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ON ISOMORPHISMS OF CONNECTED CAYLEY GRAPHS, III

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For a finite group G and a subset S of G which does not contain the identity of G, we use $\mathrm{Cay}(G,S)$ to denote the Cayley graph of G with respect to S. For a positive integer m, the group G is called a (connected) m-DCI-group if for any (connected) Cayley graphs $\mathrm{Cay}(G,S)$ and $\mathrm{Cay}(G,T)$ of out-valency at most m, $S^\sigma = T$ for some $\sigma \in \mathrm{Aut}(G)$ whenever $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$. Let $p(G)$ be the smallest prime divisor of $|G|$. It was previously shown that each finite group G is a connected m-DCI-group for $m \leqslant p(G)-1$ but this is not necessarily true for $m = p(G)$. This leads to a natural question: which groups G are connected $p(G)$-DCI-groups? Here we conjecture that the answer of this question is positive for finite simple groups, that is, finite simple groups are all connected 2-DCI-groups. We verify this conjecture for the linear groups $\mathrm{PSL}(2,q)$. Then we prove that a nonabelian simple group G is a 2-DCI-group if and only if $G = \mathrm{A}_5$.

1. INTRODUCTION

For a finite group G and a subset S of G which does not contain the identity of G, we define the *Cayley graph* of G with respect to S to be the directed graph $\mathrm{Cay}(G,S)$ with vertex set G and edge set $\{(a,b) | a,b \in G,\ ba^{-1} \in S\}$. A Cayley graph $\mathrm{Cay}(G,S)$ is called a *CI-graph* of G if, for any $T \subseteq G$, $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$ implies $S^\sigma = T$ for some $\sigma \in \mathrm{Aut}(G)$. (CI stands for *Cayley Isomorphism.*) Further, for a finite group G and a positive integer m, if every connected Cayley graph of G of out-valency at most m is a CI-graph, we call G a *connected m-DCI-group*; while if every Cayley graph of G of out-valency at most m is a CI-graph, we call G an *m-DCI-group*. This paper is a contribution to characterising (connected) m-DCI-groups.

There has been a lot of study on the problem of determining m-DCI-groups in the literature, see the surveys in [**1, 14, 18, 19**]. In particular, dependent on the finite simple group classification, a good description of 1-DCI-groups was obtained by Zhang [**24**], and further, a classification of m-DCI-groups for $m \geqslant 2$ was obtained by Praeger, Xu and the author [**16**]. One of reasons for investigating m-DCI-groups is to decide isomorphic

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classes of Cayley graphs. It is in general very difficult to determine whether or not two given Cayley graphs are isomorphic. However, if G is an m-DCI-group then two Cayley graphs $\mathrm{Cay}(G,S)$ and $\mathrm{Cay}(G,T)$ of valency at most m are isomorphic if and only if S and T are conjugate under $\mathrm{Aut}(G)$. The latter is often much easier to determine than the former. By the definition, a Cayley graph $\mathrm{Cay}(G,S)$ is connected if and only if S is a generating subset of G, that is, $\langle S\rangle = G$. Since a disconnected Cayley graph is a vertex disjoint union of isomorphic connected Cayley graphs of smaller order, in some sense only the isomorphism problem for connected Cayley graphs needs to be considered (see for example [**22**]). There have been some results on the problem of determining connected m-DCI-groups. Delorme, Favaron and Maheo [**6**] proved that all Abelian groups are connected 2-DCI-groups; Xu and Meng [**23**] completely classified the Abelian connected 3-DCI-groups. Some more general results were obtained by the author [**11, 12**]. For a finite group G, let $p(G)$ denote the smallest prime divisor of $|G|$. It was shown in [**11**] that

(1) *if G is a finite group then G is a connected m-DCI-group for $m \leqslant p(G) - 1$.*

Further, it was also proved that the conclusion (1) can be extended to $m = p(G)$ if G is Abelian. But the author [**13**] constructed an infinite family of groups which are not connected 2-DCI-groups, so (1) cannot be extended to $m = p(G)$ for arbitrary groups. A natural question arises here:

QUESTION 1.1. Which groups G are connected $p(G)$-DCI-groups?

We are inclined to believe that the answer of this question is positive for most groups, and in particular for simple groups, namely,

CONJECTURE 1.2. Finite simple groups are all connected 2-DCI-groups.

We show that this conjecture is true for the simple groups $\mathrm{PSL}(2,q)$.

THEOREM 1.3. *Let $G = \mathrm{PSL}(2,q)$, and let Γ be a connected Cayley graph of G of valency 2. Then*

(i) $G \trianglelefteq \mathrm{Aut}\,\Gamma$;

(ii) *G is a connected 2-DCI-group.*

The argument in the proof of this theorem given in Section 3 might be modified to prove that Conjecture 1.2 holds for some other families of nonabelian simple groups, for example, Suzuki groups $\mathrm{Sz}(q)$. But we do not think it can be used to prove Conjecture 1.2 completely. The next result gives a complete classification of simple 2-DCI-groups.

THEOREM 1.4. *A finite nonabelian simple group G is a 2-DCI-group if and only if $G = \mathrm{A}_5$.*

It is proved in [**16**, Theorem 1.2] that if G is an insoluble 2-DCI-group then $G = U \times V$ where $V = \mathrm{A}_5$ or $\mathrm{PSL}(2,8)$ such that $(|U|, |V|) = 1$ and all Sylow subgroups of U are homocyclic. Further, it is easy to see that any characteristic subgroup of G is also a 2-DCI-group. Thus we have an improvement on a result of [**16**]:

COROLLARY 1.5. *Suppose that G is an insoluble 2-DCI-group. Then $G = U \times A_5$ for some Abelian group U such that $(|U|, 2.3.5) = 1$ and all Sylow subgroups of U are homocyclic.*

After we draw together some preliminary results in Section 2, we prove Theorem 1.3 in Section 3. Then in Section 4, we prove Theorem 1.4.

2. PRELIMINARIES

This section collects several preliminary results which will be used. The notation and terminology used in the paper are standard (see, for example, [**4**]). For convenience, we list some of them here. Let $\mathbb{Z}_n$ denote the cyclic group of order n. For a group G and its a subgroup H, $|G : H|$ $(= |G| / |H|)$ denotes the index of H in G. For two groups G and H, $G \rtimes H$ denotes a semidirect product of G by H. Let Γ be a graph and let $\mathrm{Aut}\,\Gamma$ be the full automorphism group of Γ. Let $\Gamma(\alpha)$ be the neighbourhood of the vertex α of Γ, namely the set of all vertices of Γ which are joined to α. For a subgroup G of $\mathrm{Aut}\,\Gamma$, the graph Γ is said to be *G-vertex-transitive* or *G-arc-transitive* if G acts transitively on the set of vertices or on the set of arcs of Γ, respectively. For $\alpha \in V\Gamma$, let G_α denote the stabiliser of α in G, and let $G_\alpha^{\Gamma(\alpha)}$ denote the permutation group on $\Gamma(\alpha)$ induced by G_α. For a finite group G and its a subgroup H, we can construct arc-transitive graphs as follows (see [**20**] and [**17**]):

DEFINITION 2.1. Let G be a finite group. Suppose that there are a subgroup $H < G$ and an element $g \in G$. A graph $\Gamma = \Gamma(G, H, g)$ is defined as $V\Gamma = \{xH \mid x \in G\}$ and (xH, yH) is an arc of Γ if and only if $yx^{-1} \in HgH$.

Such a graph has the following properties.

LEMMA 2.2. ([**20**] and [**17**, Theorems 1 and 2].) *Let $\Gamma = \Gamma(G, H, g)$ be a graph defined as in Definition 2.1. Then*

(1) *$G \leqslant \mathrm{Aut}\,\Gamma$ and $G_\alpha = H$, where α is a vertex of Γ corresponding H;*
(2) *Γ is a G-arc-transitive graph of valency $|H : H \cap H^g|$;*
(3) *Γ is connected if and only if $\langle H, g \rangle = G$.*

For a Cayley graph Γ of G, the normaliser of G in $\mathrm{Aut}\,\Gamma$ is often useful for characterising the structure of Γ.

LEMMA 2.3. (See [**7**, Lemma 2.1]) *Let G be a finite group and S a subset of G, let $A := \mathrm{Aut}\,\mathrm{Cay}(G, S)$ and $\mathrm{Aut}(G, S) := \{\sigma \in \mathrm{Aut}(G) \mid S^\sigma = S\}$. Then $\mathrm{N}_A(G) = G \rtimes \mathrm{Aut}(G, S)$.*

Next we have a criterion for a Cayley graph to be a CI-graph, which was obtained by Babai [**3**], and also by Alspach and Parsons [**2**].

THEOREM 2.4. *Let Γ be a Cayley graph of a finite group G, and let $\mathrm{Sym}(G)$ denote the symmetric group on G. Then Γ is a CI-graph if and only if, for any $\tau \in \mathrm{Sym}(G)$ with $G^\tau \leqslant \mathrm{Aut}\,\Gamma$, there is $\sigma \in \mathrm{Aut}\,\Gamma$ such that $G^\sigma = G^\tau$.*

A graph Γ is called a *cover* of a graph Σ if there is a surjection ϕ from the vertex-set of Γ to the vertex-set of Σ which preserves adjacency and is such that, for each vertex α of Γ, the set $\Gamma(\alpha)$ and $\Sigma(\alpha^\phi)$ of neighbours of α and α^ϕ in Γ and Σ respectively have the same size. In this case, Σ is called a *quotient graph* of Γ. If Γ is a G-vertex-transitive graph and N is a normal subgroup of G which acts regularly on $V\Gamma$, then Γ has a quotient graph Γ_N, for which $V\Gamma_N$ is the set of all N-orbits on $V\Gamma$, and two vertices $U, V \in V\Gamma_N$ are adjacent in Γ_N if and only if there exist $\beta \in U$ and $\alpha \in V$ which are adjacent in Γ. The graph Γ_N is said to be *induced* by N.

LEMMA 2.5. *Let Γ be a connected G-vertex-transitive di-graph where $G \leqslant \mathrm{Aut}\,\Gamma$. Let N be a normal subgroup of G and $\alpha \in V\Gamma$. Assume that $G_\alpha^{\Gamma(\alpha)}$ is primitive and that N is not transitive on $V\Gamma$. Then either Γ is a cycle, or N is semi-regular on $V\Gamma$ and Γ_N is G/N-arc transitive of valency $|\Gamma(\alpha)|$.*

PROOF: Suppose that N is not semiregular on $V\Gamma$. Then $N_\alpha^{\Gamma(\alpha)} \neq 1$. Since $N_\alpha^{\Gamma(\alpha)} \trianglelefteq G_\alpha^{\Gamma(\alpha)}$, we have that $N_\alpha^{\Gamma(\alpha)}$ is transitive on $\Gamma(\alpha)$. Thus the quotient graph Γ_N is of valency 1, and so Γ is a cycle. Therefore, if Γ is not a cycle, then N is semi-regular on $V\Gamma$ and it is then easy to see that Γ_N is G/N-arc transitive of valency $|\Gamma(\alpha)|$. ∎

3. PROOF OF THEOREM 1.3

This section is devoted to proving Theorem 1.3. First we prove some simple properties of Cayley graphs of valency two.

PROPOSITION 3.1. *Let G be a finite nonabelian simple group, and let Γ be a connected G-vertex-transitive digraph of G of valency two.*

(1) *Let $A \leqslant \mathrm{Aut}\,\Gamma$ be such that $G \leqslant A$, and let $\alpha \in V\Gamma$. Let N be a normal subgroup of A. Then either N is a 2-group such that $|N| \bigm| |A_\alpha|$, or $G \leqslant N$ and if in addition N is minimal normal in A then N is nonabelian simple and is characteristic in A;*

(2) *Suppose further that Γ is a Cayley graph of G and that G is normal in $\mathrm{Aut}\,\Gamma$. Then Γ is a CI-graph.*

PROOF: (1) Since $G \cap N \trianglelefteq G$ and G is simple, either $G \cap N = 1$, or $G \leqslant N$. Assume first that $G \cap N = 1$. Then $|N|$ divides $|GN| / |G|$ which divides $|A| / |G| = |A_\alpha|$. Since the valency of Γ is 2, we have that A_α and so N is a 2-group. Assume now that $G \leqslant N$. Then N is insoluble and $|N : G| \bigm| |A : G| = |A_\alpha|$. Further assume that N is a minimal normal subgroup of A. Then $N = T_1 \times \ldots \times T_r$ where $T_1 \cong \ldots \cong T_r$ is nonabelian simple. Since $T_i \trianglelefteq N$, $T_i \cap G \trianglelefteq G$ for all $i \in \{1, \ldots, r\}$, and hence either $T_i \cap G = 1$ or $G \leqslant T_i$. If

$T_i \cap G = 1$ for some i, then $|\langle T_i, G\rangle| = |T_i|\,|G|$ divides $|A| = |GA_\alpha| = |G|\,|A_\alpha|\,/\,|G \cap A_\alpha|$, which is a contradiction since $|T_i| \nmid |A_\alpha|$. Thus $G \leqslant T_i$ for each i. It follows that $r = 1$ and so N is simple. This argument also proves that N is a unique insoluble minimal normal subgroup of A. Thus N is a characteristic subgroup of A.

(2) Let $A = \operatorname{Aut}\Gamma$. Then $A = G \rtimes A_\alpha$ such that A_α is a 2-group. Let $\tau \in \mathrm{Sym}(G)$ be such that $G^\tau \leqslant A$. If $G^\tau \neq G$ then since $G^\tau \cap G \trianglelefteq G^\tau$, we have $G^\tau \cap G = 1$. Thus $\langle G, G^\tau\rangle = G \rtimes G^\tau$, and so $|G^\tau| = |\langle G, G^\tau\rangle : G|$ divides $|A : G| = |A_\alpha|$, which is a contradiction since A_α is a 2-group. Therefore, $G^\tau = G$, and so by Theorem 2.4, Γ is a CI-graph. ∎

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3: By Proposition 3.1 (2), part (ii) follows from part (i). Thus we need only prove part (i). Suppose to the contrary that G is not normal in $\operatorname{Aut}\Gamma$. Our task in the rest is to seek a contradiction. Let $A = \operatorname{Aut}\Gamma$ and $\alpha \in V\Gamma$. Then $A = GA_\alpha$ such that $G \cap A_\alpha = 1$ and A_α is a nontrivial 2-group. In particular, G is a proper subgroup of A of a 2-power index. If A is a simple group then by a result of Guralnick [8], $(G, A) = (\mathrm{A}_{n-1}, \mathrm{A}_n)$ for some $n = 2^r$ with $r \geqslant 3$, which is not the case. Therefore, A is not a simple group.

Let M be the largest soluble normal subgroup of A. Then $M \cap G = 1$, and M is a 2-group (by Proposition 3.1). Let $\tilde{A} = A/M$, and let $\tilde{B}$ be a minimal normal subgroup of $\tilde{A}$. By the maximality of M, $\tilde{B}$ is insoluble, and therefore by Proposition 3.1 (1), $\tilde{B}$ is simple and $\tilde{G} := MG/M \leqslant \tilde{B}$. Note that $\tilde{G}$ is a subgroup of $\tilde{B}$ of 2-power index. If $\tilde{G} \neq \tilde{B}$, then by [8], we have that $(\tilde{G}, \tilde{B}) = (\mathrm{A}_{n-1}, \mathrm{A}_n)$ for some $n = 2^r$ with $r \geqslant 3$, which is not the case. Thus $\tilde{G} = \tilde{B}$. Let B be the full preimage of $\tilde{B}$ under $A \to \tilde{A}$. Then $B = M \rtimes G$ and B is characteristic in A. If G centralises M, then $B = M \times G$ and therefore, G is characteristic in B and so is normal in A, which is a contradiction. Thus G does not centralise M.

Take a (principal) series of subgroups of B:

$$1 = M_0 < M_1 < \ldots < M_t = M < B \trianglelefteq A,$$

where every $M_i \lhd B$ and M_i/M_{i-1} is a minimal normal subgroup of B/M_{i-1}. Since G does not centralise M, there exists $i \in \{0, 1, \ldots, t-1\}$ such that G centralises M_i but does not centralise M_{i+1}. We claim that $\overline{G} := M_iG/M_i$ does not centralise $\overline{M}_{i+1} := M_{i+1}/M_i$. Suppose to the contrary that $\overline{G}$ centralises $\overline{M}_{i+1}$. Let $F = \langle M_{i+1}, G\rangle$. Then $\overline{F} := F/M_i = \langle \overline{M}_{i+1}, \overline{G}\rangle = \overline{M}_{i+1} \times \overline{G}$. Thus $\overline{G}$ is characteristic in $\overline{F}$ and so the full preimage $M_i \rtimes G$ of $\overline{G}$ under $F \to \overline{F}$ is normal in F. Since G centralises M_i, we have $M_i \rtimes G = M_i \times G$. Thus G is characteristic in $M_i \rtimes G$ and so normal in F. Since $M_{i+1} \lhd F$ and $M_{i+1} \cap G = 1$, we have that $F = M_{i+1} \times G$ so G centralises M_{i+1}, which is a contradiction. Therefore, $\overline{G}$ does not centralise $\overline{M}_{i+1}$.

For convenience, write $\overline{M}_{i+1}$ as X, $\overline{G}$ as Y and the quotient graph Γ_{M_i} (induced by M_i) as Σ, respectively, and set $Z := X \rtimes Y$. It follows from Proposition 3.1 that $X = \mathbb{Z}_2^n$ for some $n \geqslant 1$. Further, X is a minimal normal subgroup of Z, Y ($\cong G$) does not centralise X and acts transitively on $V\Sigma$, and by Lemma 2.5, Σ is a Z-arc-transitive digraph of valency 2. Therefore, Y is irreducible on X. If X is not semiregular on $V\Sigma$ then the quotient graph Σ_X induced by X is a cycle and $\overline{Z} := Z/X \cong \mathrm{PSL}(2,q)$ is transitive on $V\Sigma$, which is not possible. Hence X is semiregular on $V\Sigma$. Since Y is transitive on $V\Sigma$, $|X| \mid |Y|$.

Suppose first that $q = 2^r$ for some $r \geqslant 1$. Then a Sylow 2-subgroup of Y ($\cong \mathrm{PSL}(2,q)$) is isomorphic to $\mathbb{Z}_2^r$ and any two Sylow 2-subgroups have trivial intersection (see for example [**21**, p. 417] and [**9**, p. 295]). Let $\overline{Z} = Z/X$ and $\overline{\Sigma} = \Sigma_X$. Then $\overline{Z} \cong \mathrm{PSL}(2,q)$ and $\overline{\Sigma}$ is a $\overline{Z}$-arc-transitive digraph of valency 2 (by Lemma 2.5). Thus $H := \overline{Z}_\beta$ is a 2-subgroup of $\overline{Z}$ where $\beta \in V\overline{\Sigma}$ and $\overline{Z}_\beta$ is the stabiliser of β in $\overline{Z}$. Now $\overline{Z} \cong Y$ and Y is transitive on $V\Sigma$. Since $|\overline{Z}_\beta| = |\overline{Z}| / |V\Sigma_X| = |\overline{Z}|(|V\Sigma| / |X|) = |X| \,.\, |Y| / |V\Sigma|$, we have that $|X| \mid |\overline{Z}_\beta| = |H|$. By Lemma 2.2, there exists $g \in \overline{Z}$ such that $|H : H^g \cap H| = 2$. Since any two Sylow 2-subgroups of $\overline{Z}$ have trivial intersection, $\overline{Z}$ has a unique Sylow 2-subgroup P containing H as a subgroup and either g normalises P, or $P^g \cap P = 1$. If $P^g \cap P = 1$ then $H^g \cap H = 1$ and so $|H| = 2$. Thus $X \cong \mathbb{Z}_2$ and so Y centralises X, which is a contradiction. If g normalises P, then $\overline{Z} = \langle H, g\rangle \leqslant \langle P, g\rangle < \overline{Z}$, which is also a contradiction.

Now suppose that q is odd, and let s be the order of a Sylow 2-subgroup of Y. It is easy to see that if $q \equiv 1 \pmod 4$ then s divides $q-1$, and if $q \equiv -1 \pmod 4$ then s divides $q+1$. Thus in either case $s \leqslant q+1$. Since $|X|$ divides $|Y|$, $2^n = |X| \leqslant s \leqslant q+1$. Since Y is irreducible on X, by [**10**, Theorem 5.3.9], we have $n \geqslant (q-1)/2$. Therefore, $2^{(q-1)/2} \leqslant 2^n \leqslant q+1$. Solving this inequality, we find $q \leqslant 7$, that is, $q = 5$ or 7. If $q = 5$ then $s = 4$ and so $|X| \mid 4$ but $Y \not\leqslant \mathrm{GL}(2,2)$, which is a contradiction. Therefore, we have $q = 7$ and $Z \cong \mathbb{Z}_2^3 \rtimes \mathrm{PSL}(2,7)$. Let $\overline{Z} = Z/X$ and $\overline{\Sigma} = \Sigma_X$. Then $|\overline{Z}_\beta| = 8$ where $\beta \in V\Sigma$. Thus $\overline{\Sigma}$ is of order 21 and 3-arc-transitive. By [**4**, Lemma 16.3], $\overline{\Sigma}$ is a Cayley graph, that is, $\overline{\Sigma} = \mathrm{Cay}(\mathbb{Z}_7 \rtimes \mathbb{Z}_3, S)$ for some subset S of $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$. Since $\overline{\Sigma}$ is 3-arc-transitive, it follows that S consists of elements of order 7. But such an S is such that $\langle S\rangle \cong \mathbb{Z}_7$, which is a contradiction to the connectivity of $\overline{\Sigma}$. This completes the proof of the theorem. ▯

4. Proof of Theorem 1.4

Here we shall prove that the only nonabelian simple 2-DCI-group is A_5. First we consider the alternating group A_4.

Lemma 4.1. *Let $G = A_4$. Then*

(i) *any connected Cayley graph of G of valency 2 is not 2-arc transitive;*

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(ii) *G is a 2-DCI-group.*

PROOF: It is known that $\mathrm{Aut}(G) = \mathrm{S}_4$, and so it follows that G is a 1-DCI-group and that all undirected Cayley graphs of G of valency 2 are CI-graphs. Thus we only need to consider subsets which have the form $S = \{a, b\}$ such that $|S| = 2$ and $\langle S \rangle = G$. Let $\Gamma = \mathrm{Cay}(G, S)$, and let $A = \mathrm{Aut}\,\Gamma$ and $\alpha \in V\Gamma$. If Γ is not arc-transitive, then $A_\alpha = 1$ and so Γ is a CI-graph by Theorem 2.4. If Γ is arc-transitive, then it follows that $o(a) = o(b) = 3$. Now it is easy to see that Γ is not 2-arc-transitive. Thus $|A_\alpha| = 2$ and so $|A| = |GA_\alpha| = 2\,|G|$. For any $\tau \in \mathrm{Sym}(G)$ such that $G^\tau \leqslant A$, if $G^\tau \neq G$ then $|G^\tau \cap G| = (|G|\,|G^\tau|)/\,|GG^\tau| = |G|\,/2$, which is a contradiction since G does not have a subgroup of index 2. Thus $G^\tau = G$ and Γ is a CI-graph by Theorem 2.4, so G is a 2-DCI-group. ∎

PROOF OF THEOREM 1.4: Assume that G is a nonabelian simple 2-DCI-group. By [**16**] (or [**15**]), $G = \mathrm{A}_5$ or $\mathrm{PSL}(2,8)$. Suppose that $G = \mathrm{PSL}(2,8)$. Then G has a subgroup $H \cong \mathrm{AGL}(1,2^3) = \mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$. By [**13**, Corollary 2.7], $\mathrm{AGL}(1,2^3)$ is not a connected 2-DCI-group. Hence there exist two Cayley graphs $\mathrm{Cay}(G,S)$ and $\mathrm{Cay}(G,T)$ of valency 2 such that $\langle S \rangle = \langle T \rangle = H$, $\mathrm{Cay}(H,S) \cong \mathrm{Cay}(H,T)$ and S is not conjugate in $\mathrm{Aut}(H)$ to T. It follows that $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$. If there exists $\sigma \in \mathrm{Aut}(G)$ such that $S^\sigma = T$, then $H^\sigma = H$ and so σ induces an automorphism τ of H which sends S to T, which is a contradiction. So $\mathrm{PSL}(2,8)$ is not a 2-DCI-group, and therefore, $G = \mathrm{A}_5$.

Conversely, we need to verify that A_5 is really a 2-DCI-group. Set $G := \mathrm{A}_5$, and let $\Gamma := \mathrm{Cay}(G,S)$ be of valency at most 2. By [**15**, Theorem 3], if Γ is undirected then Γ is a CI-graph, and on the other hand, by Theorem 1.3, if Γ is connected then Γ is a CI-graph. Further, since $\mathrm{Aut}(G) = \mathrm{S}_5$, any two elements of G of the same order are conjugate in $\mathrm{Aut}(G)$, so if $|S| = 1$ then Γ is a CI-graph. Thus we may assume that $S \neq S^{-1}$, $|S| = 2$ and $\langle S \rangle < G$. Let H be a maximal subgroup of G which contains S. By the Atlas [**5**], $H \cong \mathrm{A}_4$, D_{10} or S_3, and any maximal subgroup of G of order $|H|$ is conjugate to H. If $\langle S \rangle < H$ then $\langle S \rangle \cong \mathbb{Z}_5$ (since at this moment $S \neq S^{-1}$ and $|S| = 2$). Since all elements of order 5 are conjugate in $\mathrm{Aut}(G)$, it follows that Γ is a CI-graph of G. Thus we may further assume that $\langle S \rangle = H$. Then by [**3**] and Lemma 4.1, Γ is a CI-graph of H. Let $T \subset G$ be such that $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$. Then $\langle T \rangle \cong H$. Since all subgroups of G of order $|H|$ are conjugate to H, there exists $g \in G$ such that $T^g \subset H$. Thus $\mathrm{Cay}(H,S) \cong \mathrm{Cay}(H,T^g)$. Since Γ is a CI-graph of H, there exists $\alpha \in \mathrm{Aut}(H)$ such that $S^\alpha = T^g$. By the Atlas [**5**], $\mathrm{Aut}(H) \cong \mathbf{N}_{\mathrm{Aut}(G)}(H)/\mathbf{C}_{\mathrm{Aut}(G)}(H)$, and hence $S^\sigma = T^g$ for some $\sigma \in \mathbf{N}_{\mathrm{Aut}(G)}(H)$. Therefore, S is conjugate in $\mathrm{Aut}(G)$ to T, and so $\mathrm{Cay}(G,S)$ is a CI-graph of G. Hence A_5 is a 2-DCI-group, and this completes the proof of Theorem 1.4. ∎

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