Compositio Mathematica **117:** 33–43, 1999.

On the Existence of n-Harmonic Spheres

SHIGEO KAWAI[1], NOBUMITSU NAKAUCHI[2] and HIROSHI TAKEUCHI[3]
[1]*Department of Mathematics, Saga University, Saga 840, Japan.*
e-mail: kawai@kawais@cc.saga-u.ac.jf
[2]*Yamaguchi University, Yamaguchi 753, Japan*
[3]*Shikoku University, Tokushima 771-11, Japan*

(Received: 19 July 1996; accepted in final form: 2 October 1997)

Abstract. Using the bubbling argument of Sacks and Uhlenbeck, we prove the existence of n-harmonic maps from the n-sphere to Riemannian manifolds. An application is made to a problem concerning manifolds with strongly pth moment stable stochastic dynamical systems.

Mathematics Subject Classifications (1991): 58E20, 58G03.

Key words: p-harmonic map, Riemannian manifolds

1. Introduction

Let M, N be Riemannian manifolds. We assume that N is isometrically embedded in Euclidean space $\mathbb{R}^\ell$, and denote by A the second fundamental form. For $p \geqslant 2$, a *p-harmonic map* u: $M \to N$ is a map in $L^{1,p}(M, \mathbb{R}^\ell)$ such that $u(x) \in N$ a.e. and satisfies the equation

$$-\int_M \| \mathrm{d}u\|^{p-2}\, \mathrm{d}u \cdot \mathrm{d}\varphi + \int_M \| \mathrm{d}u\|^{p-2} A(u)(\mathrm{d}u,\, \mathrm{d}u)\, \varphi = 0,$$

for any $\varphi \in C_0^\infty(M, \mathbb{R}^\ell)$. This is characterized as a critical point of the *p-energy functional* $\mathrm{E}_p(u) = \int_M \| \mathrm{d}u\|^p$, in $\mathrm{L}^{1,p}(M, N) = \{u \in \mathrm{L}^{1,p}(M, \mathbb{R}^\ell) \,|\, u(x) \in N \text{ a.e.}\}$, if the value of this functional is finite. For 2-harmonic maps or harmonic maps, there are a lot of studies with applications to problems in geometry and physics (see Eells and Lemaire [3], [4]). Likewise, n-harmonic maps, where n denotes the dimension of M, seem to be objects with wide application, since the n-energy functional enjoys invariance property under conformal transformations on M.

The following existence theorem for n-harmonic maps is known.

THEOREM 0. *Let M, N be compact Riemannian manifolds. Let n denotes the dimension of M, and suppose that $\pi_n(N) = \{0\}$. Then each homotopy class of* $\mathrm{C}^1(M,\, N)$ *contains an n-harmonic map which minimizes the n-energy functional in the homotopy class.*

(Kb. 6) INTERPRINT: S.A. PIPS Nr.: 153654 MATHKAP
comp4110.tex; 4/05/1999; 11:39; v.7; p.1

Theorem 0 follows from the result of White [19] and the regularity theory. This was first proved in the unpublished paper by J. Jost [8]. He constructed, using a generalized Courant–Lebesgue lemma, a converging minimizing sequence in $\mathrm{C}^0(M,N)\cap \mathrm{L}^{1,n}(M,N)$.

When $\pi_n(N)\neq\{0\}$, we cannot expect that each homotopy class contains an n-harmonic map. In this case we need to consider the bubbling phenomena. The aim of this paper is to prove the following existence result for n-harmonic spheres using the bubbling argument of Sacks and Uhlenbeck [13]. Note that if $\pi_1(N)=\{1\}$, we can identify $\pi_n(N)$ with the totality of free homotopy classes of $\mathrm{C}^1(\mathrm{S}^n,N)$. We denote by $[u]$ for $u\in\mathrm{C}^1(\mathrm{S}^n,N)$ the free homotopy class represented by u. From now on, the word 'sphere' means the sphere with the standard metric, and we assume that the target manifold N is connected.

THEOREM 1. *Let N be a compact simply connected Riemannian manifold. Then for any $u\in\mathrm{C}^1(\mathrm{S}^n,N)$ $(n\geqslant 2)$, there exist a finite number of n-harmonic maps $u^{(1)},\dots,u^{(k)}\in\mathrm{C}^1(\mathrm{S}^n,N)$ which satisfy the following conditions:*

(1) $[u]=[u^{(1)}]+\cdots+[u^{(k)}]$.
(2) $\inf_{w\in[u]}\mathrm{E}_n(w)=\mathrm{E}_n(u^{(1)})+\cdots+\mathrm{E}_n(u^{(k)})$.
(3) $u^{(j)}$ *is a minimizer of* E_n *in* $[u^{(j)}]$ $(j=1,\dots,k)$.

Remark. If N is not simply connected, the same result holds up to the action of $\pi_1(N)$ on $\pi_n(N)$.

By an analogous method, as in Meeks and Yau [10], we get the following theorem.

THEOREM 2. *Let N be a compact simply connected Riemannian manifold. Then for every $n\geqslant 2$, there exist finite number of n-harmonic maps $f^{(1)}, f^{(2)},\dots\in\mathrm{C}^1(\mathrm{S}^n,N)$ such that*

(1) $[f^{(1)}],\ [f^{(2)}],\ \dots\neq 0$,
(2) $[f^{(1)}],\ [f^{(2)}],\ \dots$ *generate* $\pi_n(N)$,
(3) $\mathrm{E}_n(f^{(1)})=\inf\{\,\mathrm{E}_n(f)\mid [f]\neq 0\,\}$,
$\mathrm{E}_n(f^{(j)})=\inf\{\,\mathrm{E}_n(f)\mid [f]\notin\langle\,[f^{(1)}],\dots,[f^{(j-1)}]\,\rangle\,\}\quad (j\geqslant 2)$,

where the notation $\langle a,b,c,\dots\rangle$ denotes the subgroup generated by the subset $\{a,b,c,\dots\}$.

The organization of this paper is as follows: After preparing some lemmas in Section 2, we prove Theorems 1 and 2 in Section 3. In the final section, we consider a problem concerning manifolds with strongly pth moment stable stochastic dynamical systems, and give an alternative proof for a result in Elworthy and Rosenberg [5].

2. Preliminaries

In this section, we collect several properties of p-harmonic maps and n-energy which will be used later in the proofs of theorems.

PROPOSITION 1. *Let M and N be compact Riemannian manifolds, and let $p > n = \dim M$. Then each connected component of $\mathrm{C}^1(M, N)$ contains a p-harmonic map which minimizes the p-energy functional E_p in this component.*

Proof. This is a well-known fact and we only sketch the outline. By the assumption $p > n$, the Sobolev embedding from $\mathrm{L}^{1,p}(M, \mathbb{R}^\ell)$ to $\mathrm{C}^{0,\delta}(M, \mathbb{R}^\ell)$ is a compact map for some small $\delta > 0$. Hence the p-energy E_p satisfies the Palais–Smale condition (C), and it attains its minimum in every connected component of $\mathrm{L}^{1,p}(M,\ N)$. From regularity results this minimizer belongs to $\mathrm{C}^1(M,\ N)$. Since the two spaces $\mathrm{L}^{1,p}(M,\ N)$ and $\mathrm{C}^1(M,\ N)$ have the same homotopy type, we get the desired conclusion. □

PROPOSITION 2. *Let M and N be compact Riemannian manifolds. Assume that M has nonnegative Ricci curvature and $2 \leqslant p \leqslant n$ where $n = \dim M$. Then there exists a positive constant ε_1 which depend only on n, p and the Sobolev constant of M, such that if a p-harmonic map $u\colon M \to N$ satisfies $\int_M \|\,\mathrm{d}u\|^n < \varepsilon_1$, then u is a constant map.*

Proof. This is Theorem 1 in Nakauchi and Takakuwa [11]. The manifold M is assumed to be complete and noncompact with infinite volume in [11]. However, the proof can be easily modified when M is compact. (See the proof of Proposition 1 in [11].) □

PROPOSITION 3. *Let M and N be compact Riemannian manifolds, and a sequence $\{p_j\}$ satisfy $p_j > n$, $\lim_{j\to\infty} p_j = n$, where $n = \dim M$. If a sequence of p_j-harmonic maps $u_j\colon M \to N$ of class C^1 satisfies $\int_M \|\,\mathrm{d}u_j\|^n \leqslant C$, for some constant C, then there exists a (possibly empty) finite subset $\mathcal{S}$ of M and a positive constant ε_2 such that*

(1) *a subsequence of $\{u_j\}$ converges to an n-harmonic map $u_\infty\colon M \to N$ of class C^1 uniformly on any compact subset of $M \setminus \mathcal{S}$,*
(2) *each point $x \in \mathcal{S}$ is characterized by the inequality $\liminf_{j\to\infty} \int_{B_\rho(x)} \|\,\mathrm{d}u_j\|^n \geqslant \varepsilon_2$ for any $\rho > 0$.*

Proof. If p_j is a constant not greater than n, this is Theorem 2 in Nakauchi and Takakuwa [11]. We have only to note that the proof, which uses Moser's iteration method, also works well in our case. In addition, a weakly n-harmonic map u_∞ which is stated to be of class C^1 on $M \setminus \mathcal{S}$ in [11] is in fact C^1 on M by the removable singularity theorem of Duzaar and Fuchs [1]. □

In the rest of this paper, we set $\varepsilon^* = \min\{\varepsilon_1, \varepsilon_2\}$.

https://doi.org/10.1023/A:1000644910148 Published online by Cambridge University Press

LEMMA 1. *Let N be a compact manifold and n be a positive integer. Suppose that for $\{p_j\}$ with $p_j \geqslant n$ and $p_j \to n$, a sequence $u_j\colon \mathrm{S}^n \to N$ of p_j-harmonic maps with uniformly bounded n-energy is contained in a homotopy class α. If $\mathcal{S} \neq \emptyset$, then there exist a nontrivial n-harmonic map $u^{(1)}$ and a sequence $\bar{u}_j$ of C^1-maps such that for an arbitrary small constant ε_3,*

$$\alpha = [u^{(1)}] + [\bar{u}_j], \qquad \mathrm{E}_n(u^{(1)}) + \mathrm{E}_n(\bar{u}_j) \leqslant \mathrm{E}_n(u_j) + \varepsilon_3,$$

for large j.

Proof. From Proposition 3, after passing to a subsequence, we conclude that $\{u_j\}$ converges in $\mathrm{C}^1_{\mathrm{loc}}(\mathrm{S}^n \setminus \mathcal{S}, N)$ to an n-harmonic map. Since $\mathcal{S}$ is not empty, we set $\mathcal{S} = \{x_1, x_2, \ldots, x_l\}$ $(l \geqslant 1)$.

Let us take a small ball $B_{r_1}(x_1)$ of radius r_1 centered at $x_1 \in \mathcal{S}$, where r_1 is so small that $x_2, \ldots, x_l \notin B_{2r_1}(x_1)$. We set $\rho_j = \sup\{\|\,\mathrm{d}u_j\|(x) \mid x \in B_{r_1}(x_1)\}$, and take $x^{(j)} \in \overline{B_{r_1}(x_1)}$ with $\|\,\mathrm{d}u_j\|(x^{(j)}) = \rho_j$. If ρ_j is bounded, then $\|\,\mathrm{d}u_j\|$ is uniformly bounded in a neighborhood of x_1, which contradicts the fact $x_1 \in \mathcal{S}$ and the characterization of the subset $\mathcal{S}$ in Proposition 3. Hence we get, passing to a subsequence, $\lim_{j\to\infty} \rho_j = \infty$. Furthermore, we may assume $x^{(j)}$ converges to some point in $\overline{B_{r_1}(x_1)}$. Since u_j converges in C^1 topology on every compact set in $\overline{B_{r_1}(x_1)} \setminus \{x_1\}$, we have $x^{(j)} \to x_1$.

For this sequence of numbers ρ_j, we take Euclidean balls $B_{\rho_j}(O)$ in $\mathbb{R}^n$ of radius ρ_j centered at the origin O, and define a sequence of maps $\hat{u}_j \in \mathrm{C}^\infty(B_{\rho_j}(O),\, N)$ by

$$\hat{u}_j(x) = u_j\left(\exp_{x^{(j)}}\left(\frac{r_1}{\rho_j}x\right)\right),$$

where $\exp_{x^{(j)}}$ denotes the exponential map at $x^{(j)}$. Then it follows that $\hat{u}_j$ is a critical point of the functional

$$\hat{E}_j\colon \{g \in L^{1,p_j}(B_{\rho_j}(O), N) \;\big|\; g_{|\partial B_{\rho_j}(O)} = \hat{u}_{j|\partial B_{\rho_j}(O)}\} \to \mathbb{R}$$

defined by the equation $\hat{E}_j(g) = \int_{B_{\rho_j}(O)} \|\mathrm{d}g\|_j^{p_j}\,\mathrm{d}V_j$. In this definition, $\|\quad\|_j$ (resp. $\mathrm{d}V_j$) denotes the norm (resp. the volume form) in the metric $\hat{h}_j = \rho_j^2(\exp_{x^{(j)}} \cdot J)^* h$ where $J\colon B_{\rho_j}(O) \to B_{r_1}(O)$ is the map $J(x) = (r_1/\rho_j)x$, and h is the standard metric of S^n. In addition we have

$$\sup\{\|d\hat{u}_j\|_j(x) \;\big|\; x \in B_{\rho_j}(O)\} \leqslant 1, \qquad \|\mathrm{d}\hat{u}_j\|_j(O) = 1.$$

Because $\rho_j \to \infty$, the sequence of metrics $\{\hat{h}_j\}$ converges to the flat metric on every fixed disk $B_R(O)$ as $j \to \infty$. Hence this uniform estimate of $\|d\hat{u}_j\|_j$ implies that a subsequence of $\{\hat{u}_j\}$ converges to an n-harmonic map with respect to the flat metric on every fixed disk $B_R(O)$. Then we conclude by a diagonal argument that a subsequence of $\{\hat{u}_j\}$ converges to a C^1 n-harmonic map v_0 in C^1-topology on every compact subset of $\mathbb{R}^n$. This map v_0 is nontrivial because $\|d\hat{u}_j\|_j(O) = 1$.

Since the Euclidean space $\mathbb{R}^n$ is conformally diffeomorphic to S^n minus the north pole P_N, we are able to regard the map v_0 as an n-harmonic map from $S^n \setminus P_N$ to N. Since $\int_{S^n} \|dv_0\|^n < \infty$, from the removable singularity theorem of Duzaar and Fuchs [1], this map v_0 extends to a nontrivial n-harmonic map $u^{(1)}\colon S^n \to N$ of class C^1. From Proposition 2, we have $E_n(u^{(1)}) \geqslant \varepsilon^*$.

To construct C^1 maps $\bar{u}_j$, let us fix an arbitrarily small number ε_4, and proceed as follows. For a number a with $0 \leqslant a < 1$, we define a subset S_a of $S^n = \{(x_1, \ldots, x_{n+1}) \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$ by $S_a = \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} \leqslant a\}$. Then there exists a number a sufficiently close to 1 such that

$$|\,E_n(u^{(1)}{}_{|S_a}) - E_n(u^{(1)})\,| \leqslant \varepsilon_4/2, \qquad \int_{\partial S_a} \|\nabla u^{(1)}{}_{|\partial S_a}\|^n \, dV_a \leqslant \varepsilon_4/d(a),$$

where dV_a, $d(a)$ denotes the volume form on ∂S_a and the spherical distance from ∂S_a to the North Pole P_N respectively. For this fixed number a, the sequence $\{\hat{u}_j\}$, considered as mappings from $S^n \setminus P_N$ to N, converges to $u^{(1)}$ on ∂S_a in C^1-topology.

From the definition, the map $\hat{u}_j$ on ∂S_a is nearly equal to u_j on the boundary of some small geodesic disk B_j centered at x_1. We now identify two spheres, one the domain of $u^{(1)}$ and the other the domain of u_j, by conformal transformations Ψ_j which map S_a to $S^n \setminus B_j$. Then $u_j \circ \Psi_j$ converges to $u^{(1)}$ in C^1 on ∂S_a, and corresponding points of images $u^{(1)}(\partial S_a)$ and $(u_j \circ \Psi_j)(\partial S_a)$ are connected by unique shortest geodesics $c(s)$ of N for large j. In addition, we can assume that these geodesics are defined on the interval $[0, \tau_j]$ and satisfy $\|\dot{c}(s)\| \leqslant 1$. Note also that $\lim_{j\to\infty} \tau_j = 0$.

Let us choose a_j ($\geqslant a$) so that the spherical distance of two boundaries of ∂S_a and ∂S_{a_j} is equal to τ_j. We consider a C^1 map $t_j\colon S_{a_j} \setminus S_a \to N$ which satisfies

$$t_j|_{\partial S_{a_j}} = u^{(1)}|_{\partial S_a}, \qquad t_j|_{\partial S_a} = u_j \circ \Psi_j|_{\partial S_a}$$

and maps the 'meridian curves' connecting two boundaries to geodesic curves $c(s)$. Then we have $\int_{\partial S_b} \|\nabla t_{j|\partial S_b}\|^n \, dV_b \leqslant 2\,\varepsilon_4/d(a)$ for every b with $a \leqslant b \leqslant a_j$ and for large j. Hence we obtain

$$\int_{S_{a_j} \setminus S_a} \|\dot{c}(s)\|^n \leqslant \mathrm{vol}\,(S_{a_j} \setminus S_a) \leqslant \mathrm{vol}\,(S^{n-1})\,\tau_j$$

and

$$\int_{S_{a_j} \setminus S_a} \|\nabla_\partial t_j\|^n \leqslant (2\varepsilon_4/d(a))\tau_j,$$

https://doi.org/10.1023/A:1000644910148 Published online by Cambridge University Press

where $\mathrm{vol}(\,\cdot\,)$ and ∇_∂ denote the volume and the derivative in the direction of $\partial \mathrm{S}_b$ respectively. Thus the n-energy of t_j becomes small as $j \to \infty$.

Now we define $\bar{u}_j \colon \mathrm{S}^n \to N$ by the following equation:

$$\bar{u}_j = \begin{cases} u_j \circ \Psi_j & \text{on } \mathrm{S}_a, \\ t_j & \text{on } \mathrm{S}_{a_j} \setminus \mathrm{S}_a, \\ u^{(1)} \circ \Phi_j & \text{on } \mathrm{S}^n \setminus \mathrm{S}_{a_j}. \end{cases}$$

Here the map Φ_j is a conformal transformation from $\mathrm{S}^n \setminus \mathrm{S}_{a_j}$ to $\mathrm{S}^n \setminus \mathrm{S}_a$, such that the points of $\partial(\mathrm{S}^n \setminus \mathrm{S}_{a_j})$ are mapped along the 'meridian curves'. We remark that the map $u^{(1)}$ on $\partial \mathrm{S}_a$ coincide with the map $\bar{u}_j$ on $\partial(\mathrm{S}^n \setminus \mathrm{S}_{a_j})$. Then it follows that $[u^{(1)}] + [\bar{u}_j] = [u_j]$. In addition for large j, $\mathrm{E}_n(u^{(1)})$ is nearly equal to $\mathrm{E}_n(u_j|_{B_j})$, and $\mathrm{E}_n(\bar{u}_j)$ is nearly equal to $\mathrm{E}_n(u_j|_{\mathrm{S}^n \setminus B_j})$. Hence we have the estimate

$$\mathrm{E}_n(u^{(1)}) + \mathrm{E}_n(\bar{u}_j) \leqslant \mathrm{E}_n(u_j) + \varepsilon_3$$

for large j which completes the proof. □

LEMMA 2. *Let ϕ_1 and ϕ_2 be C^1 maps from S^n to a Riemannian manifold N, and ε be an arbitrary positive number. Then there exists a C^1 map $\phi \colon \mathrm{S}^n \to N$ such that $\mathrm{E}_n(\phi) < \mathrm{E}_n(\phi_1) + \mathrm{E}_n(\phi_2) + \varepsilon$, $[\phi] = [\phi_1] + [\phi_2]$ hold.*

Proof. In the beginning, we consider the connected sums of two spheres by long thin tubes, and show that they are conformally equivalent to S^n. Fix a point P of S^n and consider the polar geodesic coordinate $(r, \theta) \in \mathbb{R}_+ \times \mathrm{S}^{n-1}$ in a neighborhood of P where $\mathbb{R}_+$ and S^{n-1} denote the set of positive real numbers and $(n-1)$-dimensional unit sphere respectively. With respect to this coordinate, the standard metric g_0 on the unit sphere is written as $g_0 = \mathrm{d}r^2 + (\sin r)^2\, \mathrm{d}\theta^2$. For every small $r_0 > 0$, the metric g_1 in a neighborhood of the origin (origin deleted) defined by

$$g_1 = \left(\frac{\sin r_0}{\sin r}\right)^2 g_0 = \left(\frac{\sin r_0}{\sin r}\right)^2 \mathrm{d}r^2 + (\sin r_0)^2\, \mathrm{d}\theta^2$$

is conformally flat and isometric to the product of $\mathbb{R}$ and the sphere of radius $\sin r_0$. Indeed an isometry is given by

$$(r, \theta) \mapsto \sin r_0 \left(\int \frac{\mathrm{d}r}{\sin r}, \theta\right).$$

Let us take the following continuous function $\bar{f}$ on $\mathrm{S}^n \setminus P$:

$$\bar{f} = \begin{cases} 1 & \text{on } \mathrm{S}^n \setminus B_{r_0}(P), \\ \dfrac{\sin r_0}{\sin r} & \text{on } B_{r_0}(P) \setminus P \end{cases}$$

https://doi.org/10.1023/A:1000644910148 Published online by Cambridge University Press

and approximate it in C^0 sense by a smooth function f so that $f = \bar{f}$ on $B_{r_0}(P) \setminus P$. Then the metric $f^2 g_0$ on $\mathrm{S}^n \setminus P$ is conformally flat, and we can cut and paste the infinite tubes of these two objects to obtain a new manifold S^* which is also conformally flat. Note that we can make the tube as thin and long as possible.

Since S^* is simply connected and conformally flat, there exists a development ψ, i.e., a conformal immersion from S^* to S^n. Compactness of S^* implies that this map ψ is a conformal equivalence.

Now join the two images of ϕ_1 and ϕ_2 by a curve in N. Let us consider a map ϕ_0: $\mathrm{S}^* \to N$ which is equal to ϕ_1 and ϕ_2 on the two punctured spheres, and maps the tube along this curve. If the tube is sufficiently thin and long, we have

$$\mathrm{E}_n(\phi_0) < \mathrm{E}_n(\phi_1) + \mathrm{E}_n(\phi_2) + \varepsilon, \qquad [\phi_0] = [\phi_1] + [\phi_2].$$

This is because the norm of the derivative of ϕ_0 in the axial direction of the tube becomes arbitrarily small at each point, and the volume of the tube does not increase as we make the tube thin and long. By the conformal invariance of n-energy, we obtain the desired map $\phi = \phi_0 \circ \psi^{-1}$. □

3. Proofs of Theorems

Proof of Theorem 1. For any given $u \in \mathrm{C}^1(\mathrm{S}^n, N)$, we take a nonnegative integer m with

$$\frac{\varepsilon^*}{2} m \leqslant \inf_{v \in \alpha} \mathrm{E}_n(v) < \frac{\varepsilon^*}{2}(m+1),$$

where ε^* is the constant defined in the preceding section, and α is the homotopy class of u.

By Proposition 1, we can find a sequence $\{u_j\}_{j=1}^{\infty}$ of minimizing p_j-harmonic maps ($p_j \geqslant n,\ p_j \to n$) in the class α. This one is an E_n-minimizing sequence in α for the following reason: Let us take $v_0 \in \alpha$ with $\mathrm{E}_n(v_0) \leqslant \inf_\alpha \mathrm{E}_n + \varepsilon$, and denote by ω the volume of S^n. Then we have for large j,

$$\begin{aligned}
\mathrm{E}_n(u_j) &\leqslant (\mathrm{E}_{p_j}(u_j))^{n/p_j} \omega^{1-(n/p_j)} \\
&\leqslant (\mathrm{E}_{p_j}(v_0))^{n/p_j} \omega^{1-(n/p_j)} \\
&\leqslant (\mathrm{E}_n(v_0) + \varepsilon)^{n/p_j} \omega^{1-(n/p_j)} \\
&\leqslant (\inf_\alpha \mathrm{E}_n + 2\varepsilon)^{n/p_j} \omega^{1-(n/p_j)}.
\end{aligned}$$

Consequently $\mathrm{E}_n(u_j)$ converges to $\inf_{v\in\alpha} \mathrm{E}_n(v)$ as $j \to \infty$.

If $m = 0$, we obtain $\mathcal{S} = \emptyset$ and a subsequence converges to an n-harmonic map in $\mathrm{C}^1(\mathrm{S}^n, N)$. Indeed if $\mathcal{S} \neq \emptyset$ we get from Lemma 1, maps $u^{(1)}$ and $\bar{u}_j$ with

$$\alpha = [u^{(1)}] + [\bar{u}_j], \qquad \mathrm{E}_n(u^{(1)}) + \mathrm{E}_n(\bar{u}_j) < \varepsilon^*.$$

Since this map $u^{(1)}$ is a nontrivial n-harmonic map, we have $\mathrm{E}_n(u^{(1)}) \geqslant \varepsilon^*$ which is a contradiction. On the other hand the n-harmonic map to which a subsequence converges is a constant map from Proposition 2. Hence u is nulhomotopic and we complete the proof of Theorem 1 taking $k = 1$, $u^{(1)} \equiv$ constant.

When $m > 0$, we assume that the conclusion is true for any homotopy class γ with $\inf_{v\in\gamma} \mathrm{E}_n(v) < (\varepsilon^*/2)m$ and show that it also holds for α. For this purpose we apply Lemma 1 and obtain maps $u^{(1)}$ and $\bar{u}_j$. In the following, it is shown that the map $\bar{u}_j$ satisfies the inequality $\mathrm{E}_n(\bar{u}_j) < (\varepsilon^*/2)m$. Since the sequence $\{\bar{u}_j\}$ is contained in a fixed homotopy class, we denote this class by β. From Lemma 1, we have

$$\mathrm{E}_n(u^{(1)}) + \mathrm{E}_n(\bar{u}_j) \leqslant \mathrm{E}_n(u_j) + \varepsilon_3$$

for large j. Since $\{u_j\}$ is a minimizing sequence for E_n in α, the inequality

$$\mathrm{E}_n(u^{(1)}) + \inf_{v\in\beta} \mathrm{E}_n(v) \leqslant \inf_{v\in\alpha} \mathrm{E}_n(v)$$

holds.

If we assume that the strict inequality

$$\mathrm{E}_n(u^{(1)}) + \inf_{v\in\beta} \mathrm{E}_n(v) < \inf_{v\in\alpha} \mathrm{E}_n(v)$$

holds, then taking some $v_1 \in \beta$ and applying Lemma 2 for $u^{(1)}$ and v_1, we obtain a map $v_2 \in \alpha$ which satisfies $\mathrm{E}_n(v_2) < \inf_{v\in\alpha} \mathrm{E}_n(v)$. This is a contradiction and we have the equality

$$\mathrm{E}_n(u^{(1)}) + \inf_{v\in\beta} \mathrm{E}_n(v) = \inf_{v\in\alpha} \mathrm{E}_n(v).$$

The same argument shows that the map $u^{(1)}$ is a minimizer of E_n in $[u^{(1)}]$.

On the other hand, for sufficiently large j and small ε_3, we get

$$\begin{aligned} \mathrm{E}_n(\bar{u}_j) &\leqslant \inf_{v\in\alpha} \mathrm{E}_n(v) - \mathrm{E}_n(u^{(1)}) + \varepsilon_3 \\ &\leqslant \frac{\varepsilon^*}{2}(m+1) - \varepsilon^* + \varepsilon_3 \\ &< \frac{\varepsilon^*}{2}m. \end{aligned}$$

Consequently, by the inductive assumption, n-harmonic maps $u^{(2)}, \dots, u^{(k)}$ of class C^1 exist such that

$$\beta = [u^{(2)}] + \cdots + [u^{(k)}],$$

https://doi.org/10.1023/A:1000644910148 Published online by Cambridge University Press

$$\inf_{v\in\beta} \mathrm{E}_n(v) = \mathrm{E}_n(u^{(2)}) + \cdots + \mathrm{E}_n(u^{(k)})$$

and $u^{(i)}$ ($2 \leqslant i \leqslant k$) is a minimizer of E_n in $[u^{(i)}]$. Hence we conclude that

$$\alpha = [u^{(1)}] + \beta = [u^{(1)}] + [u^{(2)}] + \cdots + [u^{(k)}],$$

and

$$\begin{aligned} &\mathrm{E}_n(u^{(1)}) + \mathrm{E}_n(u^{(2)}) + \cdots + \mathrm{E}_n(u^{(k)}) \\ &\quad = \mathrm{E}_n(u^{(1)}) + \inf_{v\in\beta} \mathrm{E}_n(v) \\ &\quad = \inf_{v\in\alpha} \mathrm{E}_n(v). \end{aligned}$$

Thus the proof of Theorem 1 is completed. □

Proof of Theorem 2. To show the existence of $f^{(1)}$, let $\{f_j\}$ be a sequence in $C^1(\mathrm{S}^n, N)$ so that $[f_j] \neq 0$, $\mathrm{E}_n(f_j) \to \inf\{\mathrm{E}_n(f) \mid [f] \neq 0\}$. We may take a sequence $\{p_j\}$ with $p_j \geqslant n$ and $p_j \to n$, and assume that each f_j is a p_j-harmonic map which minimizes E_{p_j} in $[f_j]$. If a subsequence of $\{f_j\}$ converges in C^1-topology to an n-harmonic map, we can take this map as $f^{(1)}$. If it is not the case we obtain from Lemma 1, a sequence of maps $\{h_j\}$ in $C^1(\mathrm{S}^n, N)$ and a nonconstant n-harmonic map $g^{(1)}\colon \mathrm{S}^n \to N$ such that $[g^{(1)}] + [h_j] = [f_j]$, $\mathrm{E}_n(g^{(1)}) \geqslant \varepsilon^*$ and for every small ε_4, the inequality $\mathrm{E}_n(g^{(1)}) + \mathrm{E}_n(h_j) < \mathrm{E}_n(f_j) + \varepsilon_4$ holds for large j. Moreover the map $g^{(1)}$ gives the minimum of E_n in its free homotopy class. Clearly $[g^{(1)}] \neq 0$ and we define $f^{(1)} = g^{(1)}$.

Next we choose a sequence in $\{[f] \mid [f] \notin \langle [f^{(1)}] \rangle\}$ and repeat the same argument to define $f^{(2)}$. In the same manner, we can define $f^{(3)}$, etc. Since N is simply connected $\pi_n(N)$ is finitely generated. Hence this process has to stop in finite steps, and the proof of Theorem 2 is completed. □

4. Application

With regard to this section we refer to Elworthy and Rosenberg [5] for details.

Let $\{F_t\}_{t\geqslant 0}$ be the solution flow of a stochastic dynamical system on a compact Riemannian manifold and $\{TF_t\}$ the derivative flow. This system is said to be 'strongly pth moment stable' if the inequality

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in M} \mathbb{E}\|T_x F_t\|^p < 0$$

holds. In [5] the topological implications of the existence of such systems are considered.

THEOREM 3 (Corollary 2B1 in [5]). *If a compact manifold N admit a strongly p-th moment stable stochastic dynamical system, then $\pi_k(N) = 0$ for $k = 1, 2, \ldots, p$. In particular for $p \geqslant \dim N/2$, a pth moment stable stochastic dynamical system can only exist on homotopy spheres.*

To prove this they first consider the integral homology group. By the result of H. Federer and W. Fleming, each homology class is represented by a mass minimizing integral current. From the assumption of the theorem every integral current can be deformed to one with arbitrary small mass. Hence we have $H_k(N) = 0$ for $k = 1, 2, \ldots, p$. Then homotopy groups vanish from algebraic topology.

This can be also shown considering homotopy groups directly. Indeed the following is already proved in [5] (Theorem 1B).

If a compact Riemannian manifold N admits a strongly pth moment stable stochastic dynamical system, then every C^1 map from a Riemannian manifold to N of finite p-energy is homotopic to a map with arbitrary small p-energy.

Using this and Theorem 1, we easily get an alternative proof of Theorem 3.

References

1. Duzaar, F. and Fuchs, M.: On removable singularities of p-harmonic maps, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **7** (1990), 385–405.
2. Duzaar, F and Fuchs, M.: Existence and regularity of functions which minimize certain energies in homotopy classes of mappings, *Asymptot. Anal.* **5** (1991), 129–144.
3. Eells, J. and Lemaire, L.: A report on harmonic maps, *Bull. London Math. Soc.* **10** (1978), 1–68.
4. Eells, J. and Lemaire, L.: Another report on harmonic maps, *Bull. London Math Soc.* **20** (1988), 385–524.
5. Elworthy, K. D. and Rosenberg, S.: Homotopy and homology vanishing theorems and the stability of stochastic flows, *Geom. Funct. Anal.* **6**(1) (1996), 51–78.
6. Eells, J. and Sampson, J.H.: Harmonic mappings of Riemannian manifolds, *Amer. J. Math.* **86** (1964), 109–160.
7. Hardt, R. and Lin, F.-H.: Mappings minimizing the L^p norm of the gradient, *Comm. Pure Appl. Math.* **40** (1987), 555–588.
8. Jost, J.: A conformally invariant variational problem for mappings between Riemannian manifolds, Preprint of Centre for Math. Analysis, Australian National Univ., 1984.
9. Mou, L. and Yang, P.: Regularity for n-narmonic maps, *J. Geom. Anal.* **6**(1) (1996), 91–112.
10. Meeks, W. H. III and Yau, S.-T.: Topology of three dimensional manifolds and the embedding problems in minimal surface theory, *Ann. of Math.* **112** (1980), 441–484.
11. Nakauchi, N. and Takakuwa, S.: A remark on p-harmonic maps, *Nonlinear Anal*. **25**(2) (1995), 169–185.
12. Ochiai, T.: On properties of Douglas–Radó–Morrey solutions with some topics on existence of minimal surfaces (in Japanese). In: *Minimal Surfaces, Reports on Global Analysis* **4**(3) (1982), 1–311.
13. Sacks, J. and Uhlenbeck, K.: The existence of minimal immersions of 2-spheres, *Ann. of Math.* **113** (1981), 1–24.
14. Takeuchi, H.: Stability and Liouville theorems of P-harmonic maps, *Japan. J. Math*. **17** (1991), 317–332.
15. Takeuchi, H.: Some conformal properties of p-harmonic maps and a regularity for sphere-valued p-harmonic maps, *J. Math. Soc. Japan* **46** (1994), 217–234.

https://doi.org/10.1023/A:1000644910148 Published online by Cambridge University Press

16. Tolksdorf, P.: Everywhere-regularity for some quasilinear systems with a lack of ellipticity, *Ann. Mat. Pure Appl.* **134** (1983), 241–266.
17. Wei, S. W.: The minima of the p-energy functional, *Elliptic and Parabolic Method in Geometry* (Minneapolis, MN, 1994) pp. 171–203.
18. Wei, S. W.: On p-harmonic maps and their applications to geometry, topology, and analysis (preprint).
19. White, B.: Homotopy classes in Sobolev spaces and the existence of energy minimizing maps, *Acta Math.* **160** (1988), 1–17.