infty)$, and for sufficiently small values of M/m, the α_i comprise two complex conjugate pairs of numbers with negative real part; thus (2.28) is a sum of damped harmonic terms. With increasing (M/m) one pair of the α_i become real and negative and with further increase of M/m all the exponents become real and negative. Realistic values of M/m for the practical situation of a metal ball (density 4.0-8.0) immersed in a polymer solution of density ~ 1.0 lead to

$$0.125 < M/m < 0.25 \qquad (2.29)$$

and for this range of values the α_i are always complex (see also the incompressible results below). The only result of the compressible calculation of subsequent interest is the initial velocity with which the ball is set into motion. The detailed solution (2.28) yields

$$\dot{\xi}(t = 0+) = P_0/m \qquad (2.30)$$

in accord with the law of momentum generation.

Two limiting cases of the general theory are of interest. We note first that in the limit of low frequency behaviour

$$Lt_{\omega \to 0}Q(\omega) = -\frac{9Mc_1^2}{a^2(1 + 2\gamma^2)}$$

$$= -\frac{24\pi(1-v)\mu a}{(5-6v)}$$

in agreement with the static result (1.1). Of more interest is the limit of $(Q\omega)$

for the case of incompressibility ($v\to\frac{1}{2}$). Taking account of (2.23), (2.24) we find

$$Lt_{v\to\frac{1}{2}}Q(\omega) = M\left[-\tfrac{1}{2}\omega^2 + \frac{9i}{2}(c_2/a)\omega + \tfrac{9}{2}(c_2/a)^2\right] \qquad (2.31)$$

which result is of such algebraic simplicity that the equation resulting from substituting (2.31) into (2.19) can be inverted directly into the time domain to yield the following differential equation for $\xi(t)$

$$(m+\tfrac{1}{2}M)\ddot{\xi} + \tfrac{9}{2}(Mc_2/a)\dot{\xi} + \tfrac{9}{2}(Mc_2^2/a^2)\zeta = E(t) \qquad (2.32)$$

Equation (2.32) is susceptible to an interpretation in which the ball behaves with an effective mass $(m+\tfrac{1}{2}M)$, while the behaviour of the medium is modelled by a linear spring in series with a dashpot dissipative mechanism (the actual dissipative mechanism is the radiation of elastic energy away from the ball).

The principal curiosity in (2.32) is the result that for an initially quiescent system subject to a delta function force (2.27), the initial velocity is given by

$$\dot{\xi}(t = 0+) = P_0/(m+\tfrac{1}{2}M) \qquad (2.33)$$

in contrast to (2.30), and in seeming contradiction with the momentum law. The apparent anomaly is resolved by appealing to the limit $v\to\frac{1}{2}$ in the compressible solution (2.28). For small values of $(1-2v)$, the (negative) real parts of two of the exponents α_i are of order $O(1-2v)^{-\frac{1}{2}}$, while the associated coefficients are of order $O(1-2v)^{\frac{1}{2}}$. As $v\to\frac{1}{2}$ the pair of terms in question tend to zero for all t; however, consideration of the derivative shows that while the terms yield no contribution as $v\to\frac{1}{2}$ for $t>0$, there is a finite contribution to $\dot{\xi}$ at $t = 0$. Thus initially $\dot{\xi}(0+)$ is given by (2.30) and this is followed by an instantaneous decay to the value (2.33). Clearly the instantaneous decay is associated with the limiting process whereby $c_1\to\infty$, allowing the instantaneous transmission of momentum (and kinetic energy) to the medium. In reality this transmission occurs in a finite time of order $a/c_1 \sim 10^{-6}$ secs. for a ball of 2 mm. diameter; the corresponding time scale for the solution of (2.32) is of order

$$a/c_2 \sim 10^{-2} - 10^{-4} \text{ secs.}$$

In the context of the incompressible limit we note that for polymer solutions the bulk modulus is approximately 10^{10} dynes/cm² so that for shear moduli in the range 10^2-10^6 dynes/cm²

$$1-2v \sim 10^{-4} - 10^{-8}$$

and the incompressible approximation is excellent. From the theoretical viewpoint it is to be noted that it is not possible to introduce the limit $v\to\frac{1}{2}$ at any stage prior to (2.31) and obtain physically meaningful results.

Equation (2.32) is so elementary that there is no necessity to discuss detailed solutions beyond the case of impulsive motion defined by (2.27). For this case we find for $M/m<1.6$,

$$\xi(t) = [P_0/\beta_1(m+\tfrac{1}{2}M)]e^{-\delta_1 t}\sin(\beta_1 t) \qquad (2.34)$$

where

$$\delta_1 = c_2\delta/a, \quad \beta_1 = c_2/\beta a$$

and where

$$\delta = \frac{2\cdot25M/m}{1+\frac{1}{2}M/m}, \quad \beta = \frac{1\cdot5(2M/m-5M^2/4m^2)^{\frac{1}{2}}}{1+\frac{1}{2}M/m} \qquad (2.35)$$

For $M/m > 1\cdot6$ the solution reduces to the sum of two exponentially decaying terms, but this is a rather academic result in view of (2.29).

Typically $M/m = 0\cdot15$ and

$$\delta = 0\cdot314, \quad \beta = 0\cdot727$$

so that the motion is virtually extinguished within two cycles of the trigonometric term [see Fig. 2.]

3. Viscoelastic analysis

To solve the problem of motion in a viscoelastic solid it is only necessary to replace the Lamé constants λ and μ in the periodic elastic solutions by the corresponding complex moduli $\lambda(i\omega)$ and $\mu(i\omega)$ (defined e.g. as in Hunter (1960)).

We consider in detail the incompressible case, solved for an elastic solid by (2.26) with $Q(\omega)$ given by (2.31). The Lamé constant λ has disappeared in the limiting process while μ is retained implicitly in (2.31) through

$$c_2 = (\mu/\rho)^{\frac{1}{2}}$$

Carrying out the indicated substitution leads to the general solution

$$\xi(t) = -\int_\infty^\infty \frac{\bar{E}(\omega)e^{i\omega t}d\omega}{[(m+\frac{1}{2}M)\omega^2 - \frac{9}{2}iM\omega(\mu(i\omega)/\rho)^{\frac{1}{2}}/a - \frac{9}{2}M\mu(i\omega)/\rho a^2]} \qquad (3.1)$$

which replaces (2.32) for the viscoelastic case. It is no longer possible to express the motion in terms of a differential equation.

The evaluation of (3.1) is simplified by recalling that $\mu(i\omega)$ is the representation of a function $\mu(S)$ of the complex variable S along the imaginary axis $S = i\omega$ (e.g. see Hunter—ibid). Changing the variable from ω to S in (3.1) and considering again the case of impulsive motion, leads to the inverse Laplace transform

$$\xi(t) = (2\pi i)^{-1}P_0 \int_{k-i\infty}^{k+i\infty} \frac{e^{St}dS}{[(m+\frac{1}{2}M)S^2 + \frac{9}{2}MS(\mu(S)/\rho)^{\frac{1}{2}}/a + \frac{9}{2}M\mu(S)/\rho a^2]} \qquad (3.2)$$

where $k > 0$ and the path of integration lies to the right of any singularities of the integrand. It may be shown that the integrand in (3.2) is free from singularities for $\text{Re}(S) > 0$ and this justifies the transition from a Fourier to a Laplace integral.

Further progress is possible only by specifying in detail the transform modulus $\mu(S)$. For $\mu(S) = $ constant we recover the elastic solution (2.34). This result may be compared with experimental profiles to yield an estimate of c_2/a and hence a direct estimate of the only relevant mechanical property i.e.

$\mu = \rho c_2^2$. This approach could provide the basis for evaluating the " dynamic modulus under the conditions of the experiment " in the most naive model of a viscoelastic solid.

A more sophisticated viscoelastic model is that of a Voigt solid for which the constitutive equation relating shear stress Σ to shear strain ε is

$$\Sigma = 2\mu_0(\varepsilon + T\dot{\varepsilon}) \qquad (3.3)$$

Here μ_0 is the static shear modulus while T is a parameter of dimension time. For the Voigt solid (3.3), the transform modulus is

$$\mu(S) = \mu_0(1 + ST) \qquad (3.4)$$

and it is possible to effect the inversion (3.2) in closed form. We find

$$\xi = \frac{P_0 a e^{-t/T}}{2m(+\tfrac{1}{2}M)\beta c_0(\pi t/T)^{\frac{1}{4}}} \int_0^\infty e^{-u^2 T/4t} g(u) du \qquad (3.5)$$

where $g(u)$ is the function

$$g(u) = e^{-\lambda_1 u} \sin((\phi + \lambda_2)u) - e^{-\lambda_3 u} \sin((\phi + \lambda_4)u)$$

and the various parameters are defined by

$$c_0 = (\mu_0/\rho)^{\frac{1}{4}}, \qquad h = c_0 T/a$$

$$\lambda_1 = \tfrac{1}{2}h\delta - R\cos\phi \qquad \lambda_2 = \tfrac{1}{2}h\beta - R\sin\phi$$

$$\lambda_3 = \tfrac{1}{2}h\delta + R\cos\phi \qquad \lambda_4 = \tfrac{1}{2}h\beta + R\sin\phi$$

$$R^2 e^{2i\phi} = 1 - \tfrac{1}{2}h^2\delta(1-\delta) + \tfrac{1}{2}ih^2\delta\beta \quad (R > 0, 0 < \phi < \pi/2).$$

In these equations δ and β are as defined in (2.35) and the assumption $(M/m) < 1.6$ has been retained.

Equation (3.5) is not in a form that is useful for either indicating the effects introduced by viscoelasticity or comparison with experiment. For the latter purpose it would be necessary to invest considerable computational effort, and this would hardly be justified for the Voigt solid which provides a valid viscoelastic model for real materials only over a very limited frequency range. In view of the algebraic complexity of (3.5) it is clear that attempts to evaluate (3.2) for any finite element model (of which the Voigt is the simplest) will lead to expressions which are virtually useless to the experimentalist.

A model, much more useful for present purposes, is the one based on the assumption of constant loss angle Δ. With a number of acceptable approximations, the model leads to an expression for $\xi(t)$ which is readily susceptible to numerical analysis. Also the model provides a valid description of many real viscoelastic materials over wide ranges of frequency (e.g. see Kolsky (1956), Hunter (ibid)). The loss angle $\Delta(\omega)$ is an odd function of ω defined by

$$\tan\Delta = \mu_2/\mu_1$$

where μ_1 and μ_2 are respectively the real and imaginary parts of $\mu(i\omega)$. It is an experimental observation that over wide frequency ranges many viscoelastic

materials display a linear dependence of μ_1 with $\log(\omega)$ and concomitantly a negligible variation in $\Delta(\omega)$. Both of these results are modelled by the solid with transform modulus

$$\mu(S) = KS^n \tag{3.6}$$

where K and n are real constants. On substituting $S = \omega e^{\pm i\pi/2}$ we have immediately

$$\tan\Delta = \quad \tan(\tfrac{1}{2}n\pi), \quad \omega > 0;$$
$$= -\tan(\tfrac{1}{2}n\pi), \quad \omega < 0$$

so that for positive ω the index n is related to the loss angle by

$$n = 2\Delta/\pi$$

and Δ is precisely constant. Also (3.6) leads to

$$\mu_r(\omega) = K\cos\Delta e^{n\log\omega}$$
$$= \mu_r(\omega_0)e^{n\log(\omega/\omega_0)}$$
$$\simeq \mu_r(\omega_0)\left(1 + \frac{2\Delta}{\pi}\log(\omega/\omega_0)\right) \tag{3.7}$$

which for $\Delta \ll 1$ is a valid expansion for a wide frequency range [in practice from two to four decades]. In (3.7) ω_0 is a parameter that may be chosen arbitrarily to select an appropriate frequency range pertinent to the experimental conditions. Thus in the present application ω_0 would lie near the centre of the Fourier spectrum of $\xi(t)$. The linear dependence of μ_r on $\log\omega$ in (3.7) is in accord with the experimental observations.*

On substituting (3.6) into (3.2) and scaling appropriately the integration variable we obtain

$$\xi = \frac{P_0\tau}{(m+\tfrac{1}{2}M)}H(t/\tau) \tag{3.8}$$

where H is the contour integral

$$H(t/\tau) = (2\pi i)^{-1}\int_{k-i\infty}^{k+i\infty}\frac{e^{St/\tau}dS}{S^2 + 2\delta S^{1+\frac{1}{2}n} + 2\delta S^n} \tag{3.9}$$

and where τ the time scale is defined by

$$\tau = (\rho a^2/K)^{\frac{\frac{1}{2}}{1-\frac{1}{2}n}} \tag{3.10}$$

The integrand in (3.9) possesses simple poles at the points S_1, S_2

$$S_{1,2} = (-\delta \pm i\beta)^{\frac{1}{(1-\frac{1}{2}n)}} \tag{3.11}$$

and a branch point at $S = 0$. To evaluate $H(t/\tau)$ we complete the Bromwich

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* All of these results are known (e.g. (3.7) appears in Kolsky (1956)) but the connection of the model with (3.6) appears to have remained unnoticed. In particular the present derivation of (3.7) from (3.6) is trivial in comparison with previous analyses of the constant loss angle model.

contour as indicated in Fig. 1 and apply Cauchy's theorem. In the limi twhere the radii of C_2 and C_6 are indefinitely large and the radius of C_4 indefinitely
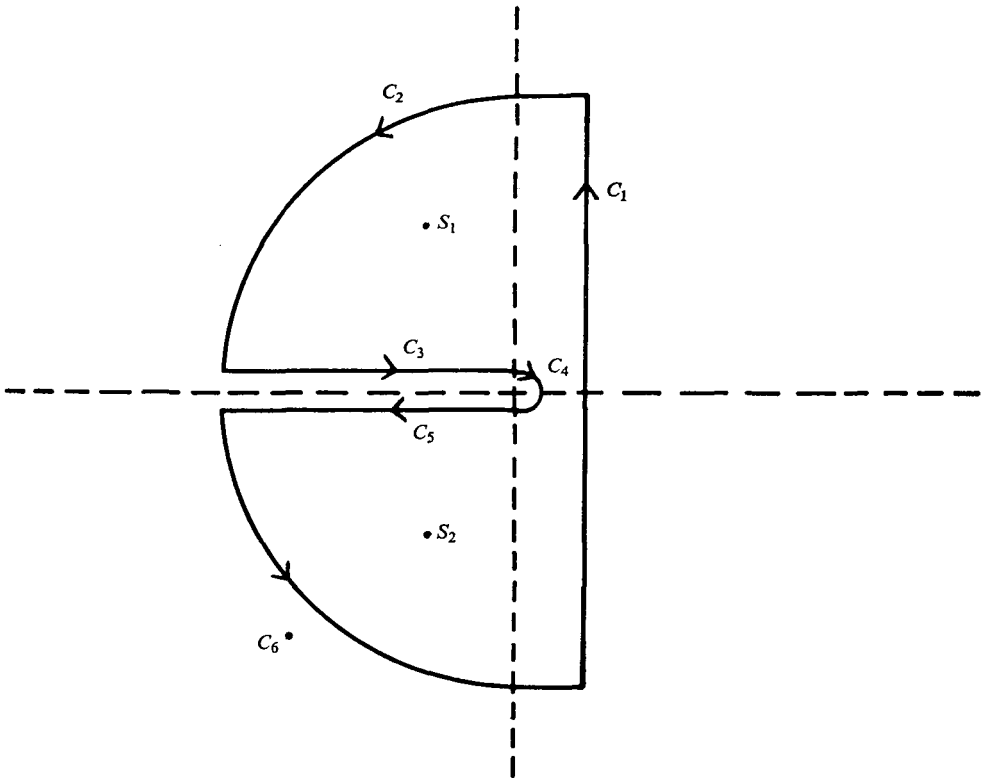

Fig. 1

Completion of Bromwich contour (C_1) for evaluation of $H(t/\tau)$

small there are no contributions from any of these arcs to the Cauchy integral. Thus in the limit we have

$$H(t/\tau) = (2\pi i)^{-1} \int_{C_1} \frac{e^{St/\tau} dS}{S^2 + 2\delta S^{1+\frac{1}{2}n} + 2\delta S^n}$$

$$= R(S_1) + R(S_2) - \left[\int_{C_3} + \int_{C_4} \frac{e^{St/\tau} dS}{S^2 + 2\delta S^{1+\frac{1}{2}n} + 2\delta S''} \right]$$

where $R(S_1)$, $R(S_2)$ denote the residues at S_1 and S_2. The residues are easily evaluated by the usual methods, while for the integrals along C_3 and C_4 we write respectively

$$S = qe^{i\pi} \quad S = qe^{-i\pi}$$

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with q a real integration variable in the range $0 < q < \infty$.   After some algebra we obtain the following formula for $H(z)$

$$H(z) = e^{-\delta'z} \left\{ [A \sin (\beta'z) - B \cos (\beta'z)] \right.$$

$$- n \left[\delta\beta^{-2}M(\beta z) + \int_0^z M(\beta y)dy - \tfrac{1}{4}\pi\beta^{-1} \right]$$

$$\left. + n\delta \int_0^\delta \frac{(2-\delta+x)}{(x^2+\beta^2)^2} e^{xz}dx \right\} \tag{3.12}$$

In this expression

$$\delta' = \delta + n[\tfrac{1}{4}\delta \log(2\delta) + \beta(\pi - \tan^{-1}(\beta/\delta)] + 0(n^2)$$
$$\beta' = \beta + n[\tfrac{1}{4}\beta \log(2\delta) - \tfrac{1}{2}\delta(\pi - \tan^{-1}(\beta/\delta))] + 0(n^2)$$
$$A = \beta^{-1}[1 + \tfrac{1}{2}n - \tfrac{1}{4}n \log(2\delta)] + 0(n^2)$$
$$B = -\tfrac{1}{2}n\beta^{-1}(\pi - \tan^{-1}(\beta/\delta)) + 0(n^2)$$
$$M(q) = \tfrac{1}{2}[1 + q\{\sin(q)ci(q) + \cos(q) si(q)\}]$$

where the logarithmic cosine and sine integrals $ci(q)$ and $si(q)$ are defined as in Erdelyi (1953).   In obtaining (3.12) we have assumed $n \ll 1$ and neglected terms of order n^2 compared to unity.

The inequality $n \ll 1$ (i.e. $\Delta \ll 1$) is characteristic of most real materials.   In the present use of the inequality we have made expansions of the type

$$(2\delta)^n \simeq 1 + n \log(2\delta)$$

and there is a limitation that δ should be neither too small nor too large.   For values of δ obtaining in the experimental situation discussed previously, the expansion is valid.

The damped harmonic terms in (3.12) derive from the residues at the poles, the remaining terms originate from the line integrals along C_3 and C_4.   The integral remaining in (3.12) possesses finite limits and is evaluable only by numerical or approximate analytic methods.   In the example considered below the evaluation was based on the result $\delta < \beta$ permitting an expansion of $(x^2 + \beta^2)^{-2}$ as a power series in x^2, and leading to integrals expressible in terms of elementary functions.

We have evaluated as a function of (t/τ) the dimensionless displacement

$$(m + \tfrac{1}{2}M)\xi/P_0\tau \equiv H(t/\tau)$$

for the case $M/m = 0.15, \quad n = 0.0637$

i.e. $\beta = 0.724, \quad \delta = 0.312, \quad \Delta = 0.1.$

The results are displayed in Fig. 2 and compared with the corresponding elastic case ($\Delta = 0$) for the same mass ratio (and hence the same values of β and δ). In the comparison we identify τ with $a \mid c_2$ for the elastic case.   In Fig. 2 the viscoelastic curve is estimated to be accurate to within 1%.
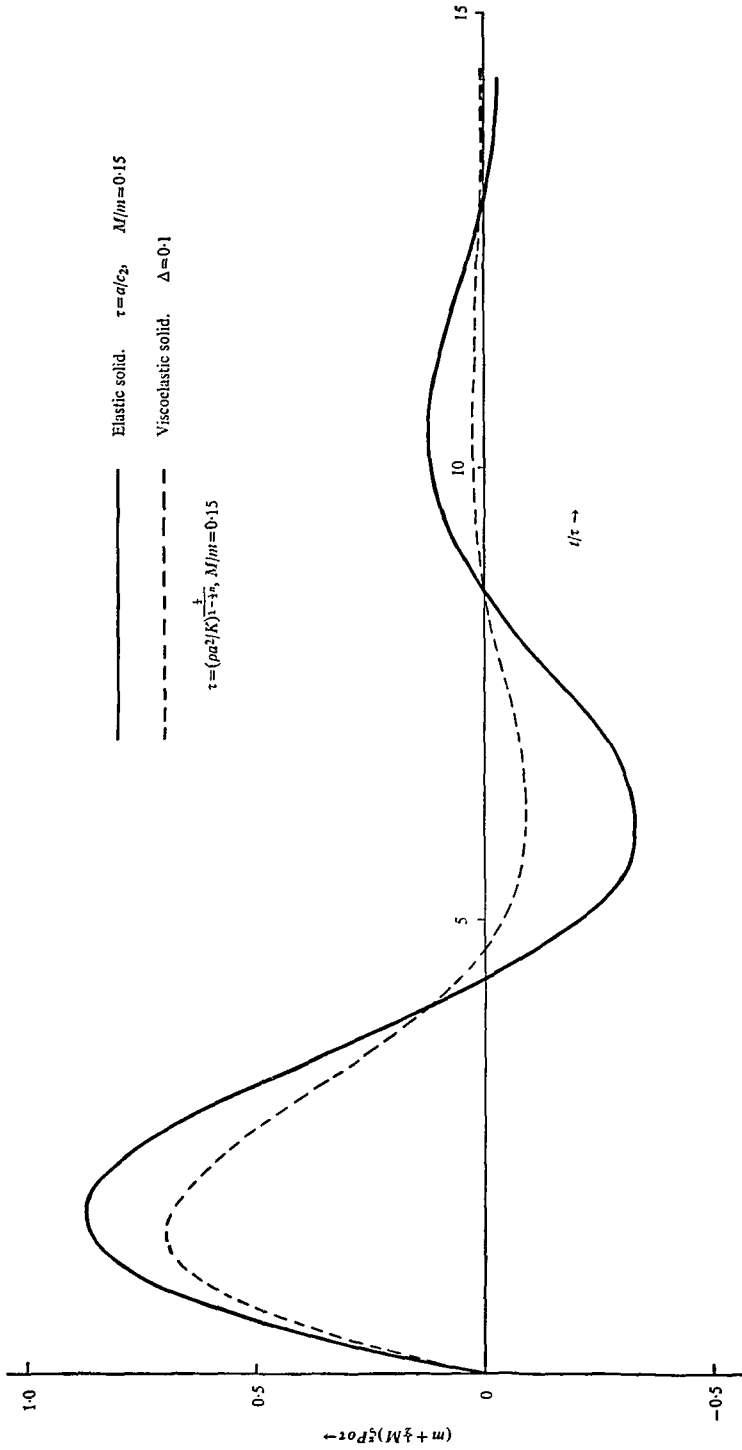
FIG. 2

Displacement-time plots for elastic and viscoelastic solid

For the mass ratio chosen for Fig. 2, viscoelastic plots for values of Δ other than 0·1 are obtainable with comparable accuracy by noting that the difference between the ordinates of the two curves is approximately a linear function of Δ.* [A different value of the mass ratio entails a new diagram with similar characteristics.]

It is of some importance to note characteristic features which would distinguish experimentally between elastic and viscoelastic materials. The most significant difference between the two curves of Fig. 2 appears to be the amplitude ratio of the first maximum to the first minimum. For $\Delta = 0·1$, the ratio is $\simeq 7·7$; for $\Delta = 0$ the ratio is $\simeq 2·6$. Thus the ratio is a sensitive measure of Δ whose determination defines completely the shape (though not the scale) of the normalised theoretical viscoelastic curve; comparison of experimental and theoretical time intervals between successive amplitude zeros (or turning points) then leads directly to a measure of the remaining viscoelastic parameter K.

A second characteristic feature of the viscoelastic curve is the dominating presence of a term of order t^{-1} for sufficiently large t. In time this term, which derives from the integral in (3.12), swamps the remaining contributions to ξ and leads to an essentially monotone asymptotic approach to the abscissa. For the viscoelastic curve of Figure 2, the term in question is dominant for $t/\tau > 12$. However it seems unlikely that this asymptotic behaviour could serve other than as a qualitative indication of viscoelastic behaviour.

Acknowledgements

Note added in proof

A recent paper by Chadwick and Trowbridge [*Proc. Camb. Phil. Soc.* **63** (1967), 1189] also solves the elastodynamic problem of Section 2 of the present paper. The approach is different and complements that given here; in particular Chadwick and Trowbridge give a more extensive discussion of the roots of the quartic equation $m\omega^2 = Q(\omega)$ together with some detailed numerical results on energy partition. As pointed out by Chadwick and Trowbridge the quadratic equation obtained from the quartic in the inconpressible limit was obtained by Lamb (*Proc. London Math. Soc. Ser.* I, **32** (1900), 120) in the study of the diffraction of elastic sheer waves by a spherical obstacle.

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* Since for small values of Δ and modest values of t/τ (3·12) may be expanded as a Taylor series in $n(\equiv 2\Delta/\pi)$.

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