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A COMMON GENERALIZATION OF FUNCTIONAL EQUATIONS CHARACTERIZING NORMED AND QUASI-INNER-PRODUCT SPACES

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ABSTRACT. We determine the general solutions of the functional equation

$$f_1(x+y)+f_2(x-y)=f_3(x)+f_4(y), \quad x,y\in G$$

for $f_i: G \longrightarrow F$ $(i = 1,2,3,4)$, where G is a 2-divisible group and F is a commutative field of characteristic different from 2. The motivation for studying this equation came from a result due to Drygas [4] where he proved a Jordan and von Neumann type characterization theorem for quasi-inner products. Also, this equation is a generalization of the quadratic functional equation investigated by several authors in connection with inner product spaces and their generalizations. Special cases of this equation include the Cauchy equation, the Jensen equation, the Pexider equation and many more. Here, we determine the general solution of this equation without any regularity assumptions on f_i.

1. **Introduction.** In this paper, we determine the general solutions of a functional equation which includes the Cauchy equation

$$\text{(CE)} \qquad f(x+y)=f(x)+f(y),$$

the Pexider equation

$$\text{(PE)} \qquad f(x+y)=g(x)+h(y),$$

the Jensen equation

$$\text{(JE)} \qquad f(x+y)+f(x-y)=2f(x),$$

the quadratic (square-norm) equation

$$\text{(QE)} \qquad f(x+y)+f(x-y)=2f(x)+2f(y)$$

and many more as special cases. The main functional equation we shall investigate is the following:

$$\text{(FE)} \qquad f_1(x+y)+f_2(x-y)=f_3(x)+f_4(y), \quad x,y\in G,$$

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where $f_i\colon G \longrightarrow F$ $(i = 1,2,3,4)$ are unknown functions, G is a 2-divisible group and F is a commutative field of characteristic different from 2. Although we shall be using addition as the group operation, G is not necessarily commutative. When F is a field of either real numbers or complex numbers, the above equation transforms into

$$\text{(KE)} \qquad T_1(x+y)T_2(x-y) = T_3(x)T_4(y)$$

if we define $T_i(x) := \exp\big(f_i(x)\big)$. The equation (KE) was investigated by Kurepa [5] and Vajzović [6] assuming the unknown functions to be differentiable and measurable respectively (among other restrictions on the unknown functions and their domains). The motivation for studying the functional equation (FE) came from a result of Drygas [4] who obtained a Jordan and von Neumann type characterization theorem for quasi-inner products. For characterizations of inner product spaces involving functional equations interested readers should refer to [2] and [3]. In Drygas' characterization of quasi-inner product the functional equation

$$\psi(x) + \psi(y) = \psi(x-y) + 2\left\{\psi\left(\frac{x+y}{2}\right) - \psi\left(\frac{x-y}{2}\right)\right\}$$

played an important role. By replacing y with $-y$ in the above equation and adding the resultant to the above equation, one obtains

$$\psi(x+y) + \psi(x-y) = 2\psi(x) + \psi(y) + \psi(-y).$$

In [4], the solution of the above functional equation was not discussed. The functional equation (FE) is also a generalization of the above equation.

A map $A\colon G \longrightarrow F$ is a *homomorphism* of G into F if it is additive, that is $A(x+y) = A(x) + A(y)$. A *symmetric bihomomorphism* $H\colon G \times G \longrightarrow F$ is a map which is additive in each variable and satisfies $H(x,y) = H(y,x)$ for all $x, y \in G$.

We exclude, once and for all, the possibility that F has characteristic 2, but otherwise F may be arbitrary.

2. **Auxiliary results.** We shall make use of the following known results concerning the functional equations (JE) and (QE). Let G be an arbitrary group and F be a commutative field (of characteristic different from 2).

LEMMA 1 [1]. *The general solution $f\colon G \longrightarrow F$ of (JE) with $f(x+y) = f(y+x)$ for all $x, y \in G$ is of the form*

$$f(x) = A(x) + b,$$

where $A\colon G \longrightarrow F$ is additive (a homomorphism) and b is an arbitrary element of F.

LEMMA 2. *The general solutions $f, g, h\colon G \longrightarrow F$ of the functional equation*

$$\text{(2.1)} \qquad f(x+y) + f(x-y) = g(x) + h(y) + h(-y), \quad x, y \in G,$$

with

$$\text{(KC)} \qquad f(x+y+z) = f(x+z+y), \quad x, y, z \in G$$

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are given by

$$\text{(2.2)}\qquad \begin{cases} f(x) = H(x,x) - \frac{1}{2}A(x) - \frac{1}{2}b \\ g(x) = 2H(x,x) - A(x) - b - a \\ h(x) + h(-x) = 2H(x,x) + a, \end{cases}$$

where $H: G \times G \longrightarrow F$ is a symmetric bihomomorphism, $A: G \longrightarrow F$ is a homomorphism and a, b are arbitrary elements of the commutative field F.

PROOF. First note that f satisfies (KC) implies $f(x+y) = f(y+x)$, for all $x, y \in G$. By letting $y = 0$ in (2.1), we get

$$\text{(2.3)}\qquad g(x) = 2f(x) - 2h(0).$$

Evidently g also satisfies (KC) and (2.1) can be written as

$$\text{(2.4)}\qquad f(x+y) + f(x-y) - 2f(x) = h(y) + h(-y) - 2h(0)$$

for all $x, y \in G$. Now, $x = 0$ in (2.4) gives

$$\text{(2.5)}\qquad h(y) + h(-y) - 2h(0) = f(y) + f(-y) - 2f(0)$$

so that (2.4) becomes

$$\text{(2.6)}\qquad f(x+y) + f(x-y) - 2f(x) = f(y) + f(-y) - 2f(0).$$

We define $H: G \times G \longrightarrow F$ by

$$\text{(2.7)}\qquad 2H(x,y) := f(x+y) - f(x) - f(y) + f(0).$$

(Remember F is of characteristic different from 2.) From (2.7), (2.6) and (KC) we obtain

$$\begin{aligned} &2H(x+u,y) + 2H(x-u,y) \\ &\quad = f(x+u+y) + f(x-u+y) - \{f(x+u) + f(x-u)\} - 2f(y) + 2f(0) \\ &\quad = f(x+y+u) + f(x+y-u) - \{2f(x) + f(u) + f(-u) - 2f(0)\} - 2f(y) + 2f(0) \\ &\quad = 2f(x+y) - 2f(x) - 2f(y) + 2f(0) \\ &\quad = 4H(x,y). \end{aligned}$$

That is, $H(\cdot, y)$ satisfies (JE). Further, $2H(x+u,y) = 2H(u+x,y)$ and $H(0,y) = 0$. Thus by Lemma 1, $H(\cdot, y)$ is additive in the first variable. Also, H defined by (2.7) is symmetric. Hence H is a symmetric bihomomorphism. Now, $y = x$ in (2.7) and (2.6) give

$$\text{(2.8)}\qquad \begin{aligned} 2H(x,x) &= f(2x) - 2f(x) + f(0) \\ &= f(x) + f(-x) - 2f(0). \end{aligned}$$

By (2.5), this gives $h(x) + h(-x)$ as asserted in (2.2) with $a = 2h(0)$.

Define $l\colon G \longrightarrow F$ by

(2.9) $$l(x) := f(x) - f(-x).$$

Since f satisfies (KC) so does l and in particular $l(x+y) = l(y+x)$. From (2.9), (2.6) and (KC) we conclude

$$\begin{aligned} l(x+y) + l(x-y) & \\ &= f(x+y) + f(x-y) - \{f(-y-x) + f(y-x)\} \\ &= 2f(x) + f(y) + f(-y) - 2f(0) - \{2f(-x) + f(y) + f(-y) - 2f(0)\} \\ &= 2l(x). \end{aligned}$$

Since $l(0) = 0$, by Lemma 1, l is additive, that is

$$l(x) = A(x)$$

where A is additive and by (2.9)

(2.10) $$f(x) - f(-x) = A(x).$$

From (2.8) and (2.10), we obtain f as in (2.2) with $b = -2f(0)$ and then from (2.3), we get g as in (2.2). This completes the proof of the Lemma 2.

REMARK. If we define $Q\colon G \longrightarrow F$ by $Q(x) := f(x) + f(-x) - 2f(0)$ and use (2.6) and (KC), we see that Q satisfies the quadratic equation (QE) and from [1] (refer to Corollary 5 also) follows (2.8).

The following corollary is obvious from the above lemma.

COROLLARY 3. *The general solution $f\colon G \longrightarrow F$ of the functional equation*

(2.11) $$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y)$$

satisfying the the condition (KC), is given by

(2.12) $$f(x) = H(x,x) - A(x)$$

where $H\colon G \times G \longrightarrow F$ is a symmetric bihomomorphism and $A\colon G \longrightarrow F$ is a homomorphism.

3. **Solution of the functional equation (FE).** Now we proceed to determine the general solution of the functional equation (FE).

THEOREM 4. *Let G be a 2-divisible group and F be a commutative field of characteristic different from 2. The general solutions $f_i\colon G \longrightarrow F$ of (FE) with f_1 and f_2 satisfying the condition (KC), are given by*

(3.1) $$\begin{cases} f_1(x) = \frac{1}{2}H(x,x) - \frac{1}{4}(A_1 - A_2)(x) + \left(a - \frac{1}{2}b\right) \\ f_2(x) = \frac{1}{2}H(x,x) - \frac{1}{4}(A_1 + A_2)(x) - \left(a + \frac{1}{2}b\right) \\ f_3(x) = H(x,x) - \frac{1}{2}A_1(x) - (b+c) \\ f_4(x) = H(x,x) + \frac{1}{2}A_2(x) + c, \end{cases}$$

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where $H: G \times G \longrightarrow F$ *is a symmetric bihomomorphism,* $A_i: G \longrightarrow F (i = 1, 2)$ *are homomorphisms and* a, b, c *are arbitrary elements of* F.

PROOF. It is easy to verify that the form of f_i's in (3.1) satisfy the functional equation (FE). Now we proceed to demonstrate that the asserted form of f_i's in (3.1) is the only solution of (FE).

Interchanging y with $-y$ in (FE), we obtain

$$f_1(x-y) + f_2(x+y) = f_3(x) + f_4(-y). \tag{3.2}$$

Adding and subtracting (3.2) to and from (FE), we obtain the following system of functional equations:

$$g(x+y) + g(x-y) = 2f_3(x) + f_4(y) + f_4(-y) \tag{3.3}$$

$$h(x+y) - h(x-y) = f_4(y) - f_4(-y), \tag{3.4}$$

for all $x, y \in G$ where $g, h: G \longrightarrow F$ are defined by

$$g := f_1 + f_2 \text{ and } h := f_1 - f_2. \tag{3.5}$$

Solving (FE) is equivalent to solving the above system of functional equations. First, we solve (3.4). Define $k: G \longrightarrow F$ by

$$k(y) := f_4(y) - f_4(-y). \tag{3.6}$$

Then (3.4) reduces to

$$h(x+y) - h(x-y) = k(y) \tag{3.7}$$

for all $y \in G$. Interchanging y with $-y$ in (3.7) we see that k is an odd function, that is $k(y) = -k(-y)$ for all $x, y \in G$. We substitute $y = x$ in (3.7) to obtain

$$h(2x) = k(x) + h(0). \tag{3.8}$$

Note $f_3(x) = g(x) + f_4(0)$ and $f_4(y) = f_1(y) + f_2(-y) - f_3(0)$. Since f_1 and f_2 satisfy condition (KC) so are g, h, k, f_4 and f_3. From (3.7) and (KC), we conclude

$$\begin{aligned} k(y+v) + k(y-v) &\\ &= h(x+y+v) - h(x-v-y) + h(x+y-v) - h(x+v-y) \\ &= h\big((x+v)+y\big) - h\big((x+v)-y\big) + h\big((x-v)+y\big) - h\big((x-v)-y\big) \\ &= 2k(y) \end{aligned}$$

that is, k satisfies (JE), $k(x+y) = k(y+x)$ and $k(0) = 0$. So, by Lemma 1, k is additive, that is

$$k(y) = A_1(y), \quad y \in G, \tag{3.9}$$

where $A_1 : G \longrightarrow F$ is a homomorphism. From (3.8) and (3.9) and 2-divisibility of G, we obtain

$$h(x) = \frac{1}{2}A_1(x) + a. \tag{3.10}$$

where $a := h(0)$. From (3.5), (3.6), (3.9) and (3.10), we obtain

$$f_1(x) - f_2(x) = \frac{1}{2}A_1(x) + a \tag{3.11}$$

and

$$f_4(y) - f_4(-y) = A_1(y). \tag{3.12}$$

Now we return to the functional equation (3.3). By Lemma 2, we get

$$f_1(x) + f_2(x) = H(x,x) - \frac{1}{2}A_2(x) - \frac{1}{2}b_1, \tag{3.13}$$

$$f_3(x) = H(x,x) - \frac{1}{2}A_2(x) - \frac{1}{2}b_1 - \frac{1}{2}a_1, \tag{3.14}$$

$$f_4(x) + f_4(-x) = 2H(x,x) + a_1. \tag{3.15}$$

From (3.11)–(3.15), we obtain (3.1) and the proof of the theorem is complete.

The following corollary is obvious from the above theorem.

COROLLARY 5. *Let G be a group and F be commutative field of characteristic different from 2. The general solution of (QE) satisfying the condition (KC) is of the form $f(x) = H(x,x)$, where $H : G \times G \longrightarrow F$ is a symmetric bihomomorphism.*

REMARK. In Theorem 4, 2-divisibility of G is needed to find $f_1(x) - f_2(x)$ in (3.11). However, if $f_1 = f_2$ we do not require the 2 divisibility of G to solve (3.3) using Lemma 2.

REFERENCES

1. J. Aczel, J. K. Chung and C. T. Ng, *Symmetric second differences in product form on groups*, In: Topics in Mathematical Analysis, (ed. Th. M. Rassias), World Scientific Publ. Co. (1989), 1–22.
2. J. Aczel and J. Dhombres, *Functional equations in several variables*, Cambridge University Press, Cambridge, 1989.
3. J. Dhombres, *Some aspects of functional equations*. Chulalongkorn University Press, Bangkok, 1979.
4. H. Drygas, *Quasi-inner products and their applications*, In: Advances in Multivariate Statistical Analysis, (ed. A. K. Gupta), D. Reidel Publishing Co. (1987), 13–30.

5. S. Kurepa, *On the functional equation:* $T_1(t+s)T_2(t-s) = T_3(t)T_4(s)$, Publ. Inst. Math. (Beograd) **16**(1962), 99–108.

6. F. Vajzović, *On the functional equation:* $T_1(t+s)T_2(t-s) = T_3(t)T_4(s)$, Publ. Inst. Math. (Beograd) **18**(1964), 21–27.

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